The Story of a Research
About the Nets of Platonic Solids with Cabri 3D: Conjectures Related to a Special Net Factor
A Window for New Researches

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Abstract: *In 2007, in the paper presented during ATCM, I showed some very strange results about the maximum of the volume of the convex hull of the folded net of a cube. This result found experimentally with Cabri 3D was validated with the use of the CAS of the Voyage 200 of Texas Instruments. The property states that this maximum is reached when the ratio between the volume of the convex hull and the given cube is 4,0141… We noticed that the angle corresponding to this maximum is given by 40,141…°. We have also proven that only the first five digits of these two numbers are the same. Several years later, I revisited this problem, trying to find experimentally some more interesting properties about this convex hull. I began to explore a possible link between this ratio and the corresponding ratio between the area of the convex hull and the area of the initial cube when the volume is maximized. Very quickly, I pointed a possible property about the value of the ratio of these ratios that could be* $\pi/2$. *It was so unexpected that I wanted to confirm this conjecture. That is the beginning of this research I want to describe. Lots of conjectures will appear experimentally that we will try to corroborate experimentally. We will study this problem analytically in order to increase the accuracy of the special number we want to discover using different CAS software. Even if the conjectures for the cube and other Platonic solids are wrong, it opens a window on a more general conjecture about the value of the special factor of a convex polyhedron in relation with the factors of the Platonic solids. This last conjecture seems to be a very difficult problem: this is a problem for those who are interested by such properties. Last remark: most of the analytic formula have been validated with Cabri 3D.*

1. Experiments, conjecture and validation with a net of cube (focus on NFC)

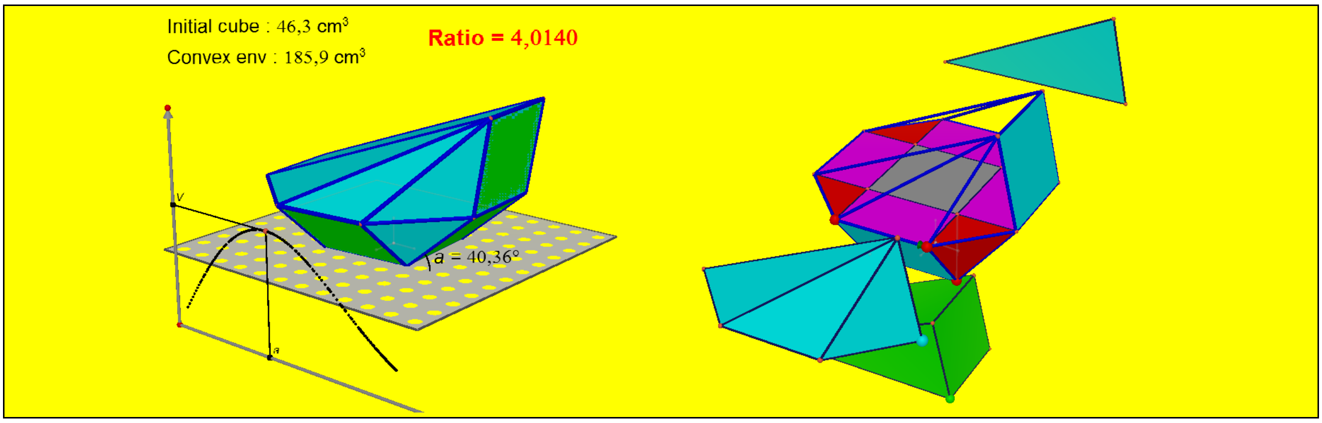

Figure 1

1.1. What we knew (ATCM Taiwan 2007, [3], Figure 1)

With Cabri 3D, we have created the net of a cube, started to unfold it and created the convex hull of this unfolded net. We have measured and displayed the volume V of the cube we first constructed and the volume $V(x)$ of the convex hull (x is the angle between the horizontal plane and the plane of one lateral face). We used the calculator of the software to evaluate the ratio $V(x)/V$ (this ratio is independent of the size of the initial cube). After that, we evaluated experimentally the maximum of this ratio and the value of x corresponding to this maximum. The best we could obtain experimentally was that the maximum of the ratio is "equal" to **4,014** for an angle x_m in degrees "equal to" **40,14°** (**Figure 1** on the left). That was surprising and unexpected. My research colleagues thought that this ratio was too complicated to be evaluated by the software and that the correct ratio might be 4 and not 4,014. So, starting with a cube of volume 1, I evaluated exactly $V(x)$ and obtained: $\boxed{V(x) = \frac{1}{6}(3+2\cos(x))^2\sin(2x)+\sin(x)}$. Using the Voyage 200 of TI, I improved the previous results in increasing the number of digits. I could confirm to my colleagues that the software was reliable for my experiments because I got with the Voyage 200:

4,014137 for the maximum of the ratio, and **40,141113°** for the angle x_m in degrees.

Only the first five digits of these numbers are the same. I tried to identify these two numbers with some known numbers with the Plouffe Inverse Symbolic Calculator (http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html) but unsuccessfully.

1.2. Similar experiments with volumes and areas

In the previous investigation, I used the model of folding-unfolding net of the software, I continued to experiment with the same one. My new idea was to explore the possible value of the ratio of this maximum and the area of the convex hull having this maximum volume. Quickly, I changed the area by the ratio of this area by the area of the cube. I conducted the same experiment but I had only to evaluate and display the ratio $\boxed{\textbf{NFC} = \frac{V(x_m)}{V}:\frac{A(x_m)}{A}\ (\textbf{NFC: NET FACTOR}\text{ of the }\textbf{CUBE})}$.

1.2.1. Unfolding the cube directly with the Cabri 3D model: the convex hull is folded and unfolded in dragging the mouse and the best we can obtain is

NFC = 1,5727818 when we maximized $\frac{V(x)}{V}$ at 4,01413693 and x_m = 40,1451562261°

1.2.2. Unfolding the cube with a net I created in changing the values of x (in degrees): we display a value of x close to the one conjectured by the previous method and we change it in a dichotomic way to increase the accuracy of the data generated by the experiment. We do it until we maximize the ratio $V(x)/V$. When we maximize it ($V(x)/V$ = 4,0141369732 and x_m = 40,141030°) we get:

NFC = 1,5707523154 which seems very close to $\pi/2$ = 1,5707963268. So, because the difference between these two numbers is less than 5.10^{-5}, **I conjecture that NFC =** $\frac{\pi}{2}$.

1.3. Analytic process of validation of this conjecture

I have chosen to evaluate NFC with a cube of unit volume (size of each side is 1).

We know that $V(x) = \frac{1}{6}(3+2\cos(x))^2\sin(2x)+\sin(x)$.

I evaluated the area of the visible faces of the convex hull for a (see below 1.4.):

$$\boldsymbol{A}(a) = 6+\cos(a)(2\sqrt{1+\sin^2(a)}+4\cos(\frac{a}{2})+\sqrt{1+\sin^2(2a)}+\sqrt{1+4\sin^2(a)}+4\cos(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})})$$

I programmed the function $\frac{V(x)}{V}:\frac{A(x)}{A} = \frac{6.V(x)}{A(x)}$ in the CAS of TI N'Spire, evaluated the value x_m of x maximizing V between 0 and $\frac{\pi}{2}$ to obtain the best approximation I could obtain with the help of technology: **NFC** = 1,5707536563781. This result was confirmed by Professor Yakoubsohn from the Paul Sabatier University of Toulouse with *Maple*.

So my conjecture was definitely rejected. This number evaluating NFC, looking like $\frac{\pi}{2}$ was not $\frac{\pi}{2}$ but the difference between NFC and this number is less than 4.10^{-5}. This result states that the results given thanks to Cabri are reliable: the difference between NFC and the number got with the Cabri experiments is less than10^{-6}. Nevertheless I was intrigued by the fact that NFC is so close to $\frac{\pi}{2}$. I decided to conduct the same experiments with the other platonic solids in order to find the value of this special ratio, hoping to be more successful.

1.4. Analytic expression of the area of the convex hull of the cube (used in1.3.)

I have chosen to evaluate NFC with a cube of volume 1 (size of each side is 1).

Using the same decomposition of the convex hull used to evaluate its volume, I evaluated its area :

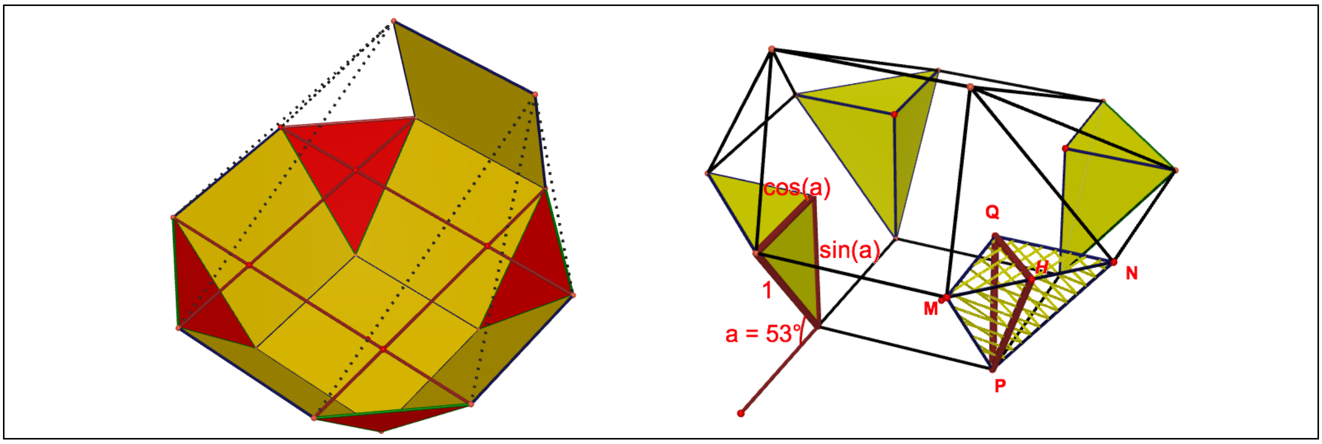

Figure 2

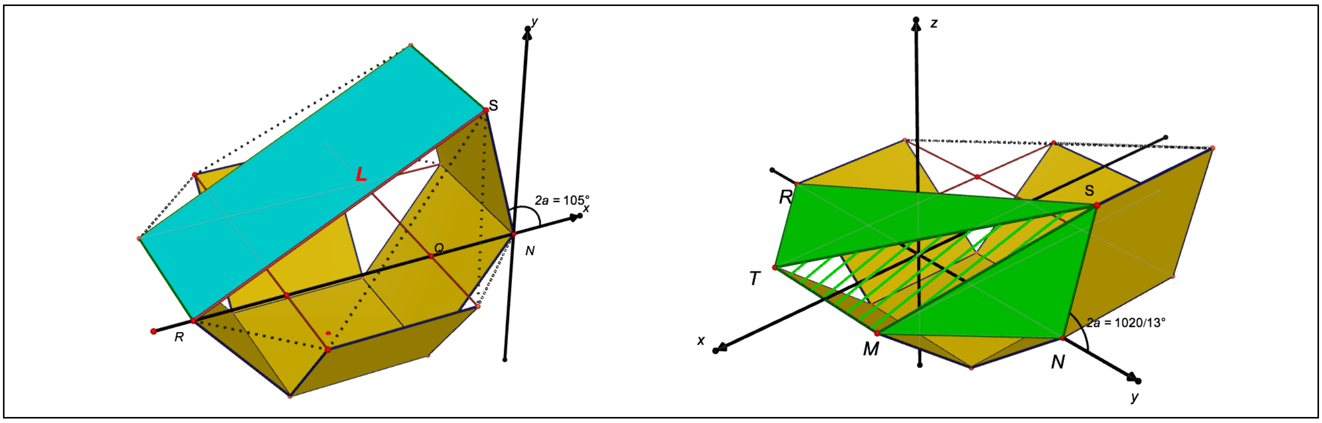

Figure 3

1.4.1. Area of the six squares (**Figure 2** on the left)

It is trivial to get the area of the 6 squares : $A_1 = 6$

1.4.2. Area of the 4 visible faces of the 4 corner pyramids (**Figure 2** on the right)

$$MN = \sqrt{2}\cos(a)\ ,\ QH = \frac{\sqrt{2}}{2}\cos(a)\ ,\ HP = \sqrt{QP^2 + QH^2} = \sqrt{(\sin^2(a) + \frac{1}{2}\cos^2(a)} = \sqrt{\frac{2\sin^2(a)+\cos^2(a)}{2}}$$

Therefore $HP = \sqrt{\dfrac{1+\sin^2(a)}{2}}$. So the area of triangle MNP is : $\frac{1}{2}$ $MN.HP$ = $\frac{1}{2}\cos(a)\sqrt{1+\sin^2(a)}$. So :

The area of the 4 visible faces of the 4 corner pyramids is given by : $A_2 = 2\cos(a)\sqrt{1+\sin^2(a)}$

1.4.3. Area of the top rectangular face (Figure 3 on the left)

In the chosen system of axes, coordinates of S and R are given by: $S\left|\begin{matrix}\cos(2a)\\ \sin(2a)\end{matrix}\right.$ et $R\left|\begin{matrix}-1-2\cos(a)\\ 0\end{matrix}\right.$,

So $L = SR = \sqrt{(1+\cos(2a)+2\cos(a))^2+\sin^2(2a)} = \sqrt{(2\cos^2(a)+2\cos(a))^2+\sin^2(2a)}$

$= \sqrt{4\cos^2(a)(1+\cos(a))^2+4\sin^2(a)\cos^2(a)} = 2\cos(a)\sqrt{(1+\cos(a))^2+\sin^2(a)}$

$= 2\cos(a)\sqrt{1+2\cos(a)+\cos^2(a)+\sin^2(a)} = 2\cos(a)\sqrt{2+2\cos(a)}$

$= 2\cos(a)\sqrt{4\cos^2(\frac{a}{2})} = 4\cos(a)\cos(\frac{a}{2})$ and finally

The area of the rectangular face is $L.1$ eqal to: $A_3 = 4\cos(a)\cos(\frac{a}{2})$

1.4.4. Area of the visible faces of the two top lateral pyramids (Figure 3 on the right)

$S\left|\begin{matrix}0\\ \frac{1}{2}+\cos(a)+\cos(2a)\\ \sin(2a)\end{matrix}\right.$, $N\left|\begin{matrix}0\\ \frac{1}{2}+\cos(a)\\ 0\end{matrix}\right.$, $M\left|\begin{matrix}\cos(a)\\ \frac{1}{2}\\ 0\end{matrix}\right.$, $T\left|\begin{matrix}\cos(a)\\ -\frac{1}{2}\\ 0\end{matrix}\right.$ et $R\left|\begin{matrix}0\\ -\frac{1}{2}-\cos(a)\\ 0\end{matrix}\right.$, then :

The area of the visible faces of one of these pyramids like SRTMN (3 visible faces for each pyramid) is the sum of the areas of triangles SNM, SMT and STR :

-Area of triangle SNM = $\frac{1}{2}\left\|\overrightarrow{SN}\wedge\overrightarrow{SM}\right\| = \frac{1}{2}\left\|\begin{bmatrix}0\\ -\cos(2a)\\ -\sin(2a)\end{bmatrix}\wedge\begin{bmatrix}\cos(a)\\ -\cos(a)-\cos(2a)\\ -\sin(2a)\end{bmatrix}\right\|$

$= \frac{1}{2}\left\|\begin{bmatrix}\cos(2a)\sin(2a)-\sin(2a)\cos(a)-\sin(2a)\cos(2a)\\ -\sin(2a)\cos(a)\\ \cos(a)\cos(2a)\end{bmatrix}\right\|$

$= \frac{1}{2}\left\|\begin{bmatrix}-\sin(2a)\cos(a)\\ -\sin(2a)\cos(a)\\ \cos(a)\cos(2a)\end{bmatrix}\right\| = \frac{1}{2}\cos(a)\sqrt{2\sin^2(2a)+\cos^2(2a)} = \boxed{\frac{1}{2}\cos(a)\sqrt{1+\sin^2(2a)}}$

-Area of triangle SMT $= \frac{1}{2}\left\|\overrightarrow{SM}\wedge\overrightarrow{ST}\right\| = \frac{1}{2}\left\|\begin{bmatrix}\cos(a)\\ -\cos(a)-\cos(2a)\\ -\sin(2a)\end{bmatrix}\wedge\begin{bmatrix}\cos(a)\\ -1-\cos(a)-\cos(2a)\\ -\sin(2a)\end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix}-\sin(2a))\\ 0\\ -\cos(a)\end{bmatrix}\right\|$

$= \frac{1}{2}\sqrt{\sin^2(2a)+\cos^2(a)} = \frac{1}{2}\sqrt{4\sin^2(a)\cos^2(a)+\cos^2(a)} = \boxed{\frac{1}{2}\cos(a)\sqrt{1+4\sin^2(a)}}$

-Area of triangle STR = $\frac{1}{2}\left\|\overrightarrow{ST}\wedge\overrightarrow{SR}\right\| = \frac{1}{2}\left\|\begin{bmatrix}\cos(a)\\-1-\cos(a)-\cos(2a)\\-\sin(2a)\end{bmatrix}\wedge\begin{bmatrix}0\\-1-2\cos(a)-\cos(2a)\\-\sin(2a)\end{bmatrix}\right\|$

$$= \frac{1}{2}\left\|\begin{bmatrix}-\sin(2a)\cos(a)\\ \sin(2a)\cos(a)\\ \cos(a)(-1-2cos(a)-\cos(2a)\end{bmatrix}\right\| = \frac{1}{2}\cos(a)\left\|\begin{bmatrix}-\sin(2a)\\ \sin(2a)\\ -1-2\cos(a)-co(2a)\end{bmatrix}\right\| = \frac{1}{2}\cos(a)\left\|\begin{bmatrix}-2\sin(a)\cos(a)\\ 2\sin(a)\cos(a)\\ -2\cos(a)-2\cos^2(a)\end{bmatrix}\right\|$$

$$=\cos^2(a)\left\|\begin{bmatrix}-\sin(a)\\ \sin(a)\\ -1-\cos(a)\end{bmatrix}\right\| = \cos^2(a)\left\|\begin{bmatrix}-2\sin(\frac{a}{2})\cos(\frac{a}{2})\\ 2\sin(\frac{a}{2})\cos(\frac{a}{2})\\ -2\cos^2(\frac{a}{2})\end{bmatrix}\right\| = 2\cos^2(a)\cos(\frac{a}{2})\left\|\begin{bmatrix}-\sin(\frac{a}{2})\\ \sin(\frac{a}{2})\\ -\cos(\frac{a}{2})\end{bmatrix}\right\|$$

$$= 2\cos^2(a)\cos(\frac{a}{2})\sqrt{2\sin^2(\frac{a}{2})+\cos^2(\frac{a}{2})} = 2\cos^2(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})} = \boxed{2\cos^2(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})}}$$

Finally the area of the 6 visible faces of these 2 pyramids is given by :

$2(\frac{1}{2}\cos(a)\sqrt{1+\sin^2(2a)}+\frac{1}{2}\cos(a)\sqrt{1+4\sin^2(a)}+2\cos^2(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})}\ )$ which is also given by

Area of the visible faces of the 2 top lateral pyramids:

$A_4=\cos(a)\sqrt{1+\sin^2(2a)}+\cos(a)\sqrt{1+4\sin^2(a)}+4\cos^2(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})}$

At last, the area of the open convex hull is given by $A_1+A_2+A_3+A_4$:

$$6+2\cos(a)\sqrt{1+\sin^2(a)}+4\cos(a)\cos(\frac{a}{2})+\cos(a)\sqrt{1+\sin^2(2a)}+\cos(a)\sqrt{1+4\sin^2(a)}+4\cos^2(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})}$$

So : $\boldsymbol{A}(x) = 6+\cos(a)(2\sqrt{1+\sin^2(a)}+4\cos(\frac{a}{2})+\sqrt{1+\sin^2(2a)}+\sqrt{1+4\sin^2(a)}+4\cos(a)\cos(\frac{a}{2})\sqrt{1+\sin^2(\frac{a}{2})})$

2. Experiments, conjecture and validation with a net of tetrahedron (NFT)

2.1. Investigations related to the Net Factor of the tetrahedron

The net factor of the tetrahedron is the value of $\frac{V(x)}{V}:\frac{A(x)}{A}$ when $x = x_m$ and where x_m is the value of x between 0 and $\frac{\pi}{2}$ maximizing $V(x)$. The notations are the same as those used for the cube.

2.2.1. Unfolding the tetrahedron directly with the Cabri 3D model (see technique in 1.2.2.)

NFT = 2.039702 when we maximized $\frac{V(x)}{V}$ at 4.94105 and x_m = 48.127953°.

2.2.2. Unfolding the tetrahedron with a net I created in changing the values of x (in degrees) (technique presented in 1.2.2). So, we maximized it ($\frac{V(x)}{V}$ = 4.9410588438 and x_m = 48.190°) and we got: NFT = 2.0403018464 which seems very close to (3/4)$\boldsymbol{e}$ = 2.0387113713. So, because the difference between these two numbers is less than 3/1000, **I conjectured that NFT** = (3/4)$\boldsymbol{e}$. At this stage of my work, I thought that this conjecture was unlikely to be correct.

2.3. Analytic process of validation of this conjecture

I have chosen to evaluate NFT with a tetrahedron which equilateral base, inscribed in a unit circle. After the work detailed below in 2.4, we obtained:

$V(x) = \dfrac{9\sqrt{3}}{32}\sin(x)(3+6\cos(x)+3\cos^2(x))$ and $V = \dfrac{\sqrt{6}}{4}$ and

$A(x) = 3\sqrt{3}+\dfrac{3\sqrt{3}}{16}(3\cos(x)+1)^2+\dfrac{9\sqrt{3}}{16}(3\cos(x)+1)\sqrt{-3\cos^2(x)-2\cos(x)+5}$ and $A = 3\sqrt{3}$

2.4. Analytic expressions of the area and the volume of the convex hull of the tetrahedron

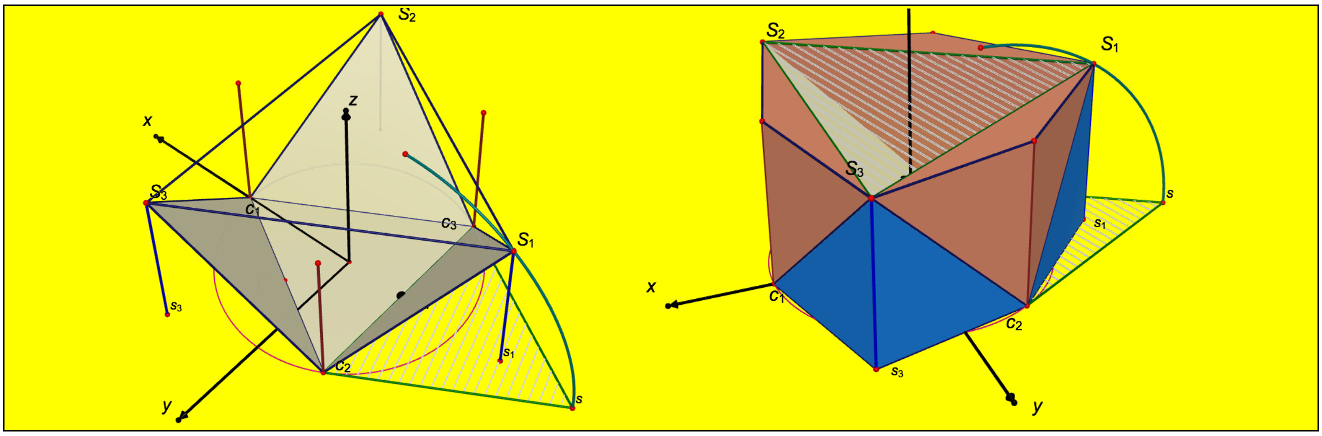

Figure 4

2.4.1. Area and volume of the initial tetrahedron (**Figure 4**)

The bottom of the tetrahedron is inscribed in a unit circle, so each side of this equilateral triangle is measured by $\sqrt{3}$ and its height by $\sqrt{2}$ The area of this equilateral triangle $\boldsymbol{c_1c_2c_3}$ is $A_b = \dfrac{3\sqrt{3}}{4}$. So the area of the initial tetrahedron is : $A = 3\sqrt{3}$. As the height of this tetrahedron is given by $\sqrt{2}$ its volume is given by $\dfrac{1}{3}\dfrac{3\sqrt{3}}{4}\sqrt{2}$ so, the volume of the initial tetrahedron is V = $\sqrt{6}/4$.

We use a system of axis where : $c_1\begin{pmatrix}1\\0\\0\end{pmatrix}$, $c_2\begin{pmatrix}-1/2\\\sqrt{3}/2\\0\end{pmatrix}$, $c_3\begin{pmatrix}-1/2\\-\sqrt{3}/2\\0\end{pmatrix}$ and $s\begin{pmatrix}-2\\0\\0\end{pmatrix}$ (see **Figure 4**).

2.4.2. Analytic expression of the volume of the convex hull of the tetrahedron

To obtain $S_1 = r(s)$ where r is the rotation around c_2c_3, having a as an angle, we translate s (vector of translation : $\overrightarrow{HO}$) to obtain s', we rotate s' around the y axis (angle of rotation : a) to obtain S'_1 ; lastly we translate this point (vector of translation : $\overrightarrow{OH}$) to obtain S_1. As the matrix of rotation r is $\begin{pmatrix}\cos(a) & 0 & \sin(a)\\ 0 & 1 & 0\\ -\sin(a) & 0 & \cos(a)\end{pmatrix}$, we get $S'_1\begin{pmatrix}-(3/2)\cos(a)\\0\\(3/2)\sin(a)\end{pmatrix}$ and $S_1\begin{pmatrix}-(3/2)\cos(a)-1/2\\0\\(3/2)\sin(a)\end{pmatrix}$. S_2 and S_3 are obtained by iteration of the rotation around the z axis (angle of rotation: $2\pi/3$) which matrix is $\begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So we get $S_2\begin{pmatrix} -(3/4)\cos(a)+1/4 \\ -(3/4)\sqrt{3}\cos(a)-\sqrt{3}/4 \\ (3/2)\sin(a) \end{pmatrix}$ and $S_3\begin{pmatrix} (3/4)\cos(a)+1/4 \\ (3/4)\sqrt{3}\cos(a)+\sqrt{3}/4 \\ (3/2)\sin(a) \end{pmatrix}$ and

$s_1\begin{pmatrix} -(3/2)\cos(a)-1/2 \\ 0 \\ 0 \end{pmatrix}$, $s_2\begin{pmatrix} -(3/4)\cos(a)+1/4 \\ -(3/4)\sqrt{3}\cos(a)-\sqrt{3}/4 \\ 0 \end{pmatrix}$ and $s_3\begin{pmatrix} (3/4)\cos(a)+1/4 \\ (3/4)\sqrt{3}\cos(a)+\sqrt{3}/4 \\ 0 \end{pmatrix}$ their projections on the *xOy* plane. So the area of triangle $c_1c_2s_3$ is given by :

$\frac{1}{2}\left\|\overrightarrow{c_1c_2}\wedge\overrightarrow{c_1s_3}\right\| = \frac{1}{2}\left\|\begin{pmatrix} 0 \\ 0 \\ -(3\sqrt{3}/2)\cos(a) \end{pmatrix}\right\| = \frac{3\sqrt{3}}{4}\cos(a)$. Therefore the area of the base of the prism enveloping the convex hull of the tetrahedron is equal to A + 3. (Area of $c_1c_2s_3$) $=(3\sqrt{3}/4)(1+3\cos(a))$

As the height of this prism is $(3/2)\sin(a)$, its volume is given by $(9\sqrt{3}/8)(1+3\cos(a))\sin(a)$ Now to obtain the volume of the convex hull, we have to subtract 3 times the volume of the pyramid $S_3c_1c_2s_3$ and 3 times the volume of the pyramid $c_2S_1S_3C_2$.

3 times the volume of pyramid $S_3c_1c_2s_3$ $= 3\frac{1}{3}\frac{3\sqrt{3}}{4}\cos(a)\frac{3}{2}\sin(a) = 9\sqrt{3}/8\cos(a)\sin(a)$

3 times the volume of pyramid $c_2S_1S_3C_2$ $= 3.\frac{1}{3}\frac{3\sqrt{3}}{16}(-3\cos^2(a)+2\cos(a)+1)\frac{3}{2}\sin(a)$ which is also $\frac{9\sqrt{3}}{32}(-3\cos^2(a)+2\cos(a)+1)\sin(a)$ (details of the proof not given here)

As the volume of the prism enveloping the convex hull is $\frac{9\sqrt{3}}{8}(1+3\cos(a))\sin(a)$, the volume of the convex hull is given by: $\frac{9\sqrt{3}}{8}(1+3\cos(a))\sin(a)-\frac{9\sqrt{3}}{8}\cos(a)\sin(a)-\frac{9\sqrt{3}}{32}(-3\cos^2(a)+2\cos(a)+1)\sin(a)$

which is $V(a) = \frac{9\sqrt{3}}{32}\sin(a)(3+6\cos(a)+3\cos^2(a))$

Here, I programmed V on TI N'Spire and the results of my previous experiments were confirmed. With 3 digits TI N'Spire gave 0.841 for x_m evaluated in degrees by 48.1857505705 (experimentally I have obtained **48.1901**). As $V=\frac{\sqrt{6}}{4}$, $\boxed{\frac{V(x)}{V} = \frac{9\sqrt{2}}{16}\sin(x)(3+6\cos(x)+3\cos^2(x))}$

For more digits, we get x_m equal to 0.841068698 in radians and 48.189686675963° in degrees. The maximum of *V(x)/V* is evaluated with 4.941058844.

2.4.3. Analytic expression of the area of the convex hull of the « tetrahedron »
We obtain it in evaluating the areas of all the faces of this convex hull)
Area of the base: $3\sqrt{3}/4$. 3 times (area of the lateral triangle like $S_1c_2c_3$): $3.(3\sqrt{3}/4)$
Area of the top ($S_1S_2S_3$) : $(3\sqrt{3}/16)(3\cos(a)+1)^2$ (proof not given here)
3 times (area of the lateral triangle like $c_1S_2S_3$): $3.(3\sqrt{3}/16)(3\cos(a)+1)\sqrt{-3\cos^2(a)-2\cos(a)+5}$ (proof not given here). Therefore the area of the convex hull is given by:

$A(x) = 3\sqrt{3} + (3\sqrt{3}/16)(3\cos(a)+1)^2 + (9\sqrt{3}/16)(3\cos(a)+1)\sqrt{-3\cos^2(a) - 2\cos(a) + 5}$

As $A = 3\sqrt{3}$, $\boxed{\frac{A(x)}{A} = 1 + \frac{1}{16}(3\cos(a)+1)^2 + \frac{3}{16}(3\cos(a)+1)\sqrt{-3\cos^2(a) - 2\cos(a) + 5}}$

2.4.4. Symbolic evaluation of NFT

NFT, with our definition, is $q(x_m)$ where $q(x) = (V(x)/V)/(A(x)/A)$ and x_m is the value of x maximizing $V(x)$ or $V(x)/V$. With TI N'Spire, for the value of x_m got with the **fmax** tool, the best we could obtain was : NFT = **2.0402987976379** instead of **2.0403028103** obtained with the Cabri experiments. Recall that (3/4).***e*** ≈ 2.0387113713. Even if the difference with this number is less than 2.10^{-3}, this second conjecture was rejected.

3. Experiments, conjectures without symbolic validation with the nets of the other Platonic solids (NFO, NFD, NFI)

3.1. Investigations related to the Net Factors of the octahedron, the icosahedron and the dodecahedron

I have decided to experiment again with Cabri 3D, to evaluate experimentally the value of the Net Factors of the other Platonic solids in using the model provided by Cabri 3D to fold and unfold the convex hulls of their nets. Here are the results of these experiments.

Octahedron : NFO =1.614090 close to the golden ratio 1.618034 (difference less than 3/1000)

Icosahedron : NFI = 1.4067545 close to $\sqrt{2}$ which value is 1.414213 (difference less than 8/1000)

Dodécahedron : NFD = 1.5174 close to a lot of known numbers with the Plouffe Inverse Symbolic Calculator. So with 2 digits, we have :

$$\boxed{\text{NFI} = 1.41 \leq \text{NFD} = 1.52 \leq \text{NFC} = 1.57 \leq \text{NFO} = 1.61 \leq \text{NFT} = 2.04}$$

3.2. Some other investigations with other model of folding-unfolding of platonic solids

Following the same process as the one followed by Imre Lakatos in *Proof and Refutation* ([1]), I became aware that this net factor depends probably from the net chosen for the experiment because it could change the expression of $V(x)$. So I have created some other models of nets and experimented with them.

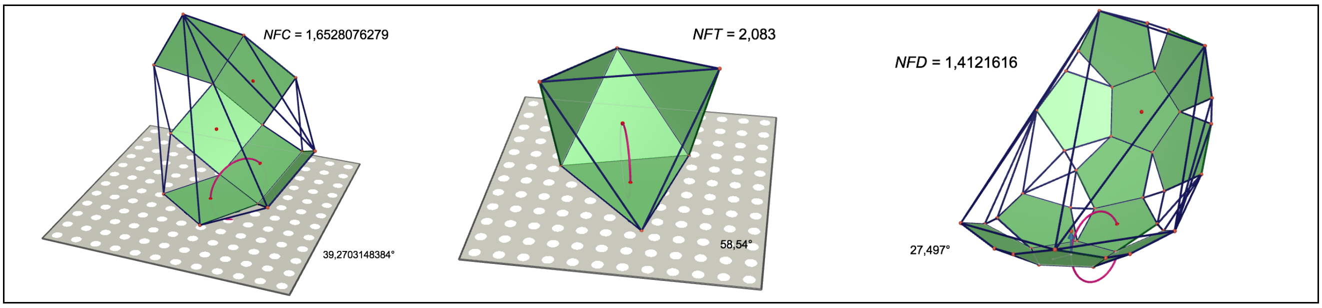

Figure 5

Here are the results of some experiments I have conducted:

NFC for another model of a net of a cube (**Figure 5** on the left) =**1.65**.

NFT for another model of a net of a tetrahedron (**Figure 5** on the middle) : **2.08** (more than the previous NFT)

NFD for another model of a net of a dodecahedron (**Figure 5** on the right) : **1.41**.

It was not surprising that the net factor changes with the net chosen but the different values of the net factor obtained for the same solid are not very different. Probably, for each solid, all the net factors must belong to a special interval which is yet to be determined : it is a work to be done in the future.

4. Experiments, conjectures without symbolic validation with the nets of various polyhedra (NFP)

4.1. Investigations related to the Net Factor of arbitrary polyhedra (NFP)

It is a well-known technique in research that, when you are not successful in a restricted domain, you try to extend this domain in order to find a more general result including the one you tried to study. So I decided to create a few polyhedra and some associated convex hulls in order to evaluate experimentally with Cabri 3D their net factors but really one for each (**Figures 6**,**7** and **8**).

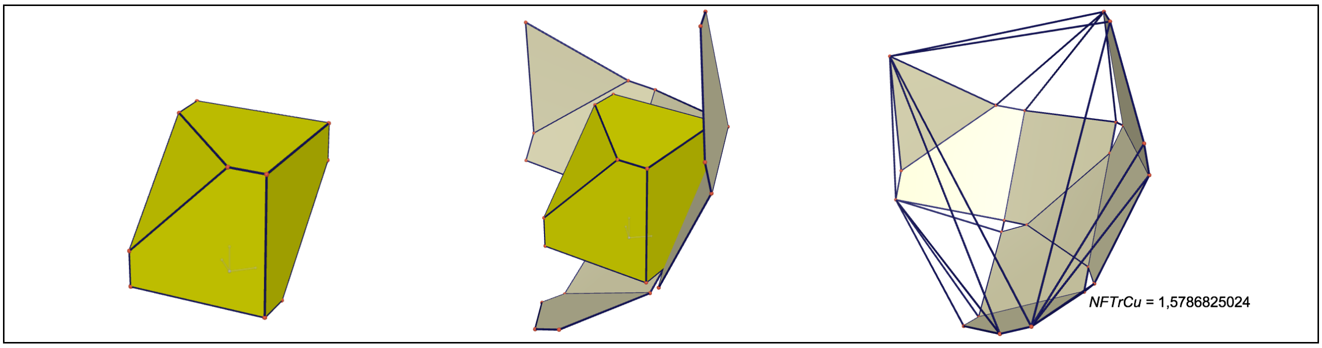

Figure 6

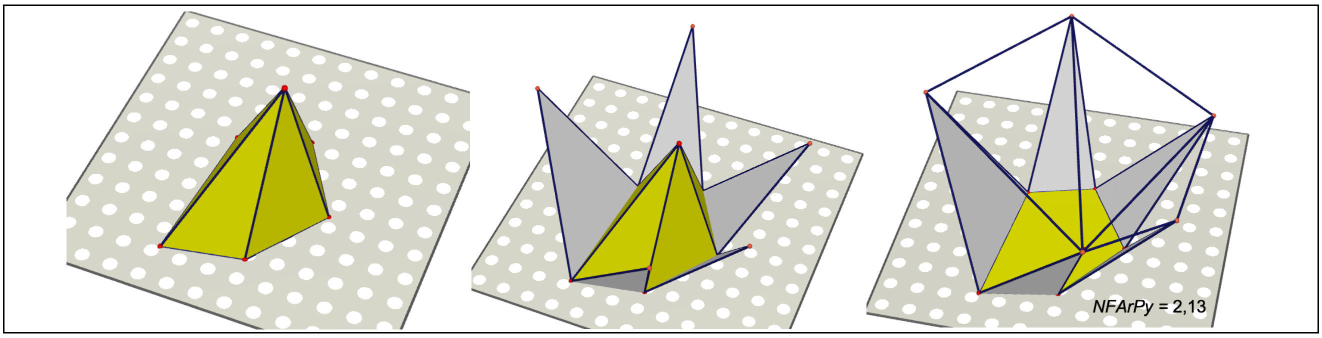

Figure 7

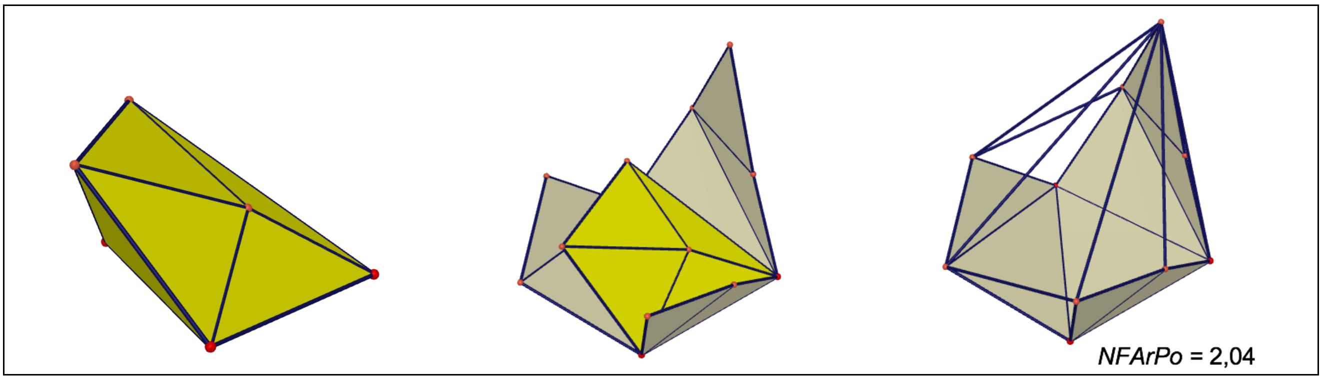

Figure 8

Here are the results of three of the experiments I have conducted:
NFP of a troncated cube: **1.58** (NFTrCu in **Figure 6**).
NFP of a pyramid: **2.13** (NFArPy in **Figure 7**).
NFP of an arbitrary polyhedron: **2.04** (NFArPo in **Figure 8**).

4.2. A double inequality related to the Net Factors of polyhedra (only conjectured)
The only property that I could notice is that the Net Factors I obtained are in the range of the net factors of the Platonic solids, that is to say:
If we call NFP one of the net factors of a convex polyhedron, it seems that:

$$\mathbf{P_m \leq NFP \leq P_M}$$

where P_m is the minimum of the possible net factors of the platonic solids and P_M the maximum of the possible net factors of the Platonic solids.

5. Conclusion

It is not clear at present whether the conjecture made is correct, and I do not have a geometric interpretation of these net factors. I think nevertheless that this work has opened a window on some possible research in 3D geometry.
The aim of this paper was not to present and justify a new result, but to demonstrate that even unsuccessful research provides new directions for exploration and new ideas to pursue.
The role of Cabri 3D software in exploratory work of this kind has to be recognized : none of the experiments I have conducted would have been possible without software of that kind
The role of a demonstration is nevertheless fundamental to confirm or reject some conjecture we could bet on. We have illustrated the importance of CAS to reject even some plausible conjecture.
Finally, I think I have illustrated the dialectic between experiments and proofs which is a core element of much research work in mathematics

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