Information Technology and Geometric Transformation
-Research on the Property of the Set Composed by Geometric Transformation via Information Technology

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Abstract

By using a square as an example, the relationship among its various transformations is investigated in this paper. In the study, the property of the set composed of the eight transformations of the square is studied by defining a new arithmetic. Based on the preliminary study, the investigation is enlarged from the symmetry of the square to the preliminary understanding of the "group". During the research, it has been found that the technology plays an important role, which shows us a clear understanding about the "transform" and the "relationship between the transformations". From the preliminary research, it reveals that there exists many properties in the set, the closure property, commutative law, single variable law, the associative law.

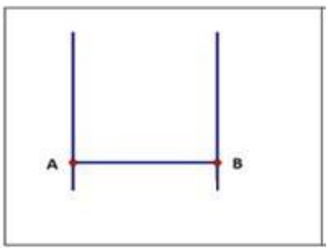
1. Introduction

It is well known that the square is almost a perfect geometry: under which there only exists one fundamental magnitude, which shows the "symmetry is a kind of realm". The square shows the symmetry properties including the axial symmetry and the central symmetry. Additionally, there are four symmetry axis and eight different geometric transformation existed in the square. Thus, the first question needs to be answer is that what is the relationship between the various transformations? To answer the question, the eight different transformation of square has been identified as as eight elements, which composes to a known mathematics concept - set. And this set shows the characteristics of the "group" actually. To simplify the study, the properties exists in the set which is composed of the eight transformation of square as the element were observed by using the Geometer's Sketchpad to carry out a series of rotation and translation transformation of square.

2. The symmetry transformation group of the square

2.1 closure property

A square was drawn by using "geometry drawing board". The procedures are described as follow:
a) composing a line segment AB (Figure2.1), and then drawing the perpendicular lines to AB through A, B respectively.



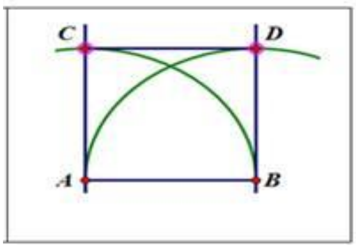


Figure 2.1 To draw the perpendicular lines to AB **Figure 2.2** To draw circles

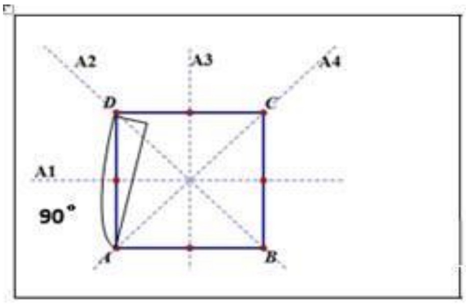
b) Drawing two circles centered on A, B, respectively, by using the length of AB as the radius, respectively, which is intersected of the perpendicular lines at C and D (Figure2.2).

c) Drawing a line between C and D. Then a square ABCD was established.

d) Finding all symmetry axis of the square from the midpoint of each side. (Figure 2.3).

e) Four symmetry transformation about the square ABCD from four symmetry axis were called A1, A2, A3, A4. And four rotation transformation about its center rotate 90 °, 180 °, 270 °, 360 ° were called A5, A6, A7, A8.

f) Establishing a set M which is composed of these eight different transformation. M = {A1, A2, A3, A4, A5, A6, A7, A8}.



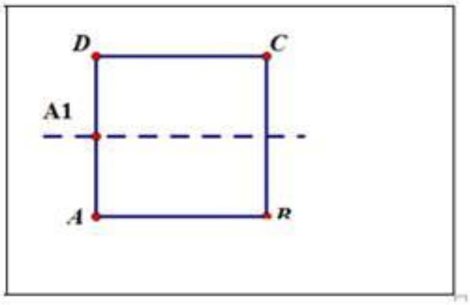


Figure 2.3 To find all symmetry axis of the square **Figure 2.4** To find the symmetry axis A1

Assumption 1: Any two elements in set M is carried out by "operation", the results of which are exactly as one element in this set (∀ Aj, Ai ∈ M, AjAi = Ak (Ak∈ M). To visualize the "operation", the geometric sketchpad was used to prove the assumption: ∀A1、A2∈M, A1A2=Ak（Ak∈M）. One example for the procedure of the visualized transformation are presented below.

a) to find the symmetry axis A1 of square ABCD (Figure 2.4)

Selecting the symmetry axis A1, through the "transformation" in the menu bar to "mark the mirror", as shown in figure 2.5.

b) to select the square ABCD again.

Selecting "reflection" (means symmetry) from "transformation menu" (Figure2.6)

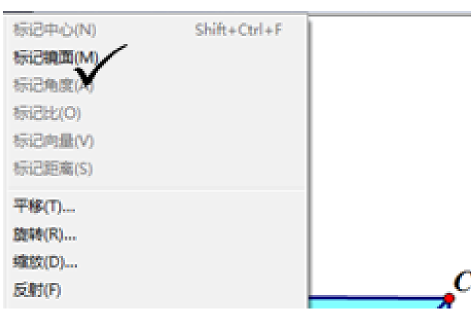


Figure 2.5 "Mark the mirror" **Figure** 2.6 Reflection

c) Finally we got the square DCBA after carrying out the transformation of the A1 (Figure2.7).

Based on the similar procedure, the transformation of A2 is shown in figure 2.8. Then transform the original image with the transformation of A5 by selecting center of square, and rotating to 90°(Figure 2.9,2.10).As shown in Figure 2.11, the graphics is the same as that after the transformation of

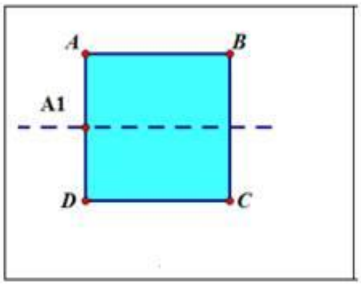


Figure 2.7 The transformation of A1

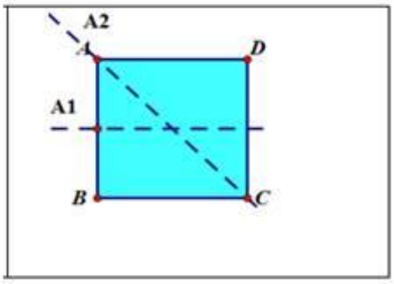


Figure 2.8 The transformation of A2

Figure 2.9 Selecting center of square

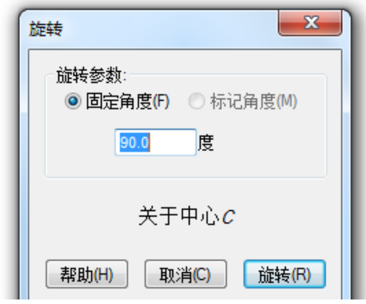


Figure 2.10 Rotating to 90 °

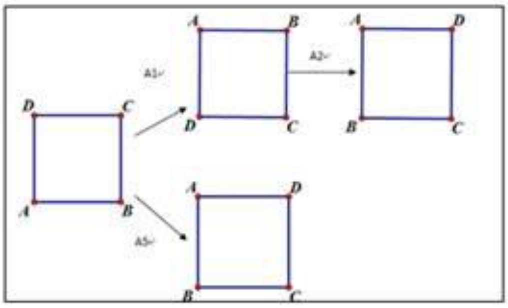


Figure 2.11 The transformation of A5

As shown in Table 2.1, similar results can be conducted under the same procedure. Therefore, the results confirmed our assumption that any two elements in set M is carried out by "operation", the results of which are exactly as one element in this set (∀ Aj, Ai ∈ M, AjAi = Ak (Ak∈ M)). Therefore, the set exhibits its closure property.

Table 2.1 closure property

/	A1	A2	A3	A4	A5	A6	A7	A8
A1	A8	A7	A6	A5	A4	A3	A2	A1
A2	A5	A8	A7	A6	A1	A4	A3	A2
A3	A6	A5	A8	A7	A2	A1	A4	A3
A4	A7	A6	A5	A8	A3	A2	A1	A4
A5	A2	A3	A4	A1	A8	A7	A6	A5
A6	A3	A4	A1	A2	A7	A8	A5	A6
A7	A4	A1	A2	A3	A6	A5	A8	A7
A8	A1	A2	A3	A4	A5	A6	A7	A8

2.2 single variable law

A new operation for any two elements in the set, which can be write as the multiplication form, such as: AiAj was defined. It was operated by carrying out the transformation of Ai first, then carry out the transformation of Aj. Assumption 2: Does an element is similar to "1" in multiplication and the result of the operation between any element and this element, is equal to themselves? (∀ Ai ∈ M, AiAn = Ai (An ∈ M)).

It can be clearly seen from the table2.1 that the compound operation of any element with A8 is equal to itself. The geometer sketchpad was also used to visualize the operation. One example is used by using A1A8. After the transformation of A8 (360°), the graphics we got is still the same as figure 2.12 (the transformation of A1). Therefore, the set fits “a single variable law” (∀ Ai∈M, AiA8=Ai).

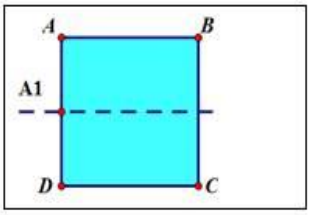


Figure 2.12 The transformation of A1(A8)

Table 2.2 Commutative law

/	A1	A2	A3	A4	A5	A6	A7	A8
A8	A1	A2	A3	A4	A5	A6	A7	A8

2.3 Commutative law

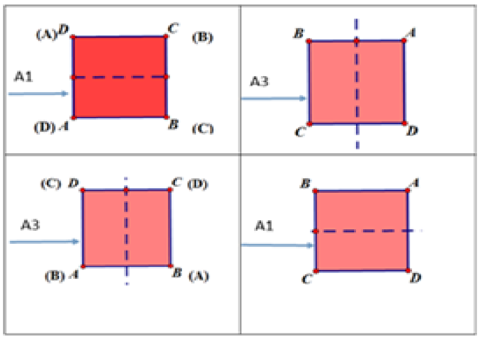
It has been found that the results of the compound operation of any element with A8, or of the compound operation of A8 and any element, are equal to itself. It is very similar to the commutative law. Therefore, it can be assumed that: ∀ Ai、Aj∈M, AiAj=AjAi. We found some successful situations as follows:

2.3.1 The symmetry axis are mutually perpendicular

In symmetry transformation, the case of the symmetry axis which are mutually perpendicular are A1, A3 and A2, A4 (Table 2.3). From the transformation by geometric sketchpad, it can be find that the transformation fits commutative law: A1A3 = A3A1, A2A4 = A4A2.Whether to carry out A1A3 first or A3A1, they are equivalent to the center of symmetry. (Figure 2.13 and 2.14). Additionally, from the result, it can be deducted that the "center symmetry" is very important.

Table 2.3 Commutative law

/	A1	A2	A3	A4
A1	A8	A7	A6	A5
A2	A5	A8	A7	A6
A3	A6	A5	A8	A7
A4	A7	A6	A5	A8



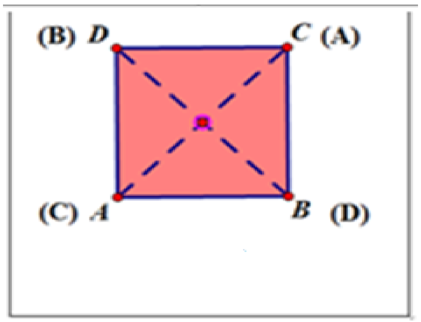


Figure 2.13 The transformation of A1A3andA3A1 **Figure 2.14** The center of symmetry

Therefore, it can be concluded that the transformation which equivalent to the center symmetry fits the exchange law in the set M.

2.3.2 Pure rotation

Table 2.4 Pure rotation

/	/	/	/	/	A5	A6	A7	A8
/	/	/	/	/	A4	A3	A2	A1
/	/	/	/	/	A1	A4	A3	A2
/	/	/	/	/	A2	A1	A4	A3
/	/	/	/	/	A3	A2	A1	A4
A5	A2	A3	A4	A1	A8	A7	A6	A5
A6	A3	A4	A1	A2	A7	A8	A5	A6
A7	A4	A1	A2	A3	A6	A5	A8	A7
A8	A1	A2	A3	A4	A5	A6	A7	A8

By observing the rotation transformation of A5 - A8, we found that they meet the commutative law (Table 2.4). One example of A5A6 and A6A5 are presented to visualize the operation. As shown in figure 2.15,2.16.By two steps: First carry out the A5A6 transformation: mark center - rotate 90 °-rotate 180 °;and then, carrying out the A6A5 transformation again: mark center - rotate 180 °-rotate 90°.

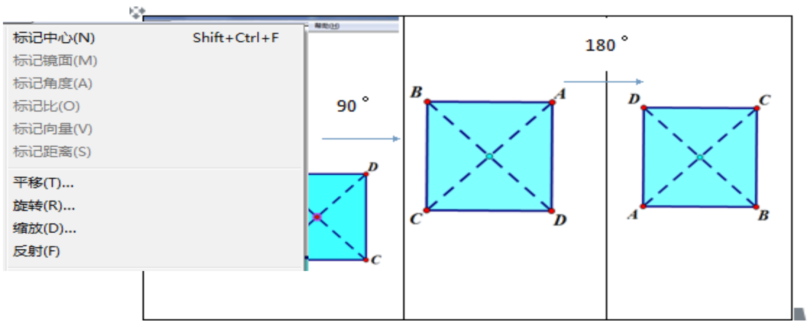


Figure 2.15 The transformation of A5A6

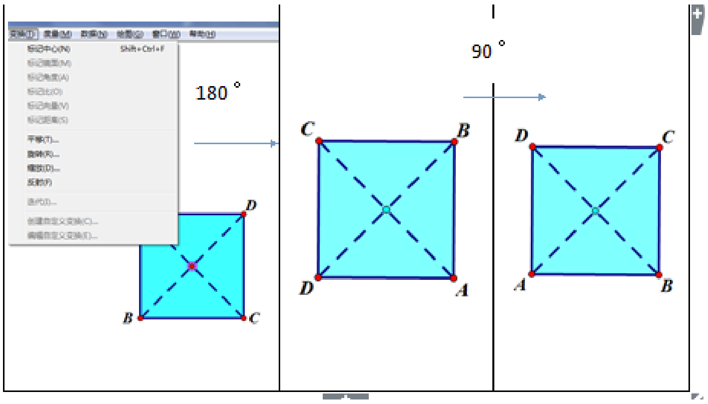


Figure 2.16 The transformation of A56A5

Turn 90 ° and turn 180 °, no matter which transformation you do first, are equivalent to turn 270 ° because rotation transformation is carried out in one direction. The results proved the “Pure rotation” meet Commutative law.

2.3.3 Single variable law meet commutative law

As discussed above, the compound operation of A8 with any elements or the compound operation of any elements with A8 are equal to any element itself. The transformation of A8 is clockwise rotate 360 °, no matter how many times it do A8 transformation, it always coincide with itself, thus, the transformation of A8 can be equal to static of the original graphics. Carrying out the transformation of Ai without any change of the original graphics, which reflects the result of the transformation of Ai. Vice versa the results proved the Single variable law meet commutative law.

2.3.4 The compound operation with A6 meet the commutative law

Table 2.5 AnA6

/	A1	A2	A3	A4	A5	A6	A7	A8
A6	A3	A4	A1	A2	A7	A8	A5	A6

Table 2.6 A6An

/	A6
A1	A3
A2	A4
A3	A1
A4	A2
A5	A7
A6	A8
A7	A5
A8	A6

As shown in Table 2.5 and 2.6, which is the compound operation of A6An and AnA6 respectively, the results of the operation are equivalent. The procedure are disrobed as below. .First carrying out the transformation of A1: selecting symmetry axis A1 - mark mirror - square - transform - reflection

(Figure 2.17). Then conducting the A6 (180 °) transform: select the center of rotation - tag center – selected graphics - transform - rotation - 180 ° (Figure 2.18). The results proves that A6An=AnA6.

Since [A6A1=(A1A3)A1]=[A1A6=A1(A1A3)]=A1(A3A1).,there is one question that (A1A3)A1=A1(A3A1) represent the operation of some elements in set M meet the associative law?

2.4 Associative law

Assumption :∀ Ai,Aj,Ak∈M,(AiAj)Ak=Ai(AjAk).
Verification by giving an example: (A1A2)A3=A1(A2A3)
Proof 1: From the data reported by table 2.1: (A1A2) = A5(A2A3) = A5; (A5A3) = A2;(A1A5) = A2;Therefore, (A1A2) A3 = A1 (A2A3).We can prove other situations in the same way.
Proof 2: using the geometry sketchpad show the transformation of symmetry and rotation, which us more intuitive. As shown in Figure 2.17, the graphs of (A1A2) A3 and A1 (A2A3) are same.

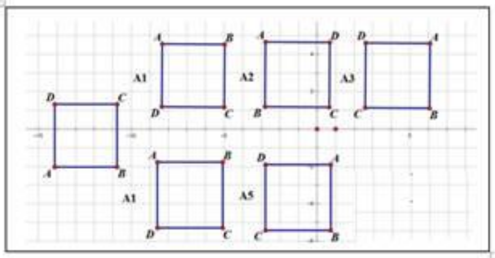


Figure 2.17 The transformation of (A1A2) A3 and A1 (A2A3)

Therefore: set M can fit the associative law. In a conclusion: set M fits the associative law, the single variable law, and shows the closure property. Part of the elements in set M meet the commutative law.

3. The symmetry transformation group of rectangle and parallelogram

3.1 Rectangle

The square is not only a special rectangular, but also a special diamond. And the rectangle and the diamond are divided to the special parallelogram. The relationship between them are expressed in Figure3.1, Figure 3.2shown the relationship between square, diamond, rectangle and parallelogram. As a result, the rectangular is only two symmetry axis than the square. Two symmetry transformations of rectangle ABCD from its two symmetry axis are recorded as B1, B2, respectively. Two rotation transformations which can coincide with itself are recorded as B3 and B4 respectively (B3 = rotate 180 ° B4 = rotate 360 °). Set N is composed of these different transformations. N = {B1, B2, B3, B4}.

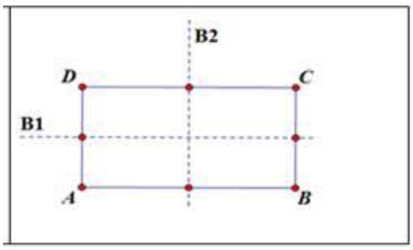


Figure 3.1 Rectangle and it's symmetry axis

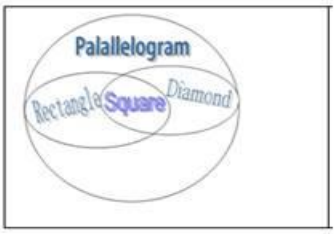


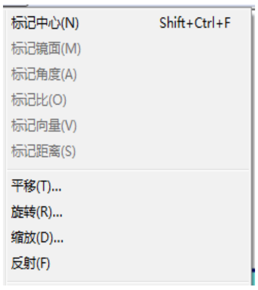
Figure 3.2 Relationship

From Table 3.1, we can verify the "single variable law": ∀ Bi ∈ N, BiBn = Bi (Bn ∈ N). The compound operation results of any element in set N with B4 are equal to any element itself. So the symmetry transformation group of rectangular also meet the "single variable law".

Table 3.1 "single variable law"

/	B1	B2	B3	B4
B1	B4	B3	B2	B1
B3	B2	B1	B4	B3

In the same way, we use the Geometric Sketchpad to visually show the change process by using B1B4 as an example (Figure 3.3) : first carrying out the transformation of B1: select the symmetry axis - transform - mark mirror reflection and obtained the graphs as figure 3.4.



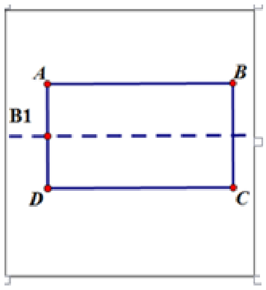
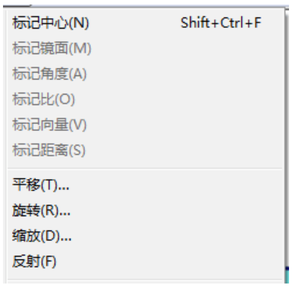


Figure 3.3 Mark mirror reflection **Figure 3.4** The transformation of B1

Then carry out the transformation of B4 (180 °): select the center of symmetry - transform - tag center - select the rectangular - transform - rotation - 180 °. As shown in figure 3.5 and 3.6 .Analog the study of symmetry transformation group of square, we can also prove that the symmetry transformation group of rectangular is closed and meet "associative law".



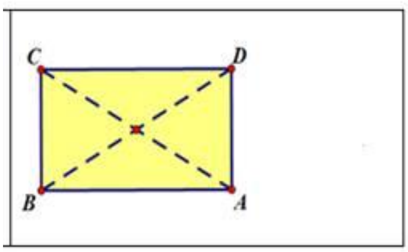


Figure 3.5 Rotation - 180 ° **Figure 3.6** The transformation of B4

3.2 Parallelogram

Parallelogram is more ordinary than rectangle, which has no symmetry axis. Therefore, the rotation transformation is the only transformation can be applied. The two rotation transformation of the parallelogram ABCD are written for C1 and C2 respectively. (C1 = rotate 180 °; C2 = rotate 360 °). They are composed of a set P. P = {C1, C2}.Obviously, the symmetry transformation group of parallelogram was much easier than that of the square. However, it also fit the "single variable law", under which the composite transform result of C2 and C1 is C1 itself, so set P is also closed. However. it didn't meet "associative law" due to its elements are less than 3.As table 3.2.

Table 3.2 single variable law

/	C1	C2
C1	C2	C1
C2	C1	C2

The achievement of the research

We have some preliminary achievements of the research about the “symmetry transformation group" by employing the square as an example:

1. Obtaining a deeper understanding of the basic concept of "Symmetry transformation": We improved the understanding from "symmetry transformation" to "group" for symmetric transformation----a set who met the "associative law" "single variable law" "closure". In this process, the technology played an important role.

2. Realizing the meaning of the operation: The operation can not only confined in the scope of the number, but also can be used in geometric elements. And the meaning of the operation is not just by combing the two elements for addition, subtraction, multiplication, and division, it can also be transformation.

3. Deepening the understanding of "identity": In symmetry transformation group, both commutative law and single variable law are good interpretations of the concept of identity and let us know the identity of the transformation.

4. Studying from static perspective to dynamic perspective: the graphs, such as, an isosceles triangle, an equilateral triangle, a square and so on, are static. We often study the properties of these "existed" graphs. And the study of "symmetry transformation", is to make these graphs alive.

5. More types of transformation can be observed in the graph with special shape.

6. The importance of technology: first of all, The Geometric Sketchpad improved the geometric accuracy and convenience. Moreover, it makes us feel the dynamic process of graphs more intuitive by using the transformation of "rotate", "symmetry" in geometric sketchpad. We can do the research on the connection between a series of transformation of the square better by using this methods. Finally, we have a visualized understanding of transformation of the relationship.

Acknowledgements The author would like to thank her mentors for their valuable comments and input.

References

[1] 《the Application of Symmetry Transformation Group, Group of Graphics》 http://www.docin.com/p-484478567.html

[2] 《Symmetry and Group》http://wenku.baidu.com/link?url=T8YiKLP03axzGeJJYKpr6ze