Computer Algebra Systems as Cognitive Tools

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Abstract

When computer algebra systems(CAS) were introduced, it was expected that it will develop to use effectively as a "tool" for mathematics education. In the 2000s, with the evolution of both software and hardware, it has become easy to use CAS in the classroom. However, the proportion of teachers who do so is still quite low. We have to consider their effective use. Humans use strategies to plan solutions, decide procedures, and solve problems. This leads to the cognitive science concept of "tool theory". We consider the guideline for structuring a class using CAS and show the practice in this paper.

1. Introduction

In the 1990s, computers were introduced to mathematics education, and the effective use of computer algebra system (CAS) technology was part of this attempt. It was expected that it will develop to use effectively as a "tool" for mathematics education will develop, and that this would lead to higher student achievement in mathematics. However, these early efforts lacked a clearly defined direction. Researchers were uncertain as to how to utilize CAS for mathematics education or how goals might be achieved.

With the continued evolution of software and hardware programs, it has become easy to use a CAS technology in the classroom. But, the proportion of teachers who use computers as teaching aids is still quite low, and we cannot yet say that technology is commonly used in the classroom. Although the reasons are complex, the value of technology as a teaching tool has not been widely recognized.

What is the purpose of using technology in mathematics education? It should be to assist the development of students' mathematical thinking. Of the various technologies available, CAS have particularly high potential for assisting the learning of mathematics. Its potential, has not previously been clearly studied, in part because CAS technology has been viewed as a tool for the expert engineer. We should consider its effective use in mathematics education. To do so, we should consider the cognitive science concept of "tool theory". This leads to new possibilities for the use of CAS technology.

2. CAS technology and cognitive science

According to the human behavior model of Rasmussen [5], automatic human actions can be classified into three levels, based on skill, rule, or knowledge.

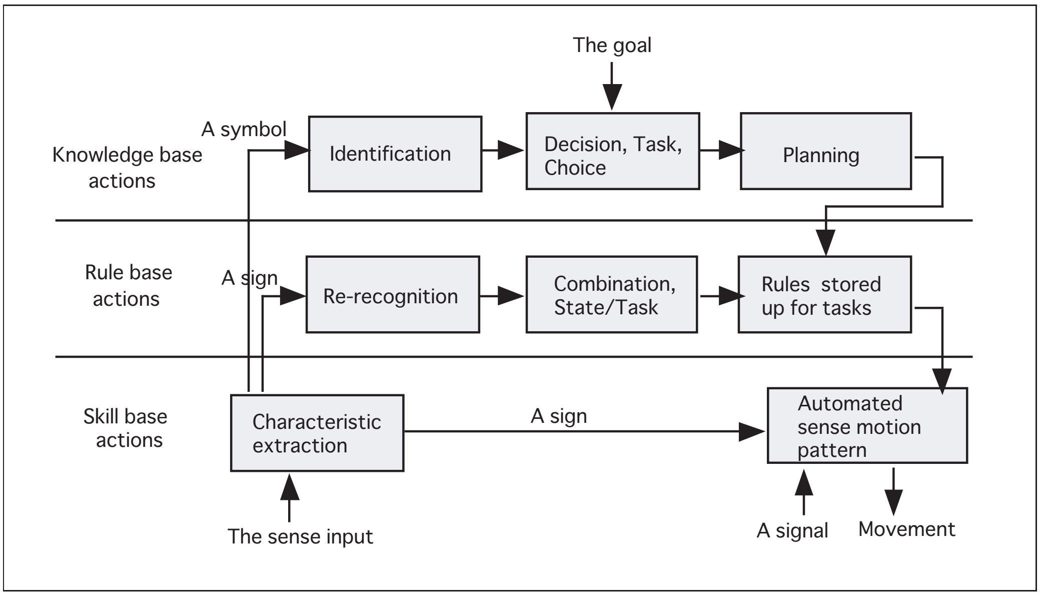


Figure 2 Three Level Model of Human Actions

A skill-based action is a response that occurs in less than one second. A chain of skill-based actions forms a rule-based action. Thinking about how to solve a problem is a knowledge-based action.

Skill-based actions are performed smoothly without intentional control. Rule-based actions require a great deal of repetitive practice in order to be transferred to the skill-based level. First, the external conditions must be recognized, then the rules for composing the act are combined with the conditions required to carry out the behavior. Knowledge-based actions require the recognition of external conditions, the interpretation of these conditions, the construction of a psychological model for considering solutions, planning, and finally, the use of the other two behavior levels to carry out the action.

This is a process model in which a behavior that requires thought is mastered and internalized to the point where it can be carried out unconsciously. Mistakes can be explained as omitted steps, such as incorrectly pushing a nearby button in a sequence of skill-based actions that are otherwise carried out smoothly. In knowledge-based actions, errors can be caused by misperceptions.

In the recent study, Rasmussen's model was used to identify which functions are essential for the smooth performance of actions and for learning. Humans act by classifying issues and their relationships by consciously combining them. Humans control themselves by constantly observing, thinking about, evaluating, and integrating their behavior in order to achieve accuracy, continuity, consistency, and normality [1].

3. Cognitive science in learning

3.1. Norman's theory

CAS technology can be used as an effective tool for calculations and for mathematical thinking. As an external tool, it can promote the development of internal tools and thus lead to improved efficiency [3].

The effective use of CAS can play an important role in the understanding of new concepts. As an external tool, CAS technology was developed to make calculations more efficient and to enable the faster manipulation of mathematical formulae. Although modern CAS packages are referred to as algebra systems, they are capable of much more than symbolic manipulation and are powerful tools for mathematics. If we hope to realize the potential of CAS technology in mathematics education, we need to develop new teaching methods, such as those based on the insights of cognitive science.

In order to do so, we consider CAS technology as an "external tool", which substitutes for some human abilities and thus creates new potential. At the same time, the use of CAS technology requires different types of ability and knowledge, and it is crucial to determine what these are.

A cognitive tool is an aspect of human perception that embodies the image of an outer object so that it appears in consciousness. Norman divided human cognition for the use of technology into two categories [2]:

- Experimental cognition: coping without conscious thought to changes in the outside world.
- Reflective cognition: thinking in order to deeply understand meaning and to refer new experiences to prior experience.

Thus, in a learning environment in which a CAS is used, we can divide human cognition into experiential and reflective. Tools for experiential cognition must be able to exploit rich sensory stimuli. Tools for reflective cognition must be supportive of the search for ideas; this requires different supports. It is not effective to combine experiential cognition with a reflective tool, nor to combine a reflective tool with experiential cognition. Therefore, teachers must distinguish which type of cognition will be used for a certain activity and then provide tools that offer appropriate support. If students simply participate when they should be reflecting, they misunderstand a reflective activity as an experiential activity.

According to Norman, in order to determine which of the two cognition modes is suitable for a particular learning activity, it is helpful to divide the activity into three categories:

- Accumulating: accumulating facts.
- Tuning: using skills involving reflection with experiential activities to adjust the acquired knowledge.
- Restructuring: using reflection to form an appropriate conceptual structure.

In many cases, accretion and tuning are seen as experiential modes, restructuring is seen as an introspective mode. To use a CAS appropriately as a cognitive tool, teachers must determine which category of activity is required.

3.2. Strategies

Strategies are used to plan solutions and decide procedures. When these procedures, in general or for the most part, obtain the correct answer, the procedure is called a heuristic; however, heuristics do not always result in a correct solution.

Strategies are used even when human beings solve mathematical problems. Recognition knowledge and experience are both used in a way that can be summarized as "doing it like this is effective in this case". The ability to rapidly reference knowledge is required for strategies based on experience. The famous book by the mathematician Polya, "How to Solve It" [4], showed the processes of mathematical problem solving; however, one cannot learn how to use heuristics in problem solving just by reading a book.

In researching problem solving, there are two contrasting concepts. The first emphasizes insight, flash, and senses, next emphasizes experiential knowledge. The former concept employs a strong tendency that strategies of thought are learned through the experience of problem solving. In other words, it is assumed that intuitive feelings and specific technical abilities can be acquired. In the latter concept, it is assumed that the ability to solve problems arises from the accumulation of rules inherent to the domain provided by an individual problem.

Such differences depend on the problem's nature, domain, and level, and the type of person involved in the learning process. In addition, it is difficult to establish clear boundary lines between these two concepts. In problem solving, experiential knowledge plays a large role. Heuristics are general ideas or algorithms (a procedure providing the correct solution), and are widely used. Heuristics are equivalent to "the logic of a thought".

4. Conclusion and practice

In the three-level model of human behavior, operations and strategies can be identified and considered in relation to human thought processes in order to facilitate error-free problem solving. In consideration of surface features and conditions, similar problems can be recognized and suitable problem-solving methods can be identified. In addition, it was found that the contents of the subconscious could be raised to the knowledge-based action level in order to support the expression process and the achievement of efficient functioning.

The ultimate goal of mathematics is mathematical reasoning. To do mathematics is to gain knowledge and solve problems by the use of reason. Researchers should appreciate the possibility of sharing cognitive level with such technology.

When students' calculations are performed with technologies such as a CAS, there is a concern that their ability to perform calculations themselves will not be developed. Students, however, can learn how to choose by themselves when to use a CAS. In other words, by educating in a way that enables students to judge in which situations to use a CAS, the concern about the loss of mathematical insight can be avoided.

As a guideline for structuring a class using a CAS, we must consider the following points:

1. To teach in a way that makes possible not only experience but also reflection.

2. To include activities that increase the mathematical skills of the students.

3. To guide students in the appropriate use of technology.

By considering these factors, it becomes possible to conduct truly effective activities for students.

There is a series of practice of Dr. Miyadera as the practice satisfying the guideline 3 mentioned above includes [6]. In contrast to them, here, we show a following practice as the practice satisfying the guideline 1 and 2 [7].

We consider the problem of maximum/minimum value of the trigonometric function.

Problem
Find the maximum/minimum value of the next function.
$\sin x - \cos\left(x - \frac{\pi}{6}\right)$.

The initial state of problem is $\sin x - \cos\left(x - \frac{\pi}{6}\right)$. And the goal state is the expression to recognize the maximum value, the minimum value. At first stage, it does not show the goal state clearly. When we consider this problem, we can rewrite it as follows step by step. The decided goal express one trigonometric function by the strategy. By explaining this process, we can let the students increase the mathematical skills of the students and experience the reflection.

① To rewrite $\sin x - \cos\left(x - \frac{\pi}{6}\right)$ into $A \sin X$.

② To expand $\cos\left(x - \frac{\pi}{6}\right)$.

③ To substitute a value for $\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{\pi}{6}\right)$.

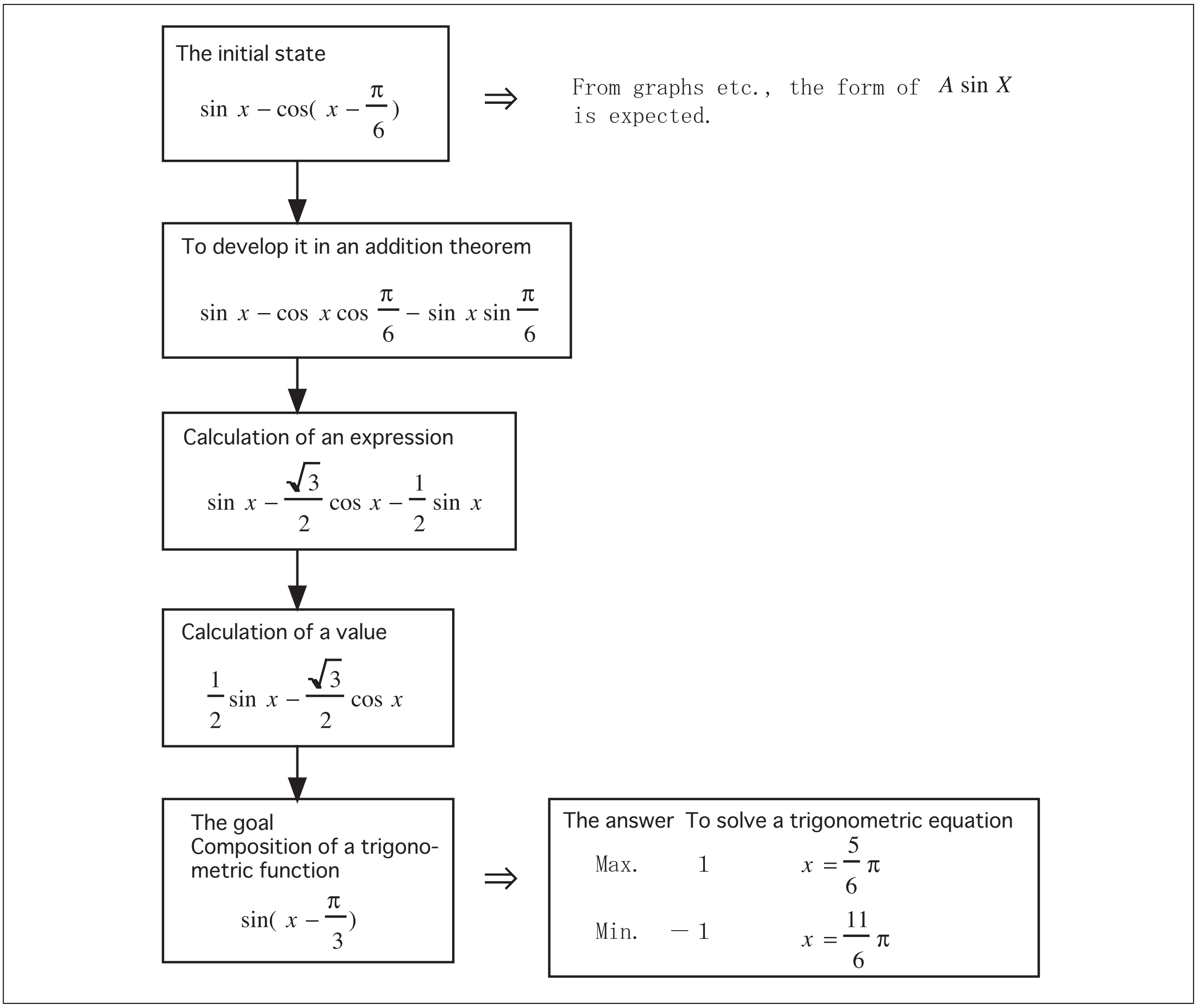
④ To together the terms of $\sin x$.

⑤ To composition of a trigonometric function.

⑥ To solve the equation $\sin\left(x - \frac{\pi}{3}\right) = 1$

⑦ To solve the equation $\sin\left(x - \frac{\pi}{3}\right) = -1$

The problem solving process of "Problem " is as the following frame.



Next, we show the example which processed this problem solving by using Mathematica.

In[1]:= `Simplify[Sin[x] - Cos[x - Pi / 6]]`

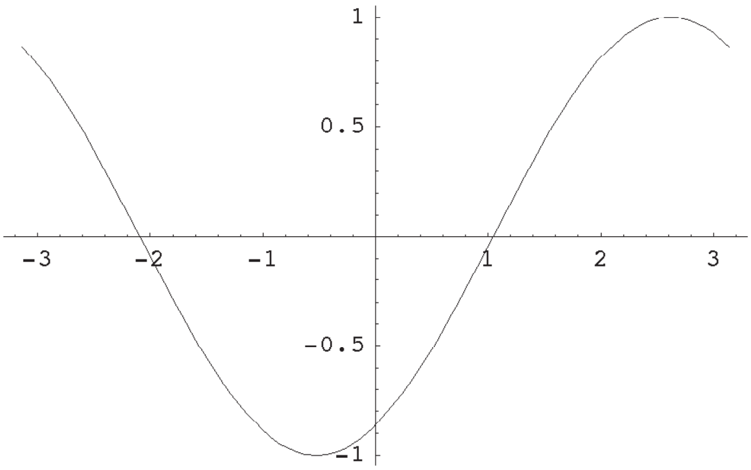
Out[1]= $\frac{1}{2}\left(-\sqrt{3}\,\mathrm{Cos}[x] + \mathrm{Sin}[x]\right)$

In[2]:= `f[a_, b_] := Sqrt[a^2 + b^2] Sin[x + ArcTan[b / a]]`

In[3]:= `1 / 2 f[1, -Sqrt[3]]`

Out[3]= $-\mathrm{Sin}\left[\frac{\pi}{3} - x\right]$

In[4]:= `Plot[%, {x, -Pi, Pi}]`



Out[4]= - Graphics -

In[5]:= $\mathbf{Solve}\left[-\mathbf{Sin}\left[\frac{\pi}{3} - x\right] == 1,\ x\right]$

Solve::ifun : 逆関数がSolve
により使用されているので，求められない解のある可能性があります．

Out[5]= $\left\{\left\{x \to \frac{5\pi}{6}\right\}\right\}$

In[6]:= $\mathbf{Solve}\left[-\mathbf{Sin}\left[\frac{\pi}{3} - x\right] == -1,\ x\right]$

Solve::ifun : 逆関数がSolve
により使用されているので，求められない解のある可能性があります．

Out[6]= $\left\{\left\{x \to -\frac{\pi}{6}\right\}\right\}$

To understand a process solving this problem, and a method to solve by using CAS is reflective learning. The strategies used to solve problems are usually not consciously considered. When strategy is analyzed, it is important to express a problem solution process in this way.

Acknowledgement

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