

Teaching Experimental Mathematics: Digital Discovery using Maple

Bill Blyth^{1,2}

bill.blyth@rmit.edu.au

¹ Australian Scientific & Engineering Solutions (ASES)

² School of Mathematical and Geospatial Sciences

RMIT University

Australia

Abstract

One of the “Recommendations (by Klaus Peters) on Jon Borwein’s LinkedIn page starts with: “Jonathan Borwein is a world-renowned research mathematician and one of the leaders in the emerging field of experimental mathematics to which he has made many original contributions. He has written several books that have helped to define the field. Examples include finding closed forms of integrals by numerically computing high precision approximations (accurate to hundreds of decimal digits) which is used to identify the answer. Knowing the correct answer facilitates the proof of its correctness.

Here this experimental mathematics approach is taken with a couple of traditional “find the maximum . . .” problems suitable for senior school or first year undergraduate students (who could even be pre-calculus students). We use the Computer Algebra System, Maple, for experimentation (and visualization) to compute high accuracy numerical approximations to the answer. Maples `identify()` command is used to identify the exact answer. Knowing the answer, we prove that it is correct - without using calculus. We use small group collaborative learning in the computer laboratory, so we parameterize the problem and recommend the use of Computer Aided Assessment (such as provided by the package MapleTA).

Students engage with the visualization, are active learners with deep learning of the concept of maximum and gain a gentle introduction to the important new field of experimental mathematics.

1 Introduction

Since the mid 1990s, we have often stated our view that all mathematics graduates from universities should be expert users of one of the two professional level Computer Algebra Systems, CAS: Maple being one of these. A recent study [9] of UK university students reported “that 59% of students find their lectures boring half the time and 30% find most or all of their lectures boring.” A range of different teaching methods were investigated: the most boring were lab sessions; computer sessions ranked as the second most boring. “Computer sessions too have

the potential to be stimulating or tedious; the findings of this study suggest that too many fall into the later category. This could be due to the manner in which sessions are conducted (e.g. are the computer tasks relevant and interesting?), . . . etc.”

These findings are not surprising given that most courses are teacher centric and use a traditional teaching approach (lectures) with “tutorials” and assignments which are mostly done without use of technology. When technology such as CAS is used (at both school and undergraduate levels), it is often a minor supplement where the usual “by hand” exercises completed by using CAS following the “by hand” approach. We suggest that innovative teaching materials which exploit the strengths of the CAS for tightly integrated e-Learning and e-Assessment, such in the Maple component of our first year course (and in higher year courses taught entirely using Maple), and a pedagogy of collaborative learning in small groups as active learners in a computer lab need to replace (at least in part) boring lectures and boring computer lab sessions.

In this paper we consider a standard calculus problem: find which cylinder inscribed in a unit sphere has the maximum volume. We provide visualization and show how to obtain an accurate decimal approximation to the radius that gives the maximum. This leads to an identification of the possible exact answer. By foregoing calculus, we give a gentle introduction to experimental mathematics: we provide several algebraic proofs that introduce important ideas in mathematics, such as approximation in the small. We also support our intuitive notion that the maximum of a graph of a function “looks like a parabola” near the maximum by algebraically calculating the quadratic approximation of the function near its maximum.

This paper is organized as follows. Section 2 introduces experimental mathematics. Section 3 has a brief overview of first semester, first year Maple sessions which provide a suitable preparation to undertake the Cylinder in a Sphere problem described here. Section 4 provides a description of the Cylinder in a Sphere problem and visualization (including multiple representation and animations). Section 5 zooms in on the plot of volume versus radius of the cylinder to approximate the r that maximizes the volume; and a possible exact answer is identified. Several proofs without using calculus are discussed: in Section 6, volume squared is maximized; in Section 7, volume is maximized. Several more proofs are indicated in Section 8.

Section 9 introduces a parameterized problem (of a Cylinder in a Spheroid) and discusses e-Marking and Computer Aided Assessment. This is followed by a brief Conclusion.

2 Experimental Mathematics

Jon Borwein is a pre-eminent research mathematician who is one of the leaders in developing and popularizing experimental mathematics. We suggest that every aspiring or practising mathematician should read some of his work: we particularly recommend the book [6] and the short article [7]. In the later, Borwein shares Polya’s view on intuition:

Intuition comes to us much earlier and with much less outside influence than formal arguments . . . Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning.
George Polya (1887–1985) [10], p.128

Borwein continued with

[Polya did go] on to reaffirm, nonetheless, that proof should certainly be taught in school. [. . .] My musings focus on the changing nature of mathematical knowledge

and in consequence ask questions such as How do we come to believe and trust pieces of mathematics?, Why do we wish to prove things? and How do we teach what and why to students?. [7]

In our experience with our own mathematics undergraduate and PhD students is that they have little appreciation of intuition in mathematics: my discussions about the importance of intuition (and the development of it) come as a surprise to them.

Borwein [7] gives an informal summary of what is meant by experimental mathematics:

- Gaining insight and intuition;
- Discovering new relationships;
- Visualising math principles;
- Testing and especially falsifying conjectures;
- Exploring a possible result to see if it merits formal proof;
- Suggesting approaches for formal proof;
- Computing replacing lengthy hand derivations;
- Confirming analytically derived results.

Another eminent mathematician, Doran Zeilberger [11] states that

The purpose of mathematical research should be the increase of mathematical knowledge, broadly defined. We should not be tied up with the antiquated notions of alleged “rigor”. A new philosophy of and attitude toward mathematics is developing, called “experimental math” (though it is derided by most of my colleagues; I often hear the phrase, “Its only experimental math”). Experimental math should trickle down to all levels of education, from professional math meetings, via grad school, all the way to kindergarten. Should that happen, Wigners “unreasonable effectiveness of math in science” would be all the more effective!

Lets start right now! A modest beginning would be to have every math major undergrad take a course in experimental mathematics.

We agree, but note that many senior school mathematics courses use CAS so we suggest “the beginning” should sensibly occur in the secondary school.

3 Preliminaries

For about a decade, we designed and ran Maple topics for our first year university “Calculus” course comprised of four traditional lectures and one Maple lab session each week. The weekly lab sessions had no lectures: students work collaboratively in small groups. For the first semester, the topics were secondary school level mathematics. Students start by taking two sessions to work through an Introduction to Maple which includes a simple assignment that is printed and submitted as a hardcopy, but all other work is completely electronic. An Animation

worksheet is followed by a major Animation Assignment: students pick a project (from a list) that is on a school level topic, see [1].

After these activities, students are expert at preparing plots and animations, so these are recurring themes. The next activity is Spot the Curve. The Maple file Spot the Curve is accompanied by another Maple binary file which contains all the procedures for randomly generating the curves, their translated curves and automatic marking procedures: this binary file is read into the student Spot the Curve file so that all of the required procedures are invisible but active in the student file. Each student group is presented with several plots of a curve and a translated copy of that curve: the task is to identify what the translation is (by using plots and animations). Students use the provided automatic marking procedure to immediately mark their result. Feedback shows that students really enjoy this activity: for details, see [2].

Another style of auto marked problem is the FishPond problem, a straightforward application of using the trapezoidal rule to approximate area. It has multiple steps with numerical answers: the marking procedure is used by students to immediately mark their work: for details, see [2]. The FishPond problem is followed by working through a Maple file of a structured Polya approach to “finding the maximum area”, followed by an assignment on finding the maximum area of the Norman Window problem. For a discussion of the Norman Window problem, multiple representations, e-Marking and Computer Assisted Assessment, see [3]. With this (or similar) background, students are prepared to investigate various solution methods (proofs) of a particular related problem, see [5].

These activities provide students with an appropriate background to undertake the maximum problem(s) described next.

4 The Cylinder of Maximum Volume inscribed in a unit Sphere

This problem is a classic one that appears in many textbooks. We introduce the experimental mathematics methodology by providing a Maple worksheet for student groups to work through. [After completing this, they are presented with a parameterized version as an assignment.]

The Problem: For a cylinder inscribed in a sphere of radius 1, find the *exact* radius of the cylinder which has the maximum possible volume.

For this problem, visualization of the unit sphere with an inscribed cylinder is wanted, but students start with plots in two dimensions and should do a lot of work with 2D plots before using plots in three dimensions (other than simplistic plots such as straightforward plots of $z = f(x, y)$). Since the details on how to produce the 3D plot are not relevant to the problem being addressed here, we recommend that the 3D plot is provided in the worksheet. It is even better to provide an animation as the radius of the cylinder, r , varies from 0 to 1: see Fig. 1 for one frame of such an animation.

Note that a cut-out of the sphere is used so that the inscribed cylinder is easily visualized. Some understanding of spherical polar coordinates is needed for this: we introduce parametrization of surfaces properly in a university third year course on Geometry of Surfaces (taught entirely in the computer lab using Maple). Students work through a worksheet on Quadric Surfaces. We introduce cut-outs of 3D objects in our course on Vector Calculus (in second or third year, also entirely in the computer lab using Maple). The use of a cut-out is very useful

r of cylinder = 11/16

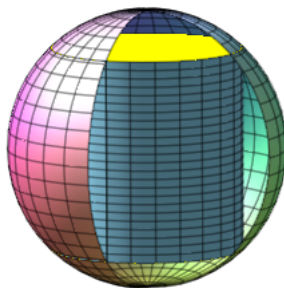


Figure 1: A cylinder inscribed in the unit sphere (with cut-out).

for visualization of “slicing” the 3D object so that triple integrals can be correctly set up. We do not advocate that students at high school or first year university be required to do this. However we always provide the Maple code since some students do want to experiment with the code. This is easy in Maple: the plot or animation should be done in a Table in which the code is hidden. Students who are interested can un-hide the code. Since graphics and animations use a lot of memory, all of the output should be removed for storage and transmission of the Maple file (greatly reducing the file size). The user can then execute the whole file or sections when working on the file.

Another “common” way of visualizing the cylinder inscribed in the sphere is to use transparency (for the sphere). This is easier than using a cut-out and so this is recommended for less experienced students. We have not included a plot in this paper since it does not transfer well to paper, but it is adequate on a computer monitor. It is relatively easy since Maple provides a package called `plottools` which plots objects such as spheres and cylinders (without the user needing to know about parameterizations of these surfaces). Using provided Maple commands, the sphere can be obtained (using the default with centre at the origin and unit radius)

```
[> with(plottools): with(plots):
```

```
[> sph := sphere( style=patchnograd, transparency=0.8):
```

This specifies the options that the grid lines are removed and a (high) transparency is used.

To define an inscribed cylinder, we need to calculate the height of the cylinder that has a given radius. It is easy to use right angle triangles to show (see below) that for $r = 11/16$ the half height is $hh = 3\sqrt{15}/16$. The coordinates of the centre of the base of the cylinder is $[0, 0, -hh]$, the radius is $11/16$ and the height is $2hh$, so the cylinder is defined by

```
[> cyl:=cylinder([0, 0, -hh], 11/16, 2*hh, strips=100, style=patchnograd):
```

To make the cylinder’s curved surface looks smooth, the number of vertical strips has been increased from the default of 24 to 100 by using the option `strips=100`. The plot seems to be more attractive if the gridlines are also removed from the cylinder, so the option `style=patchnograd` have also be used for the cylinder. The partially transparent sphere and the cylinder are displayed together by

r of cylinder = 11/16

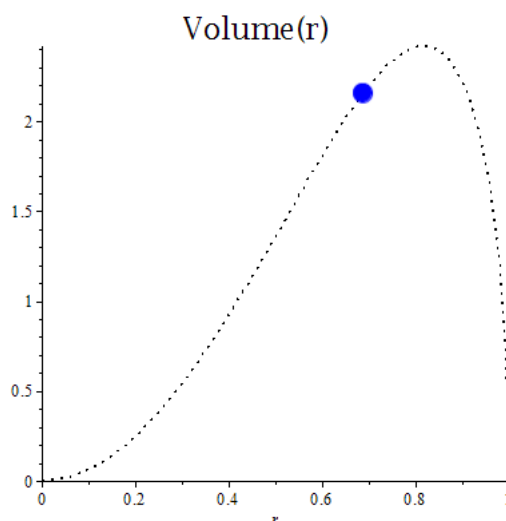
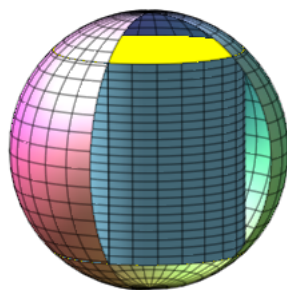


Figure 2: One frame of the animation of the multiple representation of the 3D plot of the sphere and the cylinder and the plot of the volume of the cylinder. The radius of the cylinder is the animation parameter.

```
[> display(sph, cyl, scaling=constrained, axes=none, orientation=[50,75]);
```

To give a slightly better view, the orientation of the displayed plot has been changed from the default. Note that the orientation can be changed interactively by using the mouse on the displayed plot. Students do not need much encouragement to do this and so engage as active learners which is much to be preferred to the lecturer just showing the visualization as a demonstration in a lecture.

Once students have experimented with the animation of the 3D plot, they have established their visualization of the problem. We want to help them to further develop some intuition about the problem so we provide a multiple representation where the 3D plot has placed alongside the plot of the corresponding Volume (of the inscribed cylinder) for the given radius - as an animation where the cylinder's radius is the animation parameter. See Fig. 2 for one frame of such an animation.

First year students who have followed the type of sequence of Maple lab sessions on animation; the Polya approach to problems of finding maximum area; an “Advanced” Polya approach to finding maximum area (including multiple representations), see [3], could produce multiple representation animation (using transparency for the 3D plots). However for less experienced Maple users, we recommend providing this animation.

Of course, the formula for the volume of the inscribed cylinder as a function of its radius has not yet been derived: this is done in the next section.

5 Volume of the Inscribed Cylinder and Zooming In

A cylinder of radius r is inscribed inside a sphere of radius $R=1$. To derive a formula for the volume of the cylinder, we plot a cross section obtained by cutting the sphere and the inscribed (vertical) cylinder by a plane which includes the z axis, see Fig. 3. This plot is straightforward. An even easier alternative is to use Maple's Drawing tools: insert a Canvas and use the tools

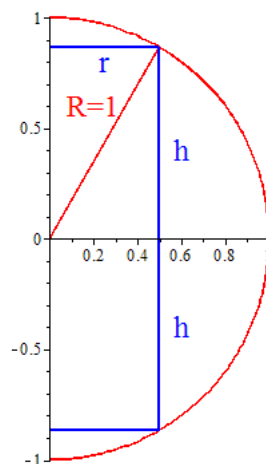


Figure 3: A cross section obtained by cutting the sphere and the inscribed (vertical) cylinder by a plane which includes the z axis.

for lines and curves. Use freehand drawing to put in arrows and variables - this is particularly easy using a stylus (or pen) on a touch screen (but harder just using a mouse!).

From the plot it is easy to see that r and the half height h of the cylinder form sides of a right angle triangle with hypotenuse 1 (for a unit sphere). Thus Volume of the cylinder is area of the circular base times the height, so

$$V = \pi r^2 \times 2h^2 = 2\pi r^2 \sqrt{1 - r^2}.$$

This is the volume as a function of r which was used to generate the plot in the right hand side of Fig. 2. The curve is skewed to the right, clearly the maximum is for r somewhere between 0.7 and 0.9 so we can obtain a better approximation by zooming in, see the plot on the left in Fig. 4.

Students at secondary schools in the State of Victoria in Australia have been using graphics calculators (and then CAS calculators) for about two decades, so students are very familiar with the idea of zooming in. This is a very useful skill, but is hardly ever used at university level (except for education studies) since calculators are hardly ever used in science and engineering courses.

From (the plot on the left in) Fig. 4, near the maximum, the curve “looks like a parabola” with a maximum (read off the graph - place cursor on plot and right click - choose Probe Info, Nearest point on line - position the cross hairs and read off the point’s coordinates) of about 0.816 or 0.817, say?

With three more zoom ins, we estimate that r is about 0.8164967. This is probably accurate enough, but (if needed) we could get higher accuracy by reading the graph better to see where the maximum is. Thus we zoom in again as well as choose a view window with restricted “ y ” values (we experiment with the y values to use):

```
[> rL, rR:=0.8164965, 0.8164968:
```

```
[> plot(V, r=rL .. rR, view=[rL .. rR, 2.4183991523122 .. 2.4183991523123] );
```

The plot is displayed on the right in Fig. 4. Hence the approximate answer is $r = 0.81649658$ accurate to about the last digit!

To try to identify the (possible) exact answer from the numeric answer by estimation from the graph:

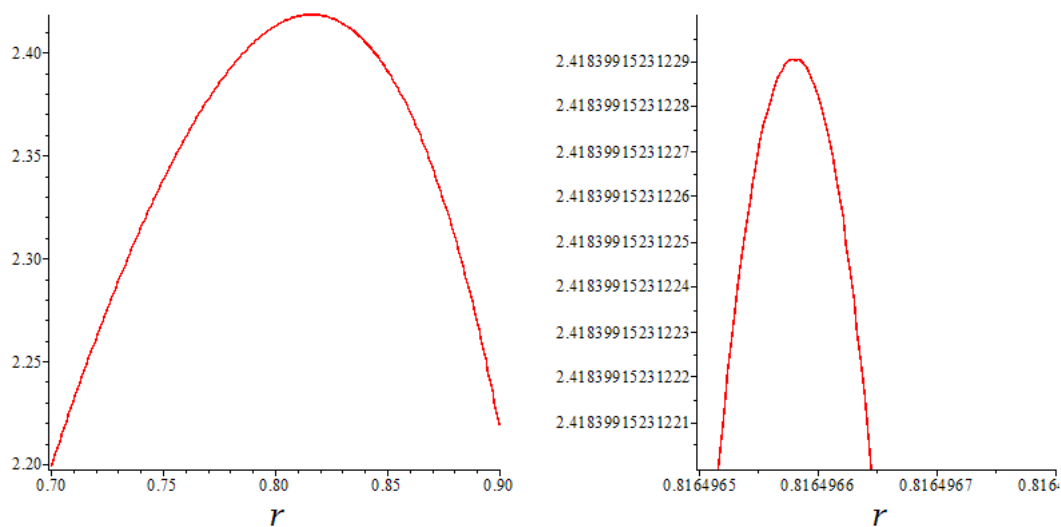


Figure 4: A zoomed in plot of the volume of the cylinder on the left; and a four times zoomed in plot on the right.

[> identify(0.8165);

This gives the result $\frac{1}{3}\sqrt{6}$. Since the exact ans is $\sqrt{\frac{2}{3}}$, this has given the correct identification of the exact answer. (We prove that this is the correct result below.)

5.1 Comparison of Maple and Mathematica

Maple and Mathematica are the only professional level CAS. For identification of numbers from floating form numbers (possibly of very high precision), the identify() command in Maple is very superior to the limited functionality provided in Mathematica. The Mathematica (version 8) commands are Rationalize[] and RootApproximate[] with option RootApproximant[x,n] which “finds an algebraic number of degree at most n that approximates x.” Note that “For degrees above 2, RootApproximant generates Root objects.”

Using Mathematica 8,

[> RootApproximate[0.8165] gives 1633/2000,

[> RootApproximate[0.816496] gives 51031/62500 and

[> RootApproximate[0.816497] gives the correct root.

Using Maple 16,

[> identify(0.8165); gives the correct result (see above).

Also, Maple correctly identifies nearby approximations: 0.81649, 0.81651 and 0.816496 are all correctly identified.

For this simple problem, Mathematica could be used, but we note that higher precision is required for the approximation. Since the approximation is obtained by graphical means, this is a disadvantage. However Mathematica is not capable of identification of transcendental numbers. For example, $1 + e \approx 3.718281828$.

In Maple: [> identify(3.718); gives the correct result of $1 + e$.

With Mathematica,

[> RootApproximate[3.718] gives 1859/500 and

[> RootApproximate[3.718282] returns its answer as the third root of a degree 9 polynomial! Mathematica cannot identify non algebraic numbers.

Research using an experimental mathematics methodology may need to compute an approximation accurate to many hundreds of decimal digits to identify a possible exact answer (which might be transcendental), which we then seek to prove is correct, see [6, 7].

6 Proof without Calculus: maximize Volume Squared

Now that we have the (possible) exact answer, we prove that it is the correct answer. By not using calculus, we engage with the experimental mathematics methodology and connect several topics in mathematics as well as enhance our understanding and visualization of how functions behave and are approximated. The objective function is Volume, V , as a function of the radius, r , where $V = 2\pi r\sqrt{1-r^2}$. The presence of the square root complicates the mathematics, but we notice that when V is a maximum, then V^2 is also a maximum. Thus we work with V^2 :

[> V_Sq:= 4π²r²(1 - r²);

Now let $r = \sqrt{2/3} + \delta$ and we postulate that $\delta = 0$ gives the maximum V^2 . Note that the graph of $V^2(\delta)$ corresponds to a horizontal translation of the graph of $V^2(r)$. We have $V^2(\delta)$:

[> V_Sq_delta:=subs(r=sqrt(2/3)+delta, V_Sq); which expands to

$$V_Sq_delta = 4\pi^2 \left(4/27 - (8/3)\delta^2 - (28/9)\sqrt{6}\delta^3 - 9\delta^4 - 2\sqrt{6}\delta^5 - \delta^6 \right).$$

This is a sixth degree polynomial. How does it behave near the origin (that is, for δ small)?

Algebraically we know that if δ is small (in magnitude) then δ^2 is smaller (than $|\delta|$). Similarly $|\delta^3|$ is even smaller. Thus, for δ small, then δ^2 is small (compared to δ , and δ^3 is small (compared to δ^2), etc. Thus we know that, for δ small, V^2 can be approximated by neglecting all terms with powers of δ higher than δ^2 : we call this approximation $V_Sq_delta_2$.

Maple has a command, `coeff(,)`, to give the coefficients of a polynomial, so it is easy to obtain $V_Sq_delta_2$:

$$V_Sq_delta_2 = \frac{16}{27}\pi^2 - \frac{32}{3}\pi^2\delta^2.$$

This is a quadratic polynomial with a maximum at $\delta = 0$ which proves our result. The maximum volume is also easy to obtain: take the square root of the constant term in $V_Sq_delta_2$ to give $4/9\sqrt{3}\pi$.

Note that we can rewrite the quadratic approximation to V^2 in terms of r by translating with the substitution of $\delta = r - \sqrt{2/3}$. The plot of the sixth degree polynomial $V^2(r)$ is displayed with the quadratic approximation (as derived above) on the left of Fig. 5. This visualization reinforces our intuition that the graph of a function “looks like a parabola” near the function’s maximum.

7 Proof without Calculus: maximize Volume

We now directly show that our value for r does give the maximum of the function $V = 2\pi r\sqrt{1-r^2}$. As mentioned in the previous section, the presence of the square root complicates the mathematics, but we indicate how this can be done. Again, we postulate that

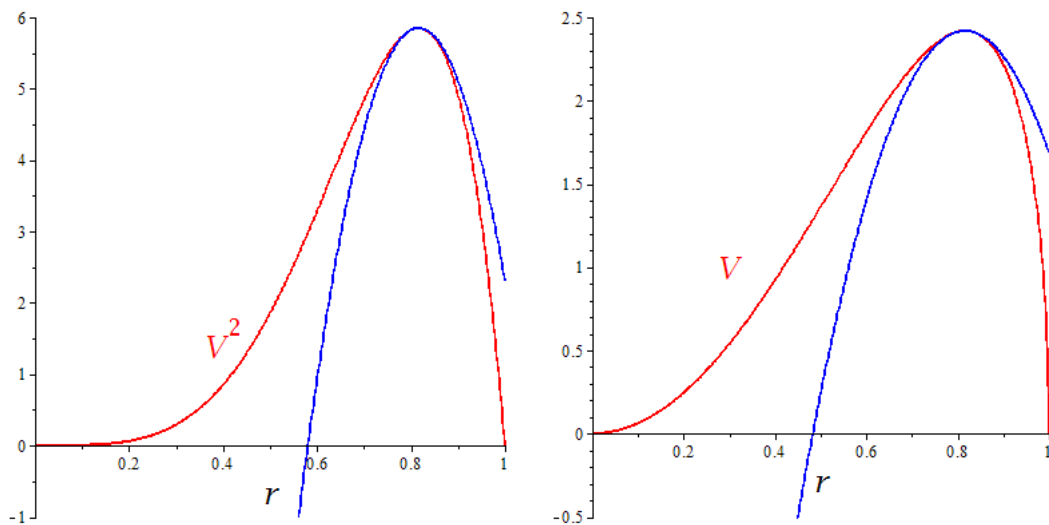


Figure 5: On the left, a plot of volume squared of the cylinder and its quadratic approximation (about the maximum) and, on the right, a plot of volume and its quadratic approximation.

$rMax = \sqrt{2/3}$ maximizes the volume. We let $r = rMax + \delta$ (initially leaving $rMax$ as undefined to avoid the algebra looking too messy) and we postulate that $\delta = 0$ gives the maximum V . Note that the graph of $V(\delta)$ corresponds to a horizontal translation of the graph of $V(r)$. We have:

$$V(\delta) = 2\pi(rMax + \delta)^2 \sqrt{1 - (rMax + \delta)^2}$$

For a maximum, the curve is approximately a parabola for δ small. So we need to approximate the square root for δ small (up to powers of 2). Now the generalized Binomial Theorem for a power of $1/2$ where $|x/a| < 1$ is

$$\sqrt{1 - x/a} = 1 - \frac{1}{2a}x - \frac{1}{8a^2}x^2 + O(x^3)$$

where $O(x^3)$ means terms with x^3 and higher powers. This is easily proven by algebra: let $\sqrt{1 - x} = a + bx + cx^2 + \dots$, square, and hence result (by equating coefficients of each power of x).

Exercise: Prove this result.

Now, rewrite: using the Binomial Theorem above and assuming δ is small (but only keeping terms up to δ^2 since higher powers of δ are negligible), it requires a little algebra to obtain the approximation as a quadratic in δ . The next step is to multiply the two quadratic expressions in the formula for $V(\delta)$, discarding all higher powers (than δ^2) to obtain the quadratic approximation. This algebra is straightforward, but a little messy, giving the approximation

$$2\pi rMax^2 \sqrt{1 - rMax^2} - \frac{2\pi rMax(3rMax^2 - 2)}{\sqrt{(1 - rMax^2)}} \delta + \frac{(6rMax^4 - 9rMax^2 + 2)\pi}{(1 - rMax^2)^{3/2}} \delta^2.$$

This could be obtained by using the high level command `series()`, then convert the result to a polynomial, followed by using the `simplify()` command:

```
[> series(V_delta, delta, 3);      (where V_delta is the V(delta) above)
[> convert(%, polynom);          (where % is the previously calculated result)
[> simplify(% )
```

Teaching Note. We support the White-Box/Black-Box Principle first articulated by Bruno Buchberger in 1989 [8]. Students using CAS should initially include every step of their working until they are secure in their understanding (the White Box), after which they should use the higher level commands provided by the CAS (the Black Box). Maple provides a command that computes the series. Students should initially master the notion and the skills to calculate the series “by hand” (either by pen on paper or step by step calculation using the CAS) before using the high level commands such as `series()`.

Now substitute our postulated value $\sqrt{2/3}$ for $rMax$ to obtain:

$$V_delta_2 = \frac{4\sqrt{3}}{9}\pi - 4\sqrt{3}\pi\delta^2.$$

This quadratic polynomial clearly has a maximum at $\delta = 0$ as postulated and hence our result that the volume has a maximum when the radius is $\sqrt{2/3}$.

The plot of $V(r)$ is displayed with the quadratic approximation (as derived above) on the right of Fig. 5. This visualization reinforces our intuition that the graph of a function “looks like a parabola” near the function’s maximum.

7.1 The quadratic approximation for V implies that for V^2 and vice versa

Knowing the quadratic approximation for V , it is simple to derive the quadratic approximation for V^2 .

Exercise. Do it.

It is also possible to derive the quadratic approximation for V from knowing the quadratic approximation for V^2 .

Exercise. Do it.

(Hint: Let the quadratic approximation for V be $c + d\delta + e\delta^2$; square this, neglecting all powers of δ greater than 2; equate to the known result and equate coefficients of powers of δ .)

8 Further Solution Methods (Proofs)

It is common for there to be several methods of solving a mathematical problem. For the “find the maximum . . .” problems common in secondary school and first year mathematics, calculus is invariably used. However it is good for students to develop their mathematical maturity by following several methods, and this is the focus in [5]. Calculus provides the most efficient method here, but does not provide the connection to (and development of) other important topics and methods. The problem here can be solved by many different methods. We indicate another two objective functions that can be obtained: they both result in cubic polynomials.

First. In Sec. 6, $V^2 = V_Sq$, we notice that only powers of r^2 appear. Thus if we consider $V^2(r^2)$, the objective function is a cubic polynomial in r^2 . It is easy to identify that the maximum is given by $r^2 = 2/3$. This is also easy to find directly since it is almost as easy to

find where the maximum (or minimum) lies for cubic polynomials as for quadratics: see Section 4 of [4].

Second. In Sec. 5, the initial volume function depends on both r and h . We choose to eliminate the h variable, but if r is eliminated instead, the volume function becomes a cubic polynomial in h , giving a problem which is much easier to solve than using $V(r)$.

We also indicate a further variation on the solution using $V(r)$, by using the approach introduced in [4]. In Sec. 7, the quadratic approximation for $V(\delta)$ was obtained with $rMax$ unspecified. We derive the value for $rMax$ by noticing that the maximum for the quadratic approximation for $V(\delta)$ must occur when the coefficient of δ is zero. Thus, we must have

$$-\frac{2\pi rMax(3 rMax^2 - 2)}{\sqrt{(1 - rMax^2)}} = 0 \quad \Rightarrow \quad 3 rMax^2 - 2 = 0$$

and hence the result.

9 A Parameterized Problem and Automatic Marking: Computer Aided Assessment, CAA

After students groups have worked through the Cylinder in Sphere problem (in one or two sessions), they are asked to complete an assignment (for assessment), where students could be asked to follow the method above to maximize the volume or the volume squared. We parameterize the assignment problem since presenting the same problem to the whole class invites copying. Our experience over many years with a variety of courses and year levels is that students consider parameterized problems to be different problems. We always try to include a lot of visualization: this requires plots and animations for which parameters cannot be arbitrary: they must have assigned values. Thus, there is not a problem with copying (even with high level courses where the assignments contribute up to 80% of the assessment).

For the Cylinder in Sphere problem, an obvious parameterization is to use the radius of the sphere R . A less obvious (but closely related) problem is to inscribe a (vertical) cylinder in a Spheroid (a special case of an ellipsoid usually used to model the Earth's surface):

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

The r to maximize the volume of the inscribed cylinder is $\sqrt{2/3}a$. Choosing the values of the parameter(s) to use is always important to ensure that some students do not encounter some unintended difficulty. It is difficult to predict what precision (that is, number of correct decimal digits) will be required to identify a given number. By experiment, we have established that higher precision is required in order to identify a rational multiple of $\sqrt{2/3}$ than a multiple of $\sqrt{2}$ or $\sqrt{3}$ (or their reciprocals). For example, if $a = 37/10$, then 23 decimal place accuracy is required to identify r that gives the maximum. Thus we recommend setting a as a rational multiple of $\sqrt{2}$ or $\sqrt{3}$. We have checked several values, including
for $a = 11\sqrt{2}$, 7 digits are required to identify r that gives the maximum,
for $a = 13/2\sqrt{2}$, 6 digits are required to identify r that gives the maximum,
for $a = 23\sqrt{2}$, 6 digits are required to identify r that gives the maximum,

for $a = 11\sqrt{3}$, 7 digits are required to identify r that gives the maximum,
for $a = 16/3\sqrt{3}$, 5 digits are required to identify r that gives the maximum.

So we recommend setting a to be multiple (integer, half integer, or third integer) times $\sqrt{2}$ or $\sqrt{3}$. The other parameter c only appears as a multiple in the formula for volume, so we recommend choosing c as a convenient rational which is not very different in value to a so that the spheroid looks like a sphere that is not deformed very much from the sphere.

With parameterized assignments, the marking load increases. Initially we used e-Marking, see [3] for a discussion and references. The basic idea is to have a Maple solution/marking file in which the student group's parameter value is entered and the Maple worksheet is executed. In this file, a Marking Section is progressively developed so that the Marking Report and the feedback comments (that are saved in the student's file that is returned to them) are saved for later use. With significant content being visualization with animations, using e-Marking (without CAA) is an attractive option, particularly for small classes.

By CAA we mean CAS-enabled CAA in which some CAS is used by the CAA system for marking: thus answers that are symbolic (as well as numeric and many other types, including multiple choice) can be marked. The only commercial CAS-enabled CAA using a professional level CAS is MapleTA which uses Maple. We could mark the Cylinder in Spheroid assignment final answer and intermediate steps (such as the quadratic approximation for volume) symbolically (writing a provided procedure within Maple, or by using MapleTA). The number of decimal places needed to correctly identify the answer could also be asked for. Using CAA would reduce the marking load of staff, which becomes a major problem for large classes. Many universities have between 1000 to 2000 students in the normal first year calculus class and are faced with having to choose between assessing less student work, or implement CAA. Similar issues arise in schools which can have large enrolments in the senior school. However many teachers might be concerned that key parts of the assignment are not marked (and so ignored by students): for example, here, a zoomed in plot is neither seen nor marked!

We emphasize visualization and believe that visualization needs some assessment: otherwise the message for students is that visualization is not important. We have recently designed a CAS-immersed CAA in which plots can be marked efficiently within an automatic marking context: this is implemented for the Norman window problem, see Section 6 of [3] and references therein.

The recent development of MapleTA's "Question Designer" has made it much easier to author questions (as we have demonstrated for the Norman Window problem in several unpublished presentations). An advantage with using MapleTA is that all of the student administration of results is dealt with automatically. Although plots are not marked, using MapleTA requires less Maple expertise than writing CAS-immersed CAA. Most academics would prefer to mark more than the final answer and provide more feedback than the answer is right or wrong. For e-Marking and CAS-immersed CAA, the extra feedback is desirable (and reasonably straightforward to achieve). However in both cases students understand that they have to use the required method (since their Maple file has to be submitted). For CAS-enabled CAA, the issue is more complicated because the student work is not available. This issue can be addressed by careful design of multiple part (or Multiple Response) questions: the new Question Designer in MapleTA (from version 8) has made the authoring of Multiple Response questions easy.

10 Conclusion

A CAS such as Maple provides a rich environment for e-Teaching and e-Marking or CAA of mathematics: innovative approaches to curriculum, pedagogy and assessment at school and university are supported. Standard problems solved in the standard way using CAS risk being another activity (such as lectures and computer sessions) that students find boring. Well designed teaching and assessment is effective, efficient and even fun. The development of these CAS based materials requires a lot of staff time, but they give high returns with staff satisfaction, student performance and attitude. With large classes, the reduced marking load using automatic marking more than repays the development cost. We recommend more collaborative learning by students and more collaborative development of CAS materials by academics.

Experimental mathematics is now an established area of mathematics: it has had its own research journal for over 20 years, and there have been some graduate level courses. Although the use of CAS is relatively common, not all use of CAS constitutes experimental mathematics: for example, routine use of CAS to execute simple evaluations (such as evaluating integrals that are typically asked for in secondary or first year undergraduate courses) are no more experimental mathematics than the evaluation of $\sqrt{2}$ with a calculator. In this paper we show that elementary optimization problems (such as “find the maximum volume of a cylinder inscribed in a sphere”) can introduce the experimental mathematics methodology and be a rich source of mathematics if the usual (and efficient) method of calculus is not used. We show advanced visualization to build intuition; a graphical method (zooming in) of obtaining a good numerical answer; the use of the Maple command `identify()` to identify a possible exact solution; and several different methods using algebra to obtain the quadratic approximation of the objective function about its maximum: hence proving the postulated maximum is correct. This approach introduces an important topic in mathematics: approximation in the small. It also provides strong support for (and visualization of) the intuitive notion that the function “looks like a parabola” near its maximum.

The Maple activities described here are all web mediated, so they could be part of an online course or a blended learning course: students have different learning styles and some do take advantage of the flexible learning that the approach here supports (since attendance in the computer lab is recommended but not required). Students enjoy the collaborative learning using Maple and learn (surprisingly quickly) to do the calculations; use and adapt advanced animations; and use automatic marking. Students are engaged, active and collaborative learners with these Maple sessions.

Acknowledgements

The author thanks Dr Asim Ghous, the Director of ASES, for support with Maple over many years (and MapleTA more recently) and ASES for financial support to prepare this paper and to be a participant at the ATCM 2014 Conference.

References

- [1] Blyth, B., “Animations using Maple in First Year”, *Quaestiones Mathematicae, Suppl.*, **1**, Supplement, 2001, 201–208

- [2] Blyth, B. and Labovic, A., “Active Learning and Fun in First Year using Maple”, in *Buffelspoort TIME2008 Peer-reviewed Conference Proceedings*, P.H. Kloppers, C.S. Joubert and M.E. Joubert, eds., Technology Integration into Mathematics Education (TIME) Conference, Buffelspoort, South Africa, September 2008, published (CD and online) ISBN: 978-0-620-43454-6, 2009, by Tshwane University of Technology, Pretoria. Available at [http://time.tut.ac.za/public/\(1\)_Peer-Reviewed_PROC_TIME2008.pdf](http://time.tut.ac.za/public/(1)_Peer-Reviewed_PROC_TIME2008.pdf)
- [3] Blyth, B., “e-Teaching and e-Assessment of Minimum-Maximum Problems using Maple”, in *Proc. 17th Asian Technology Conference in Mathematics: ATCM 2012*, W.-C. Yang, et al., eds., published by Mathematics and Technology (<http://mathandtech.org>), USA, ISBN 978-0-9821164-4-9 (hard copy), ISSN 1940-2279 (CD), ISSN 1940-4204 (online version), 2012, 353–362. Available at <http://atcm.mathandtech.org/EP2012/>
- [4] Blyth, B., “Maximum problems without calculus: Design, teaching and assessment using Maple”, in *Shining through the fog: Proceedings of the 9th DELTA conference on the teaching and learning of undergraduate mathematics and statistics*, D. King, B. Loch and L. Rylands, eds., published by The University of Western Sydney, School of Computing, Engineering and Mathematics, November, 2013. ISBN: 978-1-74108-289-0. Printed in Melbourne, Australia. 2013, 12–21. Available from <http://delta2013.net/documents/ConferenceProceedings.pdf>
- [5] Blyth, B., “Maximization using Maple: Polya approach with multiple representation with or without calculus”, in *Mathematical Association of Victoria 50th Annual Conference Proceedings*, A. McDonough, A. Downton and L.A. Bragg, eds., published by The Mathematical Association of Victoria, Melbourne, Australia, ISBN 978-1-876949-53-2, 2013, 3–12.
- [6] Borwein, J.M., and Devlin, K., *The Computer as Crucible*, A.K. Peters, MA, 2008
- [7] Borwein, J.M., “Exploratory experimentation: digitally-assisted discovery and proof”, *AustMS Gazette*, **36, 3**, (July) 2009, 166–175. Accessed May 2014 at <http://www.austms.org.au/Publ/Gazette/2009/Jul09/MathsMatters.pdf>
- [8] Buchberger, B., “Why Should Students Learn Integration Rules? (The White-Box / Black-Box Principle for Using Symbolic Computation Systems in Math Education.)”, RISC-Linz Technical Report no. 89-7.0, 1989, Univ of Linz, Austria. Available at http://www.risc.jku.at/publications/download/risc_350/1990-00-00-A.pdf
- [9] Mann, S. and Robinson, A., “Boredom in the lecture theatre: an investigation into the contributors, moderators and outcomes of boredom amongst university students”, *British Educational Research Journal* **35(2)**, April 2009, 243–258
- [10] Pólya, G., *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving*, (combined edn), John Wiley, New York, 1981
- [11] Zeilberger, D., “[Contemporary Pure] Math Is Far Less Than the Sum of Its [Too Numerous] Parts”, Opinion, Notices AMS, Dec 2013, 1431. DOI:<http://dx.doi.org/10.1090/noti1061>