The Construction of Uniform Polyhedron with the aid of GeoGebra

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Abstract

The paper deals with various constructive methods of Uniform Polyhedron with the aid of GeoGebra and explores the connections among them, summarizing some feasible methods of the construction of polyhedrons. The functions and features of GeoGebra are well utilized and fully realized especially with the customize function to construct star polyhedron.

Polyhedron is an exquisite geometric construction and its symmetrical characteristic is as good as art. If the hidden mathematical theorems can be deeply explored, the beauty and sophistication will be widely appreciated. Various methods can be applied to demonstrate the beauty of polyhedron, and polyhedron modeling is a popular one.

We used to run several workshops to construct polyhedron with multi-media technology including Gsp, Cabri 3D. Recently, the effective application of 3D function of GeoGebra helps to construct polyhedron in a way better than ever, and we would like to give a brief introduction here.

1 Platonic Solids

To construct polyhedrons, Platonic solids are the best to start with.

Definition 1 *In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex.*

It has been verified that only five platonic solids exist: Tetrahedron, Hexahedron/Cube, Octahedron, Dodecahedron and Icosahedron. All the above can be constructed directly by using the built-in function of GeoGebra. For example, select 3D Graphics, set two points A, B. Insert Cube [A,B] into input box, and we'll get a cube.

If we desire a specific position, we can insert Cube[<Point>, <Point>, <Point>] (each is the vertex of one face of the cube. The second point must be an orthogonal vertex, which means if we command Cube[B,A,C], we can guarantee AB and AC are adjacent. Command Cube[B,A, C] and Cube[C,A,B], which must reside in both sides of face ABC respectively. Such nuance will be of great significant in later operation.

2 Archimedean Solid

Definition 2 *In geometry, an Archimedean solid is a highly symmetric, semi-regular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices.*

It has been verified there are 13 Archimedean solids, whose construction is to truncate Platonic solids.

2.1 Construct Through Truncation

If a corner of Platonic solid is properly truncated, we will get a regular polygon. With accurate calculation, we will get a polygon composed of the rest part of each face. Such construction is easy to imagine and calculate, and we will explain in detail with a cube.

First, look at the cube with one corner truncated(Figure1).

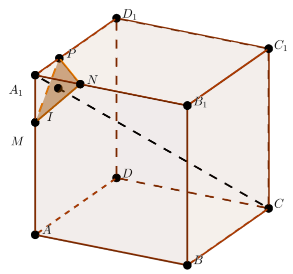

Figure 1:

If we use plane α perpendicular to body diagonal A_1C to truncate, we will get a regular triangle (if plane αkeeps moving downwards, we will get a hexagon). If each corner is truncated symmetrically, we will get a octagon. If not symmetrically, we will get a quadrilateral. Then when $A_1N = (\sqrt{2} - 1)A_1B_1$, the face of the cube will become a regular octagon.

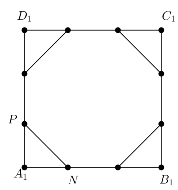

Figure 2:

And now the polyhedron becomes an Archimedean solid A_9—Truncated Cube(Figure 3).

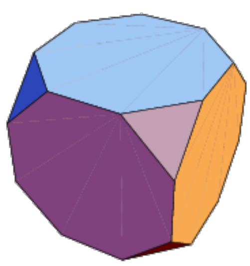

Figure 3: Truncated Cube

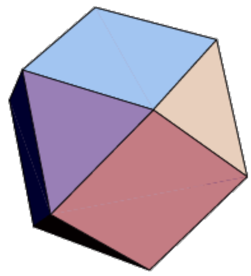

Figure 4: Cuboctahedron

If the truncation is expanded, which means changing every section into a square composed of midpoints, we will get another Archimedean solid A_1–Cuboctahedron(Figure 4).

It is named cuboctahedron because such construction is made through truncating the cube or the Octahedron. It is also the result of the intersection of the cube and Octahedron (which is the dual of cube.)

The steps are quite simple. Just truncate a corner of a cube and reflect.

1. Choose the 3D Graphics, set two points A,B, command cube[A,B], measure the length of edge AB and name it . Input m=l/(sqrt(2)+1).

2. Use “sphere with center through point” , set the center of cuboctahedron as the center of sphere with radius m, use “intersect” to get the intersection with each edge, and draw a face of a triangle.

3. Draw other faces with rotation and symmetry, and construct each face of regular octagon with the vertex of regular triangle.

Such cases will happen when we truncate other solids. We will get truncated tetrahedron, truncated octahedron, truncated icosahedron, truncated dodecahedron and icosidodecahedron.

3 Construction Through Edge Cutting and Corner Truncating

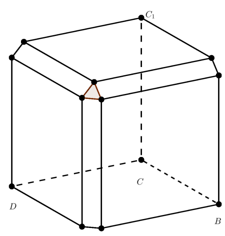

Figure 5:

First, cut the edges of the polyhedron, and truncate a corner where the edges meet. Then, we will get a Uniform Polyhedron. The design principles are as follows.

Figure 5 shows what it is like to cut the three edges meeting at one vertex. When the three edges are cut, we will get a regular triangle. If we do the same with all the eight vertices, the previous square will become a smaller one, and the vertices become regular triangles, with several rectangles in between squares and regular triangles. With proper adjustment, the rectangles will become squares. Then all the faces are regular polygons. And the adjustment is easy to be calculated. At last, we get a Small Rhombicuboctahedron(Figure 6).

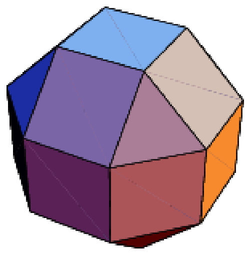

Figure 6: Small Rhombicuboctahedron

Apart from the cutting of all edges, we can also truncate corners furthermore.

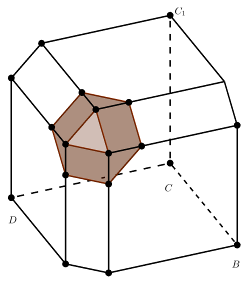

Figure 7:

After cutting the three edges of a vertex, we can truncate this geometry with a plane parallel to the triangle, and we will get a hexagon. If properly adjusted, each face of it is a regular polygon, and we can see faces with a point where three faces meet. And they are quadrilateral,hexagon and octagon. And such geometry is a great rhombicuboctahedron(Figure 8). It

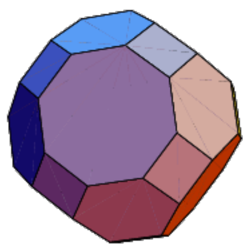

Figure 8: great rhombicuboctahedron

is worth mentioning that when we use GeoGebra to construct the above, we need not preset the exact data. Thus, we will get an adjustable dynamic graph. Afterwards, we can try calculating the specific data, which largely removes the blocks in thinking process and at the same time makes the problem easier. If we truncate a Dodecahedron, we will get truncated dodecahedron(Figure 9),small rhombicosidodecahedron(Figure 10),and great rhombicosidodecahedron(Figure 11). If we truncate an Icosahedron, we will get icosidodecahedron(Figure12)

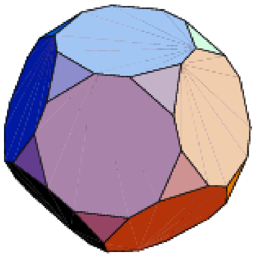

Figure 9: truncated dodecahedron

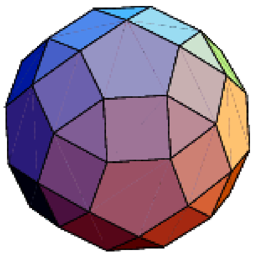

Figure 10: rhombic osidodecahedron

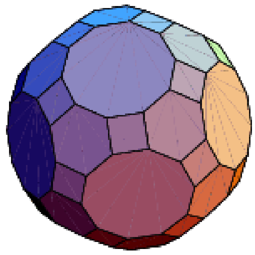

Figure 11: great rhombic osidodecahedron

and truncated icosahedron(Figure13). The above methods do not include the construction of

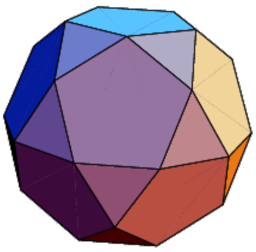

Figure 12: icosidodecahedron

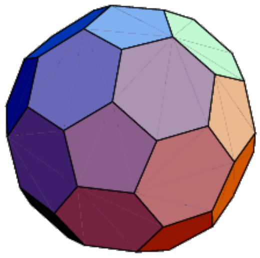

Figure 13: truncated icosahedron

all 13 Archimedean solids. To construct snub cube and snub dodecahedron, it is rather complicated if we adopt the truncating and cutting method. It is worth mentioning that the Chinese names of such solids embody truncating and cutting, and at the same time indicate where and how much they are truncated and cut, which is reflective of the construction process.

4 Catalan Solid

Definition 3 *In geometry, polyhedra are associated into pairs called duals, where the vertices of one correspond to the faces of the other. Starting with any given polyhedron, the dual of its dual is the original polyhedron. The dual of an isogonal polyhedron, having equivalent vertices, is one which is isohedral, having equivalent faces, and of one which is isotoxal, having equivalent edges, is also isotoxal.*

In mathematics, a Catalan solid is a dual polyhedron to an Archimedean solid.

It is easy to construct a Catalan solid with the aid of GeoGebra. In fact, from each Archimedean solid, we can construct a midsphere tangent to every edge of the polyhedron. Let's take cuboctahedron for example.

1. Construct a cuboctahedron and its center , which is the sphere center of the midsphere to be constructed. Construct a sphere with the midpoint of any edge, which is the midsphere of cuboctahedron.

2. Use "perpendicular plane" to construct a plane crossing and perpendicular to OM, another plane perpendicular to M edge. Construct the intersection of these two planes.

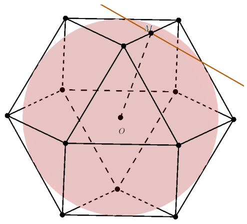

Figure 14:

3. Use the same method to construct the other three on the other three edges of the vertices. The intersections of these four lines make up a face of the daul solid.

4. Rotate and get other faces, and then we will get the dual solid of cuboctahedron—rhombic dodecahedron

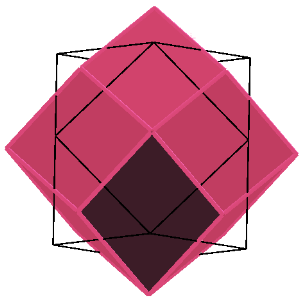

Figure 15: rhombic dodecahedron

By employing this method, we can construct 13 types of Catalan solids.

Of course, such method applies to the construction of dual polyhedron of Platonic solid. But it would be more convenient and practical to construct the dual polyhedron of Platonic solid by connecting the midpoints of each face.

5 Stellation

Definition 4 *In geometry, stellation is the process of extending a polyhedron in three dimensions, to form a new figure. Starting with an original figure, the process extends specific elements such as its face planes, usually in a symmetrical way, until they meet each other again to form the closed boundary of a new figure. The new figure is a stellation of the original.*

5.1 Constructing Polyhedron through Extending

Extending planes is an easy way to construdt a Stellation and let's take for example the Small Stellated Dodecahedron. This polyhedron is an extended Dodecahedron. Each face of the

Dodecahedron is a regular pentagon. When we extend each face into a regular pentagram, we get a Small Stellated Dodecahedron. The detailed process is as follows.

1. Create a Graphics file and a 3D Graphics file. The first Graphics is the xOy plane of 3D Graphics.

2. Set 5 points A,B,C,D,E in the xOy plane of the 3D Graphics, and enter the Graphics perspective. Connect AB,BC,CD,DE,EA into five lines and get the intersections of each two: M, N, P, Q, R. Then we get a pentagram, which is an extended pentagon.

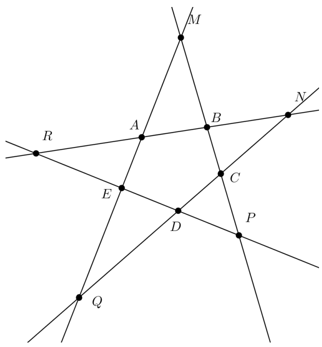

Figure 16:

3. Connect all the vertices and we get ten lines. Click "create new tool", choose the ten lines and vertices, and delete the five lines in step 2. Name the tool "pentagram". This tool is not customized in 3D Graphics, but can be used in 3D Graphics. When customizing, it is not limited to regular pentagram. It can also be used elsewhere.

4. Create 2 points P, Q in 3D Graphics, command Dodecahedron[P,Q] and we get a Dodecahedron. Use the tool"pentagram", select every face of the Dodecahedron and extend them, then we get a pentagram. It would be easier to select the points if we use Algebra perspective to construct each of the twelve pentagrams, then we get a Small Stellated Dodecahedron. Below is the graph of the pentagram with two faces constructed.

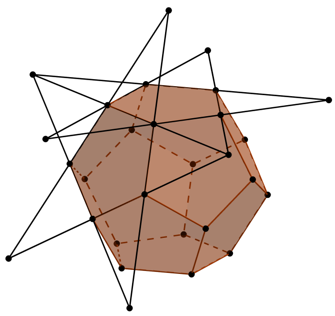

Figure 17:

Pentagram is not the only construction when every face of pentagon is extended. Figure17 is a construction of extended regular pentagon. We can see that the pentagram is a symmetrical choice of this graph. In this graph, other types of Stellation can be constructed as well if we adopt other symmetrical choices. Great Stellated Dodecahedron can be constructed by selecting the five sosceles triangles in the figure 18.

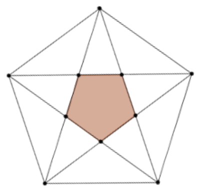

Figure 18:

It would more interesting to construct Icosahedron. We can extend a regular triangle into a fairly complicated graph as figure19.

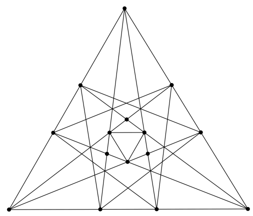

Figure 19:

If we select symmetrical polygons from figure19 and extend them, we can get 59 types of Stellated Icosahedra . The floor plan is shown in figure20. To make them into solid, we can

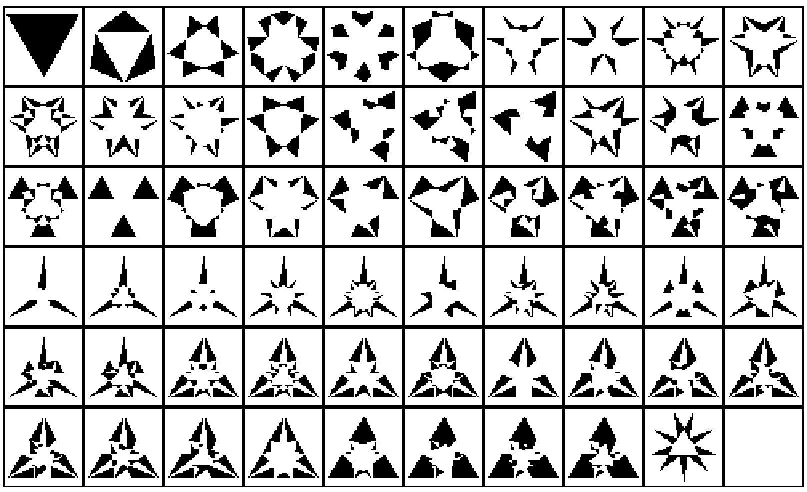

Figure 20:

employ the same method mentioned above by customizing tools.

5.2 Construction by Mutual Containing

Platonic solids can contain each other. For example, in cube $ABCD - A_1B_1C_1D_1$, we can construct a Tetrahedron with the vertices of A, C, B_1, D_1.

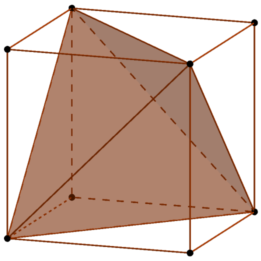

Figure 21: Tetrahedron Contained in Cube

The remaining vertices can also be used to construct a Tetrahedron. When these two Tetrahedrons intersect, we get a Stella Octangula, whose Chinese name is "Meiwending- konglinzong solid".

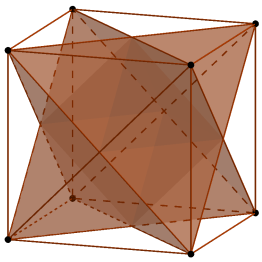

Figure 22: Stella Octangula

Actually, this is not a Uniform Polyhedron but a polyhedron compound. But the process of its construction is correlated with the properties of Uniform Polyhedron.

Below is how we demonstrate cube5-compound is a polyhedron compound consisting of the arrangement of five cubes in the polyhedron vertices of a dodecahedron (or the centers of the faces of the icosahedron).

The process is similar to the previously mentioned cube inserted with Tetrahedron, only a bit more complex.

1. Choose 3D Graphics and construct a dodecahedron with dodecahedron[A,B] command.
2. Find an edge (the blue one in figure 23), and find the two diagonals from the two pentagons meeting at the blue edge and the diagnoals must be parallel to the blue edge (see the red diagonals in figure 23). Construct a cube with the two red lines as the opposite sides of the bottom square. This cube is contained in the dodecahedron. Construct the cube with Cube[<point>,<point>,<point>] command.
3. Construct the other 4 cubes and we get cube 5-compound

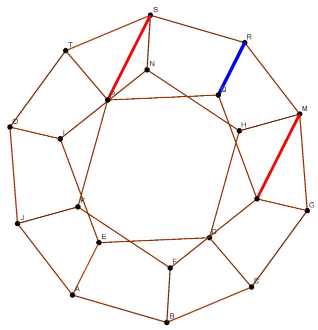

Figure 23:

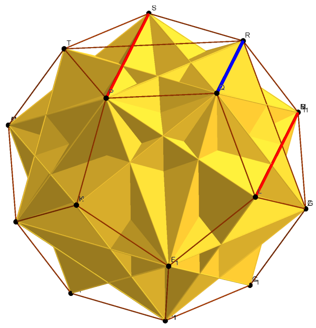

Figure 24: cube 5-compound

With several simple commands, we can easily complete the construction, which is fairly convenient for promotion and exploration.

The construction of other complicated polyhedrons like Rhombic Hexecontahedron and other Johnson Solids is made rather convenient with the aid of GeoGebra. This software provides intuitive geometric transformation and incredible calculating function. Combining theorems with graphs at the same time, it satisfies multi-layer requirements from both theorem researchers and model enthusiasts with its practical tools and commands . Therefore, it is fair to say that GeoGebra is an amazing tool in the research of polyhedron.

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