e-Teaching and e-Assessment of Minimum-Maximum Problems using Maple

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Abstract

Computer Algebra Systems, CAS, are now mature. For years, we've used Maple, a leading CAS, in weekly computer laboratory sessions: a component of an otherwise traditional first semester calculus course at university. The Maple topics come from the senior school curriculum, but with innovative approaches to curriculum, pedagogy and assessment. A major objective is that students have a positive attitude to using Maple.

Some of first year calculus repeats senior school calculus: including "Word problems" which have always been difficult for students. This paper focuses on Maximum problems. Our Maple topics have no lectures: students work collaboratively in small groups.

We introduce an explicit Polya approach to a maximizing an area problem, with an assignment on the Norman window problem that's individualized for each student group. We discuss a variety of assessment methods: paper submission, marked by hand; Maple file submission, marked by annotating each Maple file; automatic marking of the answer using Maple - which MapleTA can also do (without student use of Maple for the solution). Within Maple, we have implemented a procedure to mark plots (which no other Computer Aided Assessment can do). Surveys show students really like immediate automatic marking.

We discuss a following session (on maximizing area) that introduces multiple representations and multiple solution methods: graphical (zoom-in), animation, proof without calculus and calculus; with an accompanying parameterized assignment.

Students are engaged, active and collaborative learners with these Maple sessions.

1   Introduction

We don't want our first year students to be bored, but first year calculus does have an overlap with calculus studied in secondary school. This is deliberate, in part because many students have not understood their secondary level calculus to the level that is required at university. Students have seen some of the first year work before and so lectures may well contribute to boredom, as found in a recent study [7] of UK university students. It was reported "that 59%

of students find their lectures boring half the time and 30% find most or all of their lectures boring." A range of different teaching methods were investigated: the most boring were lab sessions; computer sessions ranked as the second most boring. "Computer sessions too have the potential to be stimulating or tedious; the findings of this study suggest that too many fall into the later category. This could be due to the manner in which sessions are conducted (e.g. are the computer tasks relevant and interesting?), ...etc."

We are not surprised by these findings: most courses use a traditional teaching approach (lectures) with "tutorials" and assignments which are mostly done without use of technology. When technology such as CAS is used, it is often just a supplement where the usual "by hand" exercises are completed by using CAS with little or no deviation from the "by hand" approach. Development of innovative teaching materials which exploit the strengths of the CAS for tightly integrated e-Learning and e-Assessment, such in the Maple component of our first year course generally, and in the Maximum problem as discussed in detail below, can be used to produce active and engaged learners of mathematics and the technology.

This paper is organized as follows. Section 2 has an overview of first semester, first year Maple sessions preceding the main topics of this paper. Section 3 outlines how Polya's problem solving approach is applied to a classic maximum area problem, a fenced area alongside a river. Section 4 describes the Norman window assignment which is parameterized (that is "individualized" for different student groups): students are required to follow all of the Polya steps to complete their assignment. An e-Marking subsection discusses assessment: by hand; e-Marking by inserting text in the student Maple file; and e-Marking by hand writing (Digital Ink) in the student Maple file. Subsection 4.2 makes some preliminary remarks about Computer Aided Assessment, CAA (which is continued later in Section 6).

Section 5 discusses multiple representations (graphical zooming-in and animations) and solution without using calculus. Section 6 outlines the use of Maple to mark plots (as well as numeric and symbolic results) and automatically generate external Marking Reports and a Marks List. This is followed by a brief Conclusion.

2   Prelude to finding the maximum area

Our Maple topics for the weekly lab sessions have no lectures: students work collaboratively in small groups. Students start by taking two sessions to work through an Introduction to Maple. A simple assignment is printed and submitted as a hardcopy, but all other work is completely electronic. There is an Animation worksheet followed by a major Animation Assignment: students pick a project that is on a school level topic, for further details see [1].

After this, the students are expert at preparing plots and animations, so these are the major elements of the next activity: Spot the Curve. The Maple file Spot the Curve is accompanied by another Maple binary file which contains all the procedures for randomly generating the curves, their translated curves and automatic marking procedures: this binary file is read into the student Spot the Curve file so that all of the required procedures are invisible but active in the student file. Each student group is presented with several plots of a curve and a translated copy of that curve: the task is to identify what the translation is. Visualization is used to identify the translation by plotting the student's guess on the same plot as the original curve and the translated curve. Also, animation is used to dynamically translate a curve to hopefully see where the dynamically translated curve coincides with the given translated curve. When the
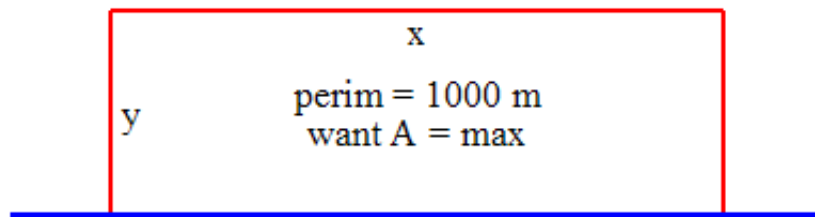
Figure 1: The diagram for the fenced area to be maximized.

student group has identified the original translation, the student can use the provided automatic marking procedure to immediately mark their result. The activity cannot be marked by any other CAA: for details, see [4].

Another style of auto marked problem is the FishPond problem, a straightforward application of using the trapezoidal rule to approximate area. It has multiple steps with numerical answers: the marking procedure is used by students to immediately mark their work: for details, see [4]. (The FishPond can also be done with CAA such as MapleTA.) Students have completed all of these activities before undertaking the maximum problem(s) described next.

3 The Polya approach to finding the maximum area

The students are now confident with basic numerical and symbolic operations using Maple, and are "expert" with graphics and animation. We introduce an explicit Polya approach to maximizing an area problem. This worksheet, max&min.mw, is basic (no animation).

The sections in the max&min.mw file are:
General preamble
The problem: fencing some land alongside a river
Step 0: Read the problem
Step 1: Draw a diagram
Step 2: Write the problem/information mathematically
Step 3: What is to be maximised ...
Step 4: What is the range for the one variable?
Step 5: Find the max
Step 6: Check
Step 7: The solution is ...
Step 8: Update the diagram ?

The problem, of some fenced land alongside a river, is:
A farmer has 1000 m. of fencing which can be used to fence off a rectangular plot of land which is alongside a river. No fencing is needed along the river. What are the dimensions of the plot that can be fenced which has the maximum area?

The labeled diagram is in Figure 1. This problem is easy for both secondary school students and undergraduates. The file provided includes all details: students just work through this as

preparation for undertaking the Norman window assignment.

4 The Norman window assignment

This assignment is a classic one that appears in many textbooks: it is modified here by the introduction of a parameter so that the assignment is individualized for each student group (of size 2 or 3, preferentially, but 4 is allowed).

The "individualized" **Norman window problem** is:

Preliminary

Work through the worksheet max_min.mw before undertaking the following (which can be done by "editing" the max_min.mw file). The problem solving steps illustrated in the max_min.mw file need to be followed in the solution of the following problem.

Define your group number, gpN, as the average value of the last nonzero digit of each of your student numbers (for example, if a group of 2 students has student numbers mm...m80 and nn...n5, then $gpN = \frac{8+5}{2}$.

Note: do NOT evaluate your gpN as a floating point number ...so that gpN remains an EXACT fraction.

Define perim (the perimeter) to be equal to this number (as an EXACT rational number).
        perim := gpN; (= 13/2 in this example).

The problem

A Norman window is constructed from a rectangular sheet of glass with a semi-circular sheet of glass fitted on top (of the rectangular piece). The amount of light that passes through the window is proportional to the area of the glass. What are the dimensions of the Norman window that has the maximum amount of light transmitted if it has a perimeter of perim = gpN meters.

4.1 e-Marking

Note that the parameter used in the Norman window problem is the perimeter of the window, so the solution is just scaled and marking the assignment, even by hand, is easy. The general solution, where r is the radius and y the height of the rectangular piece, is

$$r = \frac{perim}{4 + \pi} = y \ .$$

In 2002, students printed their Maple solution and this was marked by hand. From 2003 to 2006, students submitted their Maple file to a special proxy email account: this was to avoid student emails swamping our normal email account. From 2007, the Learning Management System, LMS, Black Board was used for students to download their Maple files: the teaching materials and the assignments; they also uploaded their assignment solution files. The Maple files were marked by viewing each file and inserting comments (in another colour - we chose dark green) and a full marking report (near the top of the student file) with all the part marks and the total marks listed. These annotated files were returned to students, initially by email but later by using Black Board. The marks were entered into Black Board. The use of Black Board was more efficient and better for both students and staff.

Note that the assignment is individualized and so may be reused for several years. The marking comments and marking report that is inserted in student files are progressively collected in the staff solution & marking file for subsequent re-use by copy, paste and possibly editing. So the marking rate (of about 5 per hour, possibly a little higher) is acceptable: the students are getting a lot of detailed feedback, but staff resources are higher than we'd like. We refer to this as e-Marking. For many details of our early work on e-Marking, including sample comments, sample marking reports, course statistics on performance and student feedback, see [3].

Recent versions of Maple support handwriting (usually called Digital Ink), but it is difficult writing with a mouse. However with a TabletPC, or a cheap external tablet device, it is easy to insert handwriting in a Maple file: in any language. The handwriting is done within a rectangular Canvas by placing the cursor in the desired position and using the menu, choose Insert, Canvas. The canvas is now ready to use with a menu of tools such as pen, eraser, pen colour and pen width. It is quick and easy to use so e-Marking within Maple is an attractive option. This is a good option for the Norman window problem since it is a relatively easy and straightforward problem which the students do well: e-Marking with Digital Ink within Maple is an appropriate and efficient choice (for small to medium size classes).

We suggest that the use of e-Marking will increase. The Open University has recently been working with this and we quote from [2] (which has a comprehensive discussion of many issues including the use of various hardware and software for Digital Ink and CAA):

> In 2009, Ben Mestle's Mathematics Online team at the Open University in the UK used 32 tutors covering 9 courses in their MSc program to develop and trial an electronic tutor-marked assignment, eTMA, system for mathematics [6]. e-Submission was optional, but about 20% of student elected to use eTMAs and these students responded positively about the eTMAs. Tutors mostly chose to use a TabletPC and PDF Annotator and were provided with hardware, software and professional development sessions. Alternatively, tutors could choose to use a tablet device or Word 2007. Interestingly, over 90% of students who used e-Submission also prepared their work electronically, with LaTeX being the most popular, followed by Word. Feedback from tutors was positive and comments indicated that a time overhead (compared to paper marking) would probably be about 10% in the long run. The conclusions in [6] finish with:
>
> > Although e-marking of mathematics is still in its infancy, its use is likely to grow as universities become increasingly digitised and students come to expect and demand electronic submission of their assignments.

Copying does not appear to be a problem. The marks allocated for this part of the first year course are low so motivation to cheat is not as high as in high stakes testing. Also, the activity is fundamentally about collaborative learning and help can be obtained from other members of the group, other groups or the tutor. It is clear (from many years experience with a variety of courses and year levels) that students consider parameterized problems to be different problems. Perhaps it is also relevant that we always try to include a lot of visualization: this requires plots and animations for which parameters cannot be arbitrary: they must have assigned values. Thus we've found that, even in higher level courses in which the assignments contribute to 50% to 80% of the assessment, copying has not been a problem.
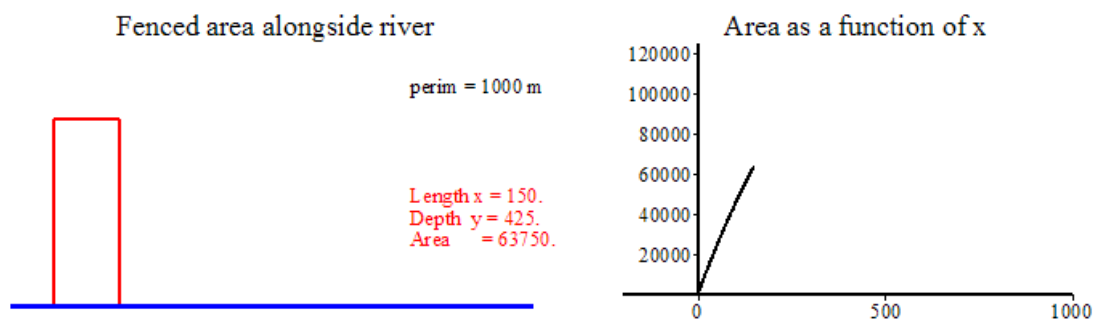
Figure 2: One of the frames in the animation of the diagram and the area function, for the fenced area to be maximized.

4.2   Automatic Marking: Computer Aided Assessment, CAA

By CAA we mean CAS-enabled CAA in which some CAS is used by the CAA system for marking: thus answers that are symbolic (as well as numeric and many other types, including multiple choice) can be marked. We could mark the Norman window final answer symbolically (writing a provided procedure within Maple, or by using MapleTA). We could also identify some intermediate steps, such as writing a formula for the area in terms of one variable, and use CAA to mark this type of intermediate step as well as the final answer. Using some CAA to do this would reduce the marking load of staff, which becomes a major problem for large classes. Many universities have between 1000 to 2000 students in the normal first year calculus class and are faced with having to choose between having a lesser amount of student work which is assessed, or implement CAA. However many academics might be concerned that key parts of the assignment are not marked (and so ignored by students): for example in the Norman window problem, a labeled diagram is neither seen nor marked!

We emphasize visualization and believe that visualization needs some assessment: otherwise the message for students is that visualization is not important. We have recently designed a CAS-immersed CAA in which plots can be marked efficiently within an automatic marking context: this is implemented for the Norman window problem, see Section 6.

5   "Advanced" Polya approach to find maximum area

The problem of finding the max area fenced in alongside a river (see above) can be continued as described here. Firstly, an animation of the diagram with text giving the perim, length, depth and area is used to run (slowly) which clearly indicates values of the parameters for the maximum area. For higher accuracy, the maximum area function is plotted and zooming-in is used. Note that our students are very comfortable with zooming-in from the school use of graphics calculators (and now CAS calculators).

Multiple representations are introduced using animations in two ways: a plot of the area function is plotted together with the diagram; and side by side (using an Array) – see Figure 2 for one frame of these plots side by side. By now, the students have worked with several visualizations. The numeric results from the plots and animations all give (approximately) the value obtained by calculus. We now obtain the exact answer without calculus. We notice that

the Area, as a function of x (the length along the river), is a quadratic function, so we can use the properties of quadratics. In particular we complete the square to find the axis of symmetry and hence find the exact value of x that gives the maximum, where the sign of the coefficient of x^2 proves whether there is a maximum or a minimum. Students can now follow the same procedures with the Norman window problem. With so much content being visualization with animations, we recommend using e-Marking (without CAA) as discussed above.

6 CAS-immersed CAA: marking the Norman window plot

We have written Maple files, as described below (and available from the author), to automatically mark the symbolic answers; to efficiently display the plot (alongside the correct plot), allowing the tutor to enter a mark for the plot (and an optional comment); to automatically generate the Marking Report and display it for the tutor to view; add the mark to an external text file (for uploading to an LMS such as Black Board when all the marking is completed); and generate an external text file copy of the Marking Report (that can be returned to the students via the LMS).

This process requires the assignment author to design several Maple files:

1. The student problem file (template): students complete their assignment within this Maple template assignment file. This file defines the problem and is sufficiently prescriptive that results and plots to be marked are saved to specified names.

2. The author's solution file: this is ensure that the problem is well designed and that the parameter values to be used by students will be appropriate.

3. The tutor automatic marking file.

   (a) Design a Maple solution and marking file (for tutor use) which reads the student Maple file (so all results are in memory which can be accessed by the marking file); automatically reads and uses the student parameter value(s) to compute (but not display) all the correct results and plots.

   (b) Decide whether all (or any) of the answers for automatically marked work (numeric or symbolic) is to be displayed (to view, but not mark, by the tutor).

   (c) Design the tutor interactive parts for the tutor to enter a mark by viewing the correct comment or plot side by side: with an option for the tutor to provide some text (to be included in the marking report) for feedback to the student.

   (d) Edit the procedure (executed by the tutor) to complete the automatic marking and generate and display (for the tutor to view) the Marking Report that is provided to the students.

   (e) Edit the procedure to print the mark to a marks list (as a text file that can be read by the LMS such as Black Board) and another procedure to print the marking report to a text file that can be sent to the students via the LMS.

4. A student problem file completed (as an example, to test the marking file).
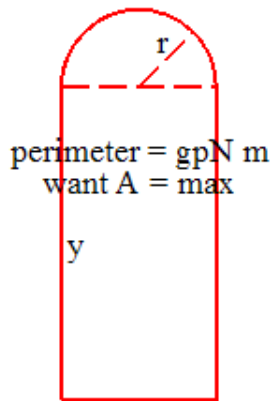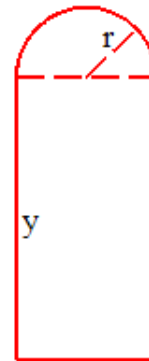
Figure 3: The correct diagram, followed by the student diagram for the Norman window area to be maximized.

The tutor downloads the student files, and for each student template file: the student file is executed (without viewing the file); in shared kernel mode, the tutor file is executed; the tutor interactive elements (that is, plots or comments) are viewed side by side with the correct element; the tutor enters the mark (and optionally can add or edit a comment). When all interactive elements are marked, the tutor executes the procedure to generate (and display) the full marking report. If no further adjustments or reviews are required by the tutor (which is usually the case), the next section is executed: this generates the entry in the external marks list text file and generates the external text file Marking Report. The tutor sends the Marking Report to students via the LMS (such as Black Board) and when all marking is completed, the tutor uploads the marks list to the LMS.

For the Norman window problem, there is only one interactive element that the tutor needs to view and assign a mark. So, the tutor views the correct plot and the student plot (which are Part (a) of the assignment), see Figure 3: the marking file automatically places these plots side by side (in a Table).

Immediately below the two plots, there is the following Maple code. The only operation required by the tutor is to edit the first statement aMark:= 2 to give the correct mark. Optionally, but rarely required, the tutor can edit the automatically provided marking comments.

```
   aMark:= 2 : #out of 3
if aMark=3 then
   aMark_Comment:="Your part(a) plot ...        3 marks (out of 3 marks)":
elif aMark=2 then
   aMark_Comment:="Your part(a) plot ...        2 marks (out of 3 marks)\n
You need to include text displaying
the given information and what is to be found.":
elif aMark=1 then
   aMark_Comment:="Your part(a) plot ...        1 mark (out of 3 marks)\n
You need to include text displaying the variables;
the given information and what is to be found.":
else
```

```
    aMark_Comment:="Your part(e) comment ...        0 marks (out of 3 marks)":
end if:
printf("  %s\n", aMark_Comment);
```

The tutor now executes the marking command, and if the displayed marking report looks correct (as it usually does), the tutor executes the final two procedures (to enter the marks into the marks list and to generate the marking report as an external text file. A sample output Marking Report is:

```
Marking report for student numbers:
                         3274878, 3284750
Student names are:   Joe Blow, Jane Blewett


The group number used was: gpN = 13/2


Total Marks: 9 (out of 10)

   Your labeled diagram ...            2 marks (out of 3 marks)
You need to include text displaying
the given information and what is to be found.
   Your Area as A(r) is correct;      3 marks (out of 3 marks)
   Your r max in domain is correct;   1 mark (out of 1 mark)
   Your r for A max is correct;       2 marks (out of 2 marks)
   Your y for A max is correct;       1 mark (out of 1 mark)
```

Students feedback and surveys show that they like the automatic marking as provided above. If automatic marking is immediate, they behave like gamers trying to "clock" the game, see the Fish Pond problem [4]. With no interactive elements to be marked, the students read in a marking file which can immediately mark their work. With any interactive elements, such as plots, see the Pendulum Problem [5] which includes an extra step to the above. Another Maple file is written with an automatic marking procedure to mark all the non-interactive elements: this procedure is saved in Maple's binary format and is automatically read into the students template. When the students have completed the assignment, they execute the marking procedure which immediately displays (within Maple) a Marking Report "so far". Since our main objective is to encourage learning, students can correct errors without penalty.

7 Conclusion

A CAS such as Maple provides a rich environment for e-Teaching and e-Marking of mathematics: innovative approaches to curriculum, pedagogy and assessment at school and university are supported. CAS solution of standard problems in the standard way risk being another activity (such as lectures and computer sessions) that students find boring. Well designed teaching and assessment is effective, efficient and even fun. The staff resources required to develop these CAS based materials are high, but do give high returns with staff satisfaction, student performance and attitude. With large classes, the reduced marking load using automatic marking more than

repays the development cost. We recommend more collaborative learning by students and more collaborative development of e-Teaching and e-Assessment by academics.

The Maple sessions described here, particularly on the Maximum problem, could be part of an online course: students are very comfortable with using the technology (more so than the staff), and taking advantage of the flexible learning that it supports. Students enjoy the collaborative learning using Maple to do the calculations and learn (surprisingly quickly) to use and adapt advanced animations; and use automatic marking. Students are engaged, active and collaborative learners with these Maple sessions.

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