Technological tools have enhanced our teaching, learning and doing mathematics, what is next?

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Abstract

In this paper, we give an overview why technological tools have advanced so fast since ATCM 1995, and yet the adoption of exploration activities have not been widely implemented in many mathematics curricula. We also give examples to demonstrate the types of exploration that require the integration of CAS and DGS. We outline the components needed when developing an interactive online system, which is crucial for communications and collaborations on mathematical fields now and in the near future.

1 Introduction

Several speakers at various ATCM occasions have shown that technological tools not only can lead learners to deeper understandings on abstract and complex concepts of mathematics, but also encourage learners to discover more mathematical applications in real life. Indeed, with current technological tools, a mathematical problem or a project can be explored with the help of a computer algebra system (CAS), a dynamic geometry system (DGS) or combinations of both. We have discovered that new concepts or knowledge can be acquired by exploring mathematics with rapidly evolving technological tools. On one hand, many researches have recognized the positive impacts of proper adoption of technological tools in enhancing teaching, learning and doing mathematics and agree that curriculum heavily based on teaching for the test will hinder students' creative thinking skills and opportunities for innovation. However, because mathematics curriculum in many countries still examinations driven, the adoption of technological tools in teaching, learning and research is still far from desired.

In section 2, we describe typical difficulties for teachers when adopting technological tools in a classroom and emphasize the importance of creating video clips. In section 3, we use examples to demonstrate a mathematical problem can be made interesting and yet challenging at the same time. We also give examples to demonstrate why integrating the knowledge of both CAS and DGS is crucial for training students in current environment. Finally in section 4, we outline the needs of an online learning tutorial system which allow students to explore mathematics online, allow students to do online practice problems and receive instant feedbacks through a server. We urge educators, researchers, software and hardware developers to work cooperatively to make doing mathematics online a reality someday. Only when an online interactive mathematics learning environment becomes available, can we deliver dynamic interactive contents to users, and encourage instant discussions on mathematics at anytime and anywhere.

2 Supporting materials for teachers

The benefits of implementing technological tools in classrooms have been long discussed since the first ATCM in 1995. We have seen many governmental initiatives to boost the use of technological tools in math and sciences areas around the world ever since. However, we still see many teachers simply use computers or technological devices for PowerPoint presentations in a regular classroom. The problem exists because many teachers feel uncomfortable using technological tools for discussing or exploring mathematics in a classroom. Simply, it is still a non-trivial task for beginners learning how to use a technological device. So what are the possible reasons?

1. For hand-held devices, there are different brands and different keystrokes, author had always dreamed about all brands of calculators have the same *basic keyboard display.* This way teachers and students can get familiar with a calculator very quickly without worrying about how to manipulate a device. The place where different manufacturing companies may compete is how their computation engines reach an answer or how additional functionality of a calculator are being offered. Just like driving a rental car, consumers are allowed to choose different brands of cars, styles and options but the renter can easily start driving a car without worrying about what the brand of such car is.

2. For computer software packages with different capabilities, we have a lot to choose from, for complete introduction regarding this, we refer readers to *[10]*. In short, we can group them into software packages that can do CAS, DGS, statistics and etc. Any software packages have their own learning curves, it takes time to train teachers to be comfortable using them; not to mention nowadays, we need teachers to be able to integrate different capacities of software together, for example it is important to know how to integrate CAS with DGS. We will see examples in section 3.

The professional trainings for teachers to be competent of utilizing a hand-held device and computer software packages should be an ongoing process but it takes time and resources to attain predetermined objectives. It is known that video clips provide effective way of delivering lectures anywhere anytime such as those from *[7]* Video clips are useful for learners to recap content being covered in a lecture or to review steps when learning how to use a technological tool. It is worth mentioning a project from South Africa directed by Professor Werner Oliver, which contains all learning and teaching contents, to improve the quality of students learning and teachers teaching in school environments where internet bandwidth is not sufficient and also not affordable. His projects are aimed at providing off-line 21st century scaffolding support platforms for learners and teachers that are curriculum aligned and flexibly accessible. One of the aims of the projects is to improve the national pass rate of students who obtain more than of 50% for their final Mathematics school exams as this is normally the minimum requirement for access to science, engineering and technology related study programmes at higher education in this country(currently this rate is only 6%).

The project utilizes syllabus aligned video lessons, student workbooks, cell phone technology for formative assessment and peer tutoring with tablets for also improving teaching and learning mathematics and physical sciences subjects, see Figures 1(a) and 1(b) below:

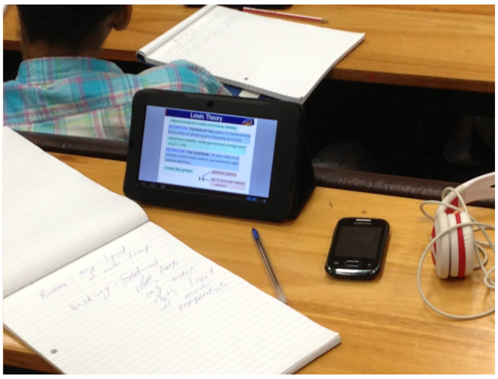

Figure 1(a) Students bring tablets to classrooms

Figure 1(b) Teachers and students have access to contents

Since internet is not widely available currently in South Africa, the project is using an independent offline package which runs in a local Windows or Android browser environment. A local web browser menu system of hyperlinks allow user-friendly and flexible access to (see Figure 2) all math and sciences content components (video lessons, PowerPoint slides, student workbooks and solutions) by merely a click of a mouse or touch of a screen. The content video lessons consist of narrated concept explanations, sets of related examples, tutorial exercise problems and corresponding solutions. It also contains some interactive applets. Although the resource materials of this project is Windows or Android based, we believe this is still very valuable for those countries where internet is not widely available. Moreover, all the content video clips are narrated and animated using dynamic graphical software to enhance visualization of mathematics or science concepts and problems. This makes the model even more attractive as it provides a wonderful teaching and learning tool for teachers and students that is complete and freely accessible on a local device - anywhere, anytime.

Figure 2 Intranet webpages which contains topics in math and sciences

3 Skills needed for students and teachers

Many educators and researchers believe that creativity does not come from rote or repetitive learning but from exploration and play. Those open-ended or out of textbook real-life problems are excellent resources for students to develop their creative thinking skills. The existence of a solution often is not enough. Instead, we may ask how we can approximate a solution. Can we use a technological tool such as DGS to simulate or conjecture what a possible solution may be? Do we have real-life applications for the open-ended project? However, we often see mathematics curriculum (in most part of the world) is geared toward exam-based. Too much emphasis on testing alone when measuring a student's understanding in mathematics will hinder student's interests in mathematics and their abilities for innovation. Here we note that Finnish students do not take a national, standardized high-stakes test until they matriculate secondary school and then only if they intend to enter higher education. Instead, the purpose of assessment in Finland is to improve learning; it is "encouraging and supportive by nature" (Finnish National Board of Education, 2010, "Encouraging Assessment and Evaluation, para. 1). Therefore, we may say that allowing time for exploring is essential for students' learning and success in mathematics for Finnish system. We have seen many presenters showing that evolving technological tools in recent years have allowed learners to increase and enhance ones' mathematics content knowledge. In addition, math can be approached as Fun, Accessible, Challenging and then Theoretical. Learners can expand their knowledge horizons with technology in various stages. Exploration with technology is the key, examination is only one way to measure students' understandings.

We use the following example (see Figure 3) to demonstrate how mathematics can be made interesting and yet challenging at the same time. In the mean time, it also demonstrated some basic skills needed for students and teachers when it comes to solving an open-ended exploratory type of activities in current mathematics current. The following graph is contributed by V.SHELOMOVSKII using *[5]* and its animation can be found at *[6]*.

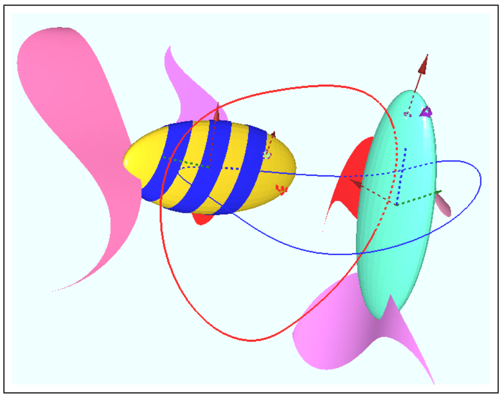

Figure 3 Floating fish with their respective space curves

We can ask the following simple questions to those students who have learned (multivariable) calculus:

1. Please describe how computer program draw a 3D fish in this case. Do we need a two or three variables function or parametric equation?

2. How can we make a 3D fish swim in a space? How many variables do we need in this case?

3. Please describe, in mathematics term, how two fish can swim without running into each other.

The following is a simple example showing basic skills of manipulating a CAS is needed both teachers and students when solving a problem.

Example 1 *Consider* $r = \sin 2\theta$ *versus* $r = \sin 160\theta$*, where* $\theta \in [0, 2\pi]$*. (a) What do you see the differences between these two polar graphs? (b) Would you like to make your conjecture base on what you see? [Do you see the graph of* $r = \sin 160\theta$*, where* $\theta \in [0, 2\pi]$ *almost fill up the whole circle of* $r = 1$*?] (c) Can you prove if your conjecture is true or false? Some may conjecture the area of* $r = \sin n\theta$ *when* $n \to \theta$*, where* $\theta \in [0, 2\pi]$*, is equal to that of* $r = 1$*, can you prove or disapprove this? It is a fun problem for those who just learned the plot of a polar graph, and using a graphics calculator can even trace the direction of the curve such as* $r = \sin 2\theta$*. The Figure 4(a) shows the plot of* $r = \sin 2\theta$ *for* $\theta \in [0, 2\pi]$ *and Figure 4(b) shows the plot of* $r = \sin 160\theta$*, for* $\theta \in [0, 2\pi]$*. Some may ask if we let* $n \to \infty$*, the graph of* $r = \sin n\theta$ *may fill out the whole circle and thus they may conjecture that the area of* $r = \sin n\theta$*, for* $\theta \in [0, 2\pi]$ *and when* $n \to \infty$*, will approach that of the circle of* $r = 1$*, which is* π*. However, we know that the area of* $r = \sin n\theta$*, for* $\theta \in [0, 2\pi]$ *is* $\frac{1}{2}\int_0^{2\pi} (\sin n\theta)^2 \, d\theta$*, and Maple returns the answer of*

$$\frac{1}{2} \frac{\left(-2 (\cos n\pi)^3 \sin(n\pi) + \sin(n\pi)\cos(n\pi) + n\pi\right)}{n}. \tag{1}$$

After taking the limit of equation (1) for $n \to \infty$ using [8] we obtain the answer of $\frac{\pi}{2}$, which means that our earlier conjecture was false.

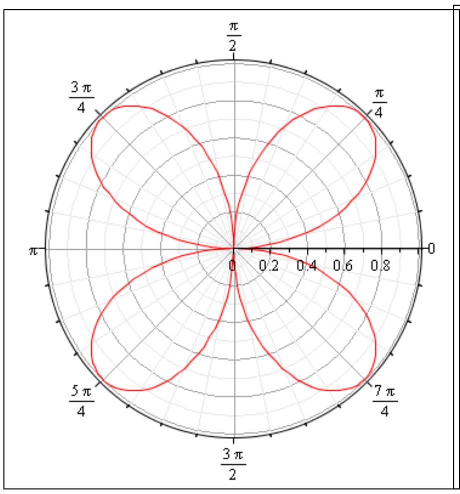

Figure 4(a) Polar graph of $y = \sin 2\theta$

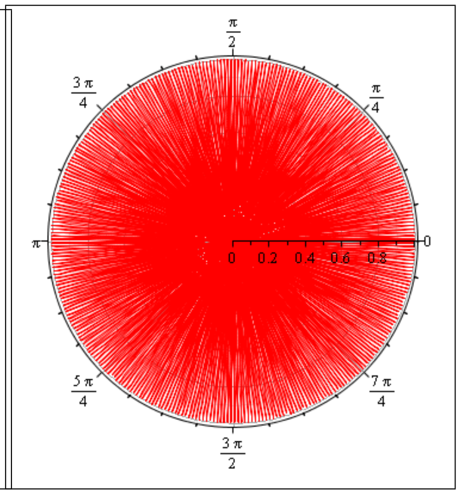

Figure 4(b) Polar graph of $y = \sin 160\theta$

The following problem was posted by Ph.D. students during my research project at Guangzhou University after discussing the problem *[15]* with them.

Example 2 *Consider the following three curves given by $C_1 : y = \sin(x), C_2 : y = x^2 + 2$, and $C_3 : (x-3)^2 + (y-3)^2 - 1 = 0$, see Figure 5(a) below. We need to find approximate points A, B and C on C_1, C_2 and C_3 respectively, so that the total distance of $AB + BC + CA$ achieves its minimum.*

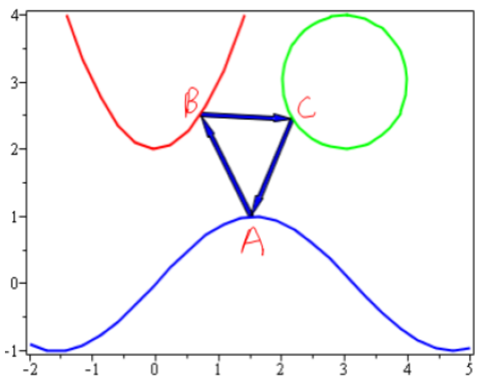

Figure 5(a) Total shortest distance obtained from a CAS

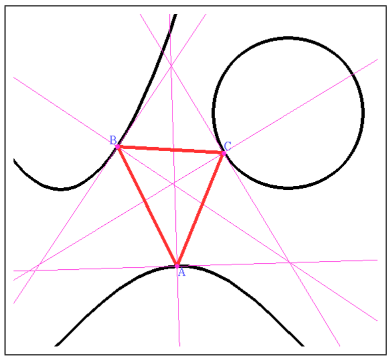

Figure 5(b) Total shortest distance obtained from a DGS

We remark that Ph.D. students have access to Chinese developed DGS so they are excellent in manipulating a DGS; however, some do not have access to a CAS or not comfortable using a CAS, so it is difficult for them to verify if their answers are correct. They first reduced their problem to a simpler case by considering three circles as follows (see Figure 6): $C_4 : (x-2)^2 +$

$(y-2)^2-1=0$, $C_5:(x+3)^2+(y+3)^2-4=0$, *and* $C_6:(x-3)^2+(y+2)^2-1/4=0$. *We need to find appropriate points* A, B *and* C *on* C_4, C_5 *and* C_6 *respectively, so that the total distance of* $AB+BC+CA$ *achieves its minimum. Students were able to use geometric constructions (will be published in a separate paper) to show that* $AB+BC+CA$ *achieves its minimum when we move* A *to* $A'=1.76761632020814, 1.0273758046571)$, B *to* $B'=(-1.20628618061282, -2.115358414871)$, *and* C *to* $C'=(2.5843451027448, -1.72209532859668)$ *accordingly in Figure 6. The normal vectors at* A', B' *and* C' *respectively, pass through the* ***incenter*** *of triangle* $A'B'C'$.

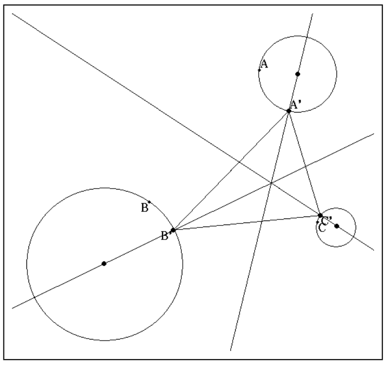

Figure 6 Solution obtained using a DGS

However, because of lacking the access to a CAS, they were not able to extend their result to the original problem (Example 2). In [16], we show that for the general case described in Example 2, the normal vectors at A, B, C *respectively should be the angle bisectors of* BAC, ABC *and* BCA *respectively and they should pass through the incenter of triangle* ABC. *We also obtain an analytical solution by using Lagrange multipliers and with the help of Maple (see Figure (a)). We confirm that the analytical solution is consistent with the geometry construction using a DGS [2] (see Figure 5(b)).*

It is natural to generalize the $2D$ problem to the following $3D$ case: We are given four surfaces shown in Figure 7, where $S_1: x^2+y^2+z^2-1=0$; $S_2: x^2+(y-3)^2+(z-1)^2-1=0$; $S_3: z-(x^2+y^2)-2=0$ and $S_4: (4(x-3)+(y-3)+(z-1))(x-3)+((x-3)+4(y-3)+(z-1))(y-3)+((x-3)+(y-3)+4(z-1))(z-1)-3=0$. We want to find points A, B, C and D on the surfaces S_1, S_2, S_3 and S_4 respectively, so that the total distance $AB+BC+CD+DA$

achieves its minimum.

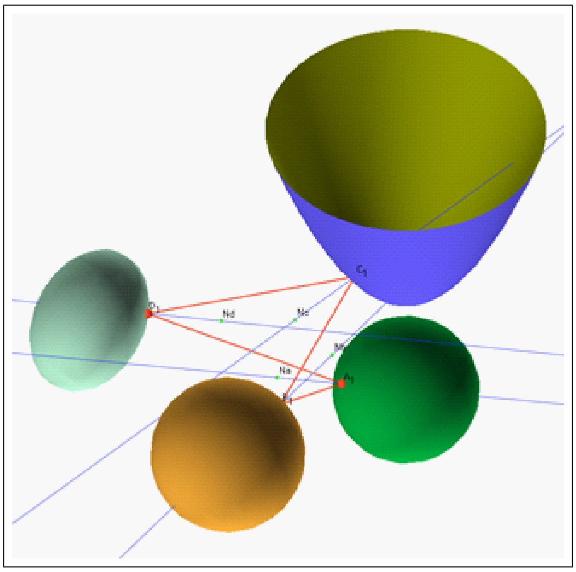

Figure 7 A total shortest distance problem in 3D

Remark: We may think of this problem as a light reflection from one surface to the other. If we think BA as an incoming light toward the point A on surface S_1 and DA as an outgoing light at A, then N_A, the normal vector at A on S_1 should be their angle bisector. Similarly, we should have the followings:

1. The line segments BC, AB and the normal vector at B, denoted by N_B, should lie on the same plane and N_B is the angle bisector for AB and BC.

2. The line segments BC, CD and the normal vector at C, denoted by N_C, should lie on the same plane and N_C is the angle bisector for BC and CD.

3. The line segments CD, DA and the normal vector at D, denoted byN_D, should lie on the same plane and N_D is the angle bisector for CD and DA.

We apply Lagrange Method with the help of Maple to find the shortest total distance $AB + BC + CD + DA$ algebraically. The desired points A, B, C and D on S_1, S_2, S_3, and S_4 respectively obtained from Maple are $A = [s_{11}, s_{12}, s_{13}], B = [s_{21}, s_{22}, s_{23}], C = [s_{31}, s_{32}, s_{33}]$ and $D = [s_{41}, s_{42}, s_{43}]$, where

$$\begin{aligned}
s_{11} &= .33389958127263692867, s_{12} = .82675923225542380177, s_{13} = 0.45274743677505224502;\\
s_{21} &= 0.13641928282913798603, s_{22} = 2.0175068585849490438, s_{23} = 1.1268739782018690446;\\
s_{31} &= 0.28488630072773836633, s_{32} = .62315636129160757197, s_{33} = 2.4694840549605319295;\\
s_{41} &= 2.3607938428457871966, s_{42} = 2.4901551912577622423, s_{43} = 1.3195995907854243548.
\end{aligned}$$

We show geometrically (see Figure 8) that the blue dotted lines are respective normal vectors at respective points A, B, C and D, and the dark red line segments are respective angle bisectors in Figure 8, they coincide with respective normal vectors when the minimum total distance $AB + BC + CD + DA$ is achieved. In other words, **our geometry constructions coincide with our algebraic analysis using CAS.**

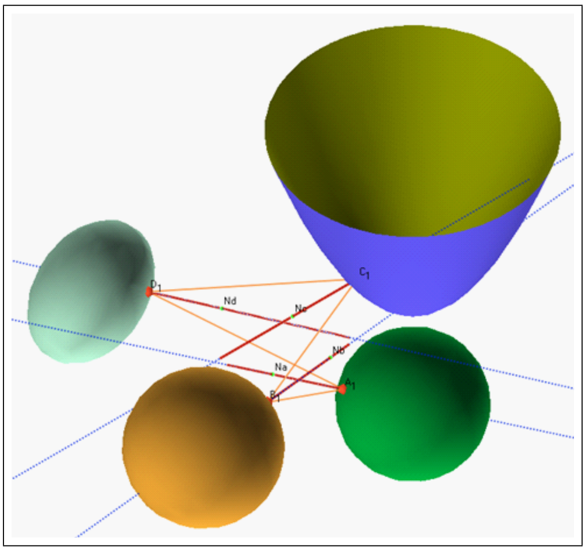

Figure 8 Total shortest distance 3D problem verified using a DGS

The preceding problem demonstrates why integrating the knowledge of both CAS and DGS is crucial for training students in current environment. The following example comes from a university entrance exam practice problem from China. We use the dynamic software *[5]* to demonstrate how the problem can be made accessible to most students and how the problem can be generalized to more challenging ones in 2D and corresponding ones in 3D. We remark that these open ended projects are excellent choices for adopting technological tools for exploring mathematics.

Example 3 *Consider the following Figure 9:*

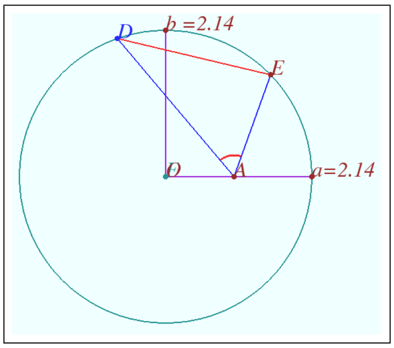

Figure 9 A college entrance practice problem from China

The circle is given with radius r, assuming the center is at $(0,0)$, and $A=(x,0)$ and $x \in [0,r]$. The points D and E are any two points on the circle such that the angle of EAD forms a fixed angle β. For the college entrance exam practice problem from China, students are asked to conjecture the maximum and minimum lengths for DE. The following values were those given at the practice college entrance problem from China: $r=2, \beta=\frac{\pi}{2}$ and $A=(1,0)$. It is not difficult for one to conjecture after exploring with a DGS that the maximum length of DE occurs when DE is perpendicular to AO and A is on the opposite of DE, see Figure 10(a), and the minimum of DE occurs when DE is perpendicular to AO but A is on the same side of DE, see Figure 10(b).

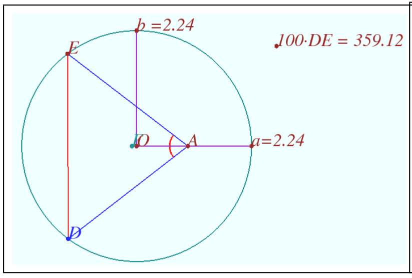

Figure 10(a) Maximum length DE

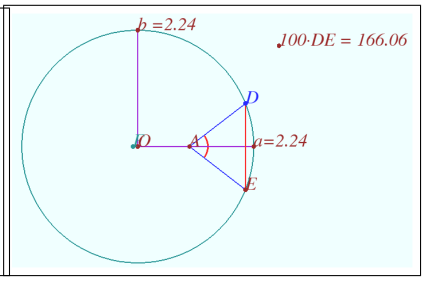

Figure 10(b) Minimum length DE

It is not difficult to generalize this problem from a circle to a sphere in this case. However, the problem should be rephrased as follows: Given a sphere of a fixed radius r of the form $x^2+y^2+z^2=r^2$and pick the point $A=(d,0,0)$, where $d \in [0,r]$. Let B be a point on the sphere and rotate AB with a fixed angle β to form a cone, see Figure 11. We want to find the maximum and minimum intersecting surface areas between the cone and the sphere.

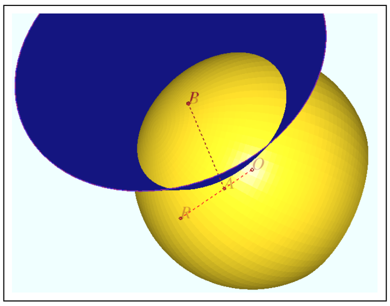

Figure 11 Generalize the circle probem to a sphere

After exploring with [5], it is not difficult to see that the maximum intersecting surface area occurs when the normal vector at B is parallel to the vector OA and A is on the opposite of B (see Figure 12(a)). The minimum intersecting surface area occurs when the normal vector at

B is parallel to the vector OA but A is on the same side of B (see Figure 12(b)).

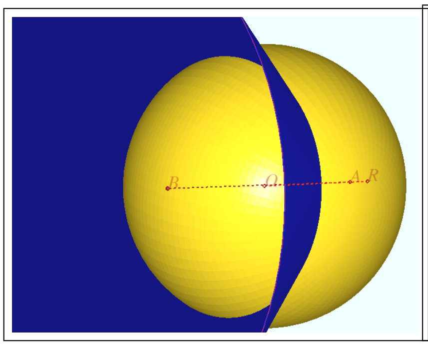

Figure 12(a) Maximum intersecting surface area

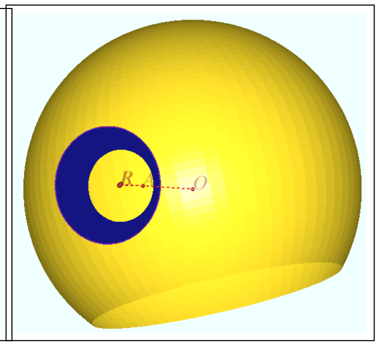

Figure 12(b) Minimum intersecting surface area

We may now ask students to extend Example 3 to an ellipse of this form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and use a DGS to make conjectures where the maximum and minimum lengths of DE could be, see Figure 13. We note that the answers will vary depending on the values of a, b, the fixed angle β and the position of the point A. Complete solutions on the ellipse case have been discovered, see *[17]*

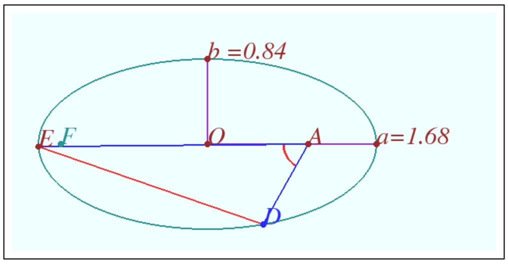

Figure 13 Generalize the 2D problem to an ellipse

The problem becomes even more challenging and is still an *open problem* if we extend the ellipse to an ellipsoid. For example, given an ellipsoid of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, let $A = (x, 0, 0)$ with $x \in [0, a]$. Pick B be on the ellipsoid and if the cone determined by rotating a fixed angle β around AB, see Figure 14 below:

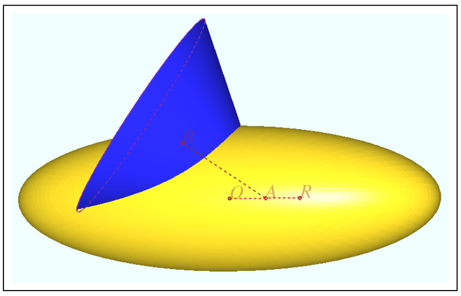

Figure 14 An ellipsoid and a cone

We may ask the question: For a fixed point A and fixed angle for the cone, find the point B that will result in the maximum or minimum intersecting surface area between the ellipsoid and the cone. Just like the case for ellipse, the answers vary depending on the values of a, b, c, the fixed angle β and the position of the point A.

We summarize a general strategy of how we may approach solving a problem and how we can generalize it to higher dimensions with the helpful of a DGS and CAS.

1. We use DGS or similar technological tool to simulate animations in 2D.
2. We make conjectures through your observations from step 1.
3. We verify our results using a DGS or CAS for 2D case.
4. We extend our observations to a 3D scenario with technologies if possible.
5. We prove our results for 3D case using a DGS or CAS.
6. We extend our results to finite dimensions or beyond if possible.

It is interesting to note the following scenarios while author conducted lectures to students of different backgrounds:

1. For students who have access to a CAS but not DGS: It is impossible for them to reproduce different animations with geometric motivations. They can only modify the CAS worksheet author provides them when validating their algebraic answers.
2. For students who have access to a DGS but no CAS: Students will use their favorite DGS to reproduce author's problems and make conjecture about the validity of the solutions; however, their conjectures cannot be verified since they have no CAS to verify solutions analytically.
3. For those students who have no access to either a CAS or DGS, they can only appreciate the graphical representations of authors' lectures, they have no available tools to experiment on their own.

4 Delivering interactive online contents

It is interesting to see that the internet technology has advanced rapidly since the first ATCM in 1995. However, delivering interactive online content still has room to improve. To develop a truly online dynamic learning system for mathematics, we need computation engines that are capable of doing both CAS and DGS. With such a system in place, we will be able to create online interactive textbooks, we can also allow open-ended exploratory type of problems such as those described in Section 3. We describe some of the desired components for online interactive learning in mathematics and related subjects so people can easily discuss mathematics.

1. Allowing instant exploration using CAS such as Wolfram|Alpha *[13]*, the system provides an excellent system where users can explore various areas using Mathematica engine online. For example, we can explore the torus knots (see Figure 15 *[14])*. Author has seen teachers from China developing teaching contents using *[13]*. More interactive approaches using *[13]* can be found in the invited speech by *[1]*. After all, this is a CAS system where users have no way of experimenting problems when a DGS system is desirable.

Figure 15 An interactive torus knots using Wolfram Alpha

2. For exploring DGS online, we direct readers to the page of *[12]*, where *[4]* is used as the DGS engine to describe a parabola Caustic Rays emanate from D, and their reflections in the parabola form a caustic (see Figure 16). The family of rays by pressing `Show`, then the caustic curve by pressing `Show` again. You can drag C and D. However, this is a DGS

based not CAS based so we cannot experiment problems when CAS are needed.

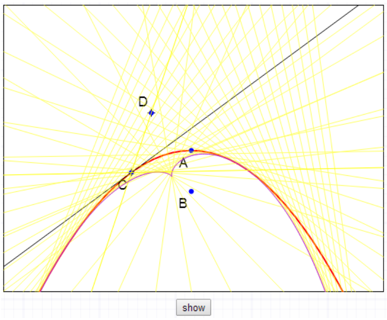

Figure 16 An online interactive caustic using Geometry Expression

3. Using applets: If we look at the webpage *[9]*, it describes an online content in Calculus subject using Java applets. For example the following Figure 17 gives a dynamic way of learning the spherical coordinate system, users can drag ρ, θ and ϕ, and can see the effects of these parameters on the position by dragging the red dot. However, we know that applets using Java script is platform dependent, which also presents a problem for developers when they consider designing interactive online contents.

$$\begin{aligned} x &= \rho \sin\phi \cos\theta, \\ y &= \rho \sin\phi \sin\theta, \\ z &= \rho \cos\phi \end{aligned} \tag{2}$$

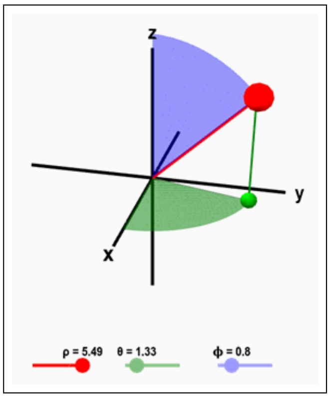

Figure 17 An interactive online content using Java applet

4. Many book publishers are interested in an online assessment system because these are what schools and universities desire for assessment purpose. Ideally a good system should

allow teachers to create randomized questions containing graphs, and students are able to see their results with feedbacks instantaneously. Therefore, a system requires an authoring tool for teachers to design problems and an users' testing system. The following shows a question generated by the Exam Builder of Scientific Notebook (see *[11]*), which is a Window-based and has been available in mid 90s. However, there is no compatible online testing system so far. We observe in Figure 18, the quantity a in $\sin ax$ can be randomized using the authoring system; accordingly the graph of $\sin ax$ will vary accordingly because there is a built-in CAS. After choosing the answers and clicking '`click to grade`', we receive a feedback from the system as seen in Figure 19. This is a Window based program which author adopted during 1997-1998 while visiting Singapore, and yet it is surprising that a comparable 'engine-based' online system is still not available as of now. The invited talk by *[3]* introduces a mathematics interactive learning environment, it remains to be seen if the system can fulfill the possibilities of online discussions on those open ended problems that are discussed in Section 3.

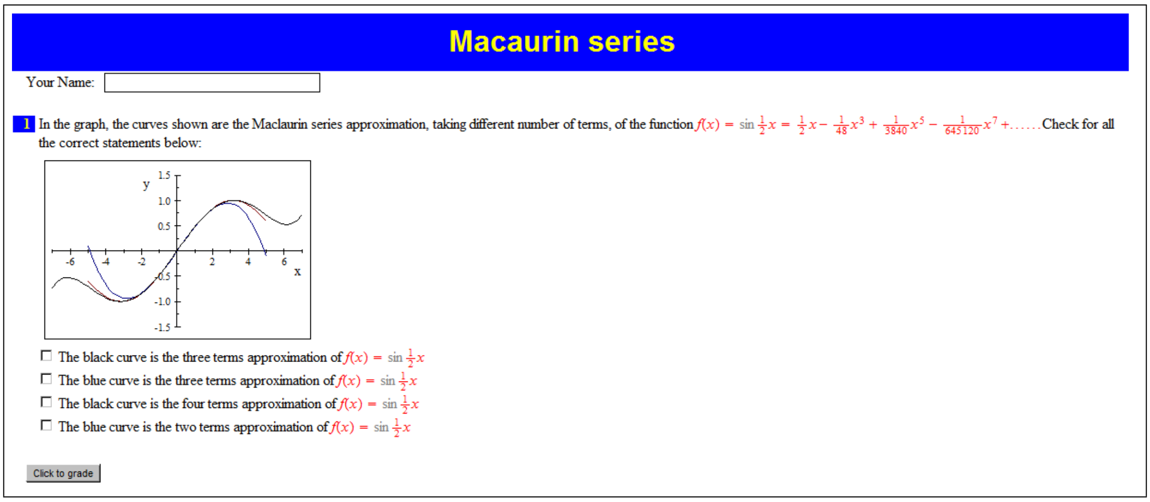

Figure 18 An authoring tool from Exam Builder

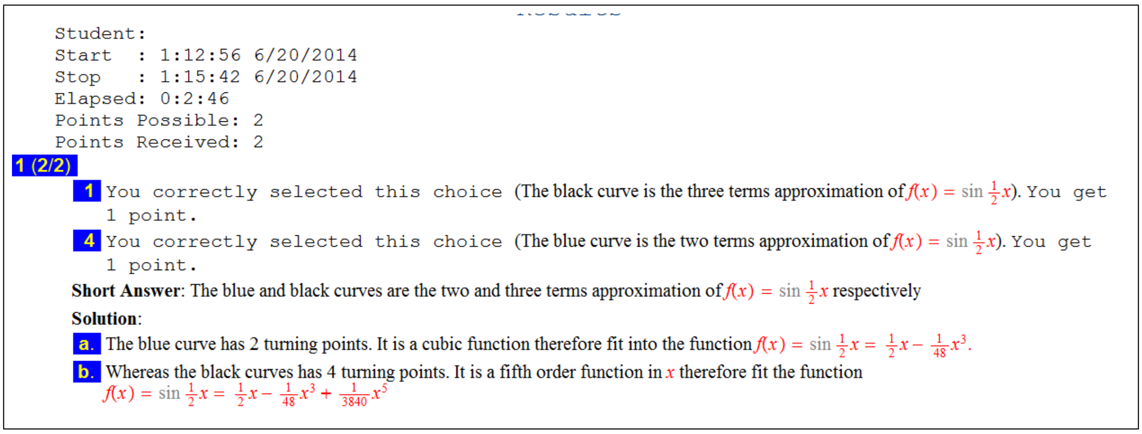

Figure 19 A feedback form using Exam Builder

5 Conclusions

It may be true that many math textbooks (including those from US) are not based on exploration, and schools buy the books that are aligned to the test questions instead of the books that promote deep mathematical understanding and creativity. This is rather a true reality but common sense tells us that teaching to the test can never promote creative thinking skills in a classroom. We know the continual efforts for all ATCM communities to address the importance and timely adoption of technological tools in teaching, learning and research can never be wrong. Therefore, we encourage ATCM communities continue creating innovative examples by adopting technological tools for teaching and research and influence your colleagues, communities and decision makers in your respective countries. We also think by developing technology based textbooks which describe contents in continuos, connected and creative fashion is important. Selecting examples that can be explored from middle to high schools, university levels or beyond, and they should be STEM related by linking mathematics to real-world applications. Allowing users to explore mathematics with technological tools online is crucial step for developing a social media of discussing mathematics through internet. Many agree that examination should not be the only way to measure students' understandings in mathematics. Technology has become a bridge to make us rethink how to make mathematics an interesting and a cross disciplinary subject. Through the advancement of technological tools, learners will be able to discover more mathematics and its applications through an interactive online learning environment.

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