# Elementary Proof of Sejfriedian Properties

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**Abstract**

*In this paper we investigate a new type of symmetry for an arbitrary triangle, so called Sejfriedian, and we show elementary proofs of selected properties of Sejfriedians. This type of symmetry was obtained by Michael Sejfried in 2008. Sejfriedian is a pair of triangles inscribed into any circle, the circle and the set of lines coming out of all vertices of the given triangle. It has many unusually interesting properties. The Sejfriedian gives students great opportunity for in-depth study of the properties of stereographic projection, spatial inversion and combining of mathematical expressions.*

*All the pictures in the electronic version of this paper are interactive. Install GInMA software from the website* ***http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx***
*click on the Figures and investigate the constructions and interactive solutions of the problems.*

## 1. Introduction

For at least 10 years I have dealt with geometrical problems related to the triangles, the circles and their interrelationships. This has allowed me to build many interesting constructions and to investigate their marvellous properties. So I have found the family of the circles, which I called "perfect circles", and the pair of the triangles inscribed in each of them called by me "amicable triangles". The family of "perfect circles" begins at the Fermat point and ends at the circumcenter, including also incircle and many other interesting circles. The incircle is here one of special cases. The well-known Soddy circles, Soddy line, Soddy center, Gergonne point, Gergonne line, Nobbs points, Oldknow point, Griffiths points and Rigby points are all based on incircle. Using "perfect circles" I generalized them. I formulated also a few theorems, where one of them called "Golden theorem" is related to golden mean. Unfortunately till today I couldn't find the equation of the locus of the centers of the "perfect circles".

For personal reasons I cannot take part in ATCM 2012. I'm very pleased that my paper about "amicable triangles" and "perfect circles", published on the ATCM 2011 in Bolu and on the ICGG 2012 in Montreal [1], aroused an interest in Vladimir Shelomovskii, who was going to develop it and will present today elegant proofs of the theorems related to "amicable triangles" and "perfect circles" using 3D-geometry and GInMa software [2].

Michael Sejfried.

## 2. Definitions and denominations

Traditionally mathematicians denominate beautiful geometric constructions and objects using the name of the person who first discovered and described them. In accordance with this tradition and [1], I denominate the objects in the paper as follows.

Given any reference $ABC$ triangle. Let's construct three pairs of lines, as shown on Figure 1. The first pair connects the vertex $A$ with the points $A_1$ and $A_2$, lying on the line $BC$ and $A_1$ is closer to $B$

than $A_2$. The second pair connects the vertex $B$ with the points $B_1$ and $B_2$, lying on the line $AC$ and $B_1$ is closer to $C$ than $B_2$. The third pair connects the vertex $C$ with the points $C_1$ and $C_2$, lying on the line $AB$ and $C_1$ is closer to $A$ than $C_2$. The lines $AA_1, BB_1,$ and $CC_1$ intersect at three points $K_1$, $L_1$ and $M_1$. The lines $AA_2, BB_2,$ and $CC_2$ intersect at three points $K, L,$ and $M$. We also construct the lines $AA_3, BB_3,$ and $CC_3$ passing through the points $J_1, J_2,$ and $J_3,$ intersection of $BB_2$ and $CC_1,$ $AA_1$ and $CC_2, AA_2$ and $BB_1,$ respectively. Points $A_3, B_3,$ and $C_3$ are located on $BC, AC,$ and $AB$ lines. The lines $AA_3, BB_3$ (and $CC_3$ ) intersect at point $G$. If points $K_1$, $L_1$, $M_1$, $K$, $L$, and $M$ belong to one circle, the pair of the triangles $KLM$ and $K_1L_1M_1$ have been called **Sejfried triangles,** the circle have been called **Sejfried circle,** the center of **Sejfried circle** have been called **Sejfried point**, the locus of **Sejfried point** have been called **Sejfried function**, and all this construction have been called **Sejfriedian**. If Sejfried circle is an incircle of $ABC$, then Sejfriedian have been called the **base Sejfriedian**.

Figure 1 and all the Figures in the electronic version of this paper are interactive. Install GInMA software from the website
**http://deoma–cmd.ru/en/Products/Geometry/GInMA.aspx**
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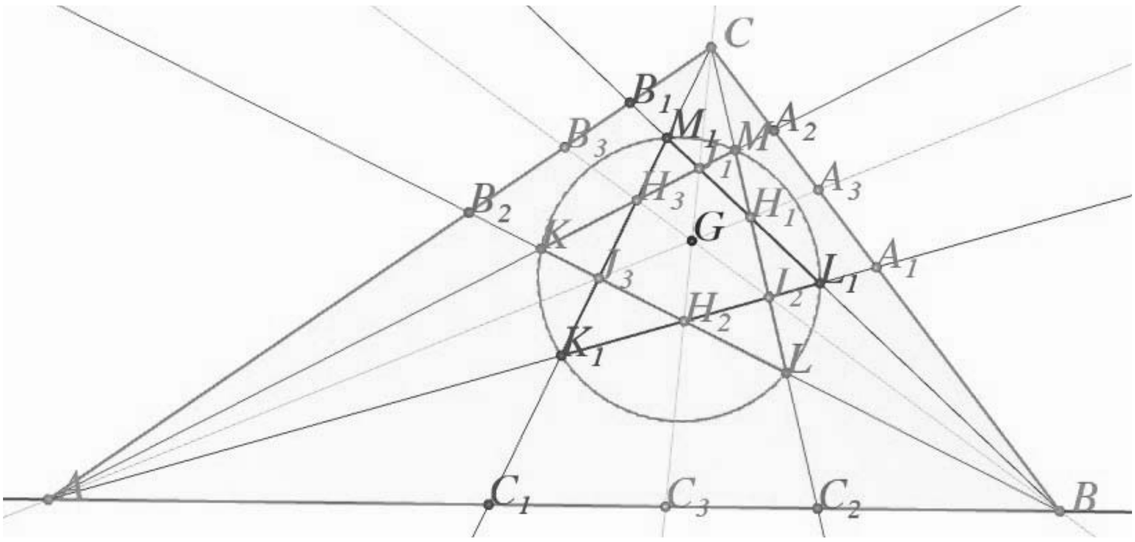


**Figure 1** Sejfriedian for arbitrary triangle

## 3. Sejfriedian for equilateral triangle

When we rotate equilateral triangle around the center at the angle 120°, it transforms into itself. Hence, $AC_1 = BA_1 = CB_1$, $AB_2 = BC_2 = CA_2$. Since the equilateral triangle is symmetric about the median, $AB_1 = AC_2$. Therefore Sejfried triangles must be equilateral triangles and the points $A_3, B_3,$ and $C_3$ must be midpoints of $BC, AC,$ and $AB$, respectively. Let the unit equilateral triangle be given and $\xi = BA_1 \leq \frac{1}{2}$. Then

$$AB_2 = AC_1 = BC_2 = BA_1 = CA_2 = CB_1 = \xi \,, \tag{1}$$

$$A_1A_3 = A_2A_3 = B_1B_3 = B_2B_3 = C_1C_3 = C_2C_3 = \frac{1}{2} - \xi \,. \qquad (2)$$

By the Pythagorean theorem, we obtain

$$AA_1 = AA_2 = BB_1 = BB_2 = CC_1 = CC_2 = \sqrt{\frac{3}{4} + \left(\frac{1}{2} - \xi\right)^2} = \sqrt{1 - \xi + \xi^2}\,. \qquad (3)$$

Let's denote $AA_1 = l$. We use the properties of crossing segments within the triangle and obtain the lengths of the parts on which the segments $AA_1$, $AA_2$, $AA_3$ and similar segments are divided

$$AK = \frac{\xi}{l},\ KH_2 = \frac{(1-2\xi)(1-\xi)}{(2-\xi)l},\ H_2J_3 = \frac{(1-2\xi)l^2}{1+\xi-\xi^2}, \qquad (4)$$

$$J_3M = \frac{(1-2\xi)\xi}{(1+\xi)l},\ MA_2 = \frac{\xi^2}{l},\ KM = \frac{1-2\xi}{l}. \qquad (5)$$

$$AJ_1 = \frac{1-\xi}{1+\xi}\frac{\sqrt{3}}{2},\ J_1G = \frac{5\xi-1}{3+3\xi}\frac{\sqrt{3}}{2},\ GH_1 = \frac{1-2\xi}{2-\xi}\frac{\sqrt{3}}{3}, \qquad (6)$$

$$H_1A_3 = \frac{\xi}{2-\xi}\frac{\sqrt{3}}{2},\ AG = \frac{\sqrt{3}}{3},\ A_3G = \frac{\sqrt{3}}{6}. \qquad (7)$$

The inradius of an equilateral unit triangle is equal to $\frac{1}{2\sqrt{3}}$. Let's denote $\lambda = 2\sqrt{3}\rho$, where $\rho$ is the radius of the Sejfried circle. We find from the condition $KM = \rho\sqrt{3}$, that

$$KM = \frac{1-2\xi}{l} = \rho\sqrt{3} = \frac{\lambda}{2}, \quad 1-2\xi = \frac{\lambda\sqrt{3}}{\sqrt{16-\lambda^2}} = \mu, \quad \xi = \frac{1}{2} - \frac{\lambda\sqrt{3}}{2\sqrt{16-\lambda^2}} = \frac{1-\mu}{2}. \qquad (8)$$

The last formula shows that the solution exists only if $\lambda < 4$. For the base Sejfriedian $\lambda = 1$, $\xi = \frac{1}{2} - \frac{1}{2\sqrt{5}}$, $\mu = \frac{1}{\sqrt{5}}$. Let's note $\frac{1-\xi}{1-2\xi} = \phi$, $\frac{1-\xi}{\xi} = \phi^2$, where $\phi = \frac{\sqrt{5}+1}{2} \approx 1{,}618$, it is golden mean. If the radius of the Sejfried circle is equal to the circumradius, then $\lambda = 2$, $\xi = 0, \mu = 1$. So Sejfried circle passes through the $ABC$ vertices, and coincides with the circumcircle of the original triangle.

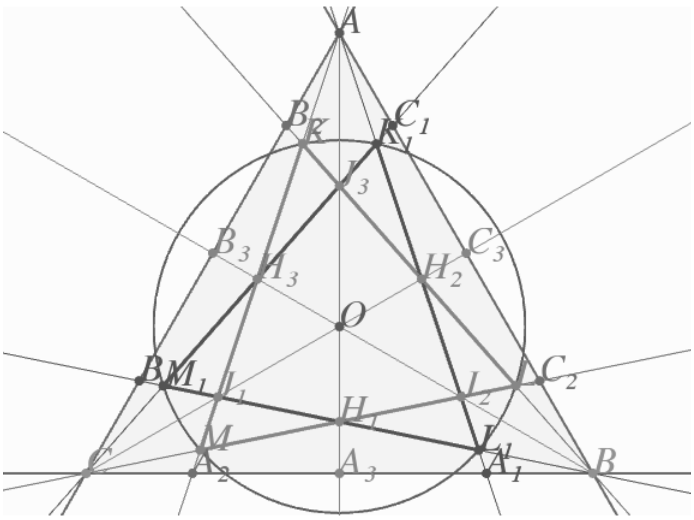


**Figure 2** Sejfriedian for equilateral triangle

## 4. Sejfriedian cross-ratios

It is known that points of intersection of three or more lines are preserved by the projective transformations. If $U, V, W,$ and $P$ are collinear, then the cross ratio is defined similarly as $(U, V; W, P) = \frac{UW \cdot VP}{UP \cdot VW}$. The cross-ratio is preserved by the projective transformations. It will be shown below that the Sejfriedian for any triangle can be constructed by using projective transformation of the Sejfriedian for equilateral triangle. Therefore, cross-ratios, calculated for the equilateral triangle, are right for any arbitrary triangle.

We calculate the cross-ratios for different sets of points. For each side of the triangle *ABC* and each of its median, the cross-ratios can be found using any four of the five defined points. For example, the side *BC* contains the points $B, C, A_1, A_2,$ and $A_3$. The segments of the form $AA_2$ contain the sets of six points. For example, there are points $A$, $K$, $H_2$, $J_3$, $M$, and $A_3$ on the line $AA_2$. The values of some cross ratios are

$$(B, A_1; A_3, C) = \frac{BA_3 \cdot A_1 C}{BC \cdot A_1 A_3} = \frac{0.5(1-\xi)}{0.5-\xi} = \frac{1-\xi}{1-2\xi} = \sqrt{\frac{4}{3\lambda^2} - \frac{1}{12}} + \frac{1}{2}. \quad (9)$$

In the case of the base Sejfriedian we have $\lambda = 1$, $(B, A_1; A_3, C) = \frac{\sqrt{5}+1}{2} = \phi \approx 1{,}618$.

We use the properties of the projections in the triangle *ABC* and get

$$\begin{aligned}(B, A_1; A_3, C) &= (C_2, J_2; H_1, C) = \\ &= (C_3, H_3; G, C) = (C_1, K'; J_1, C) = (C_1, J_1; H_2, C) = (C_2, J_2; H_1, C) = (A, B_2; B_3, C).\end{aligned} \quad (10)$$

$$(B, A_2; A_3, C) = \frac{BA_3 \cdot A_2 C}{BC \cdot A_2 A_3} = \frac{0.5\xi}{0.5-\xi} = \frac{\xi}{1-2\xi} = \sqrt{\frac{4}{3\lambda^2} - \frac{1}{12}} - \frac{1}{2}. \quad (11)$$

In the case of the base Sejfriedian $\lambda = 1$, $(B, A_2; A_3, C) = \frac{\sqrt{5}-1}{2} = \phi^{-1} \approx 0{,}618$.

$$(B, A_1; A_2, C) = \frac{BA_2 \cdot A_1 C}{BC \cdot A_1 A_2} = \frac{(1-\xi)^2}{1-2\xi} = \frac{1}{2}\left(1 + \frac{8+\lambda^2}{\lambda\sqrt{48-3\lambda^2}}\right). \quad (12)$$

In the case of the base Sejfriedian we have $(B, A_1; A_2, C) = \frac{3\sqrt{5}+5}{10} \approx 1{,}171$.

$$(B, A_2; A_1, C) = \frac{BA_1 \cdot A_2 C}{BC \cdot A_2 A_1} = \frac{\xi^2}{1-2\xi} = \frac{(1-\xi)^2}{1-2\xi} - 1 = \frac{1}{2}\left(\frac{8+\lambda^2}{\lambda\sqrt{48-3\lambda^2}} - 1\right). \quad (13)$$

In the case of the base Sejfriedian we have $(B, A_1; A_2, C) = \frac{3\sqrt{5}-5}{10} \approx 0{,}1708$.

$$(B, A_1; A_3, A_2) = \frac{BA_3 \cdot A_1 A_2}{BA_2 \cdot A_1 A_3} = \frac{1}{1-\xi} = \frac{16-\lambda^2-\lambda\sqrt{48-3\lambda^2}}{2(4-\lambda^2)}. \quad (14)$$

In the case of the base Sejfriedian we have $(B, A_1; A_3, A_2) = \frac{5-\sqrt{5}}{2} \approx 1{,}382$.

$$(A, K; M, A_2)^{-1} = \frac{AA_2 \cdot KM}{AM \cdot A_2 K} = \frac{(1-2\xi)(1-\xi+\xi^2)}{(1-\xi)^3} = 1 - \frac{\xi^3}{(1-\xi)^3} = 1 - \frac{(\sqrt{16-\lambda^2} - \lambda\sqrt{3})^3}{64(4-\lambda^2)^3}. \quad (15)$$

In the case of the base Sejfriedian we have $(A, K; M, A_2)^{-1} = 1 - \phi^{-6} \approx 0{,}9443$.

## 5. Construction of the base Sejfriedian for any reference triangle

Let's construct the base Sejfriedian for any reference triangle *ABC*. The circle ω coincides with the Sejfried circle in this case. Let *I* be the incenter, $A_3$ and $B_3$ be the points of the Gergonne triangle. Let $D_0 D_1$ be incircle diameter which contain the Gergonne point of the triangle *G*, and $D_0 G > D_1 G$. Let us make the perpendicular to the plane *ABC* through the point $D_1$ and choose the point $D$ such that $DD_1 = \frac{D_0 D_1}{\sqrt{\frac{D_0 G}{D_1 G} - 1}}$. Let the plane *P* be perpendicular to $D_0 D$. It passes through $D_1$ and crosses $D_0 D$ at point $D_0'$. When we use central projection from $D$, the circle ω transforms into the circle ω' with the diameter $D_0' D_1$ belonging to the plane *P*. Point *G* transforms into the center *O*, $D_0' O = D_1 O$ (see Appendix 3). The triangle $A'B'C'$ is the image of the triangle *ABC*. The images of the segments $AA_3, BB_3,$ and $CC_3$ connect the points of the triangle $A'B'C'$ with the points of tangency of ω' and $B'C', A'C',$ and $A'B',$ respectively. Since these images pass through the center of the circle, the triangle *A'B'C'* is equilateral (see Appendix 2).

We construct the base Sejfriedian (points $A_1', A_2', B_1', B_2', C_1', C_2'$ ) of equilateral triangle $A'B'C'$. Next, we make central projection from *D* and found the points of the base Sejfriedian for triangle *ABC*.

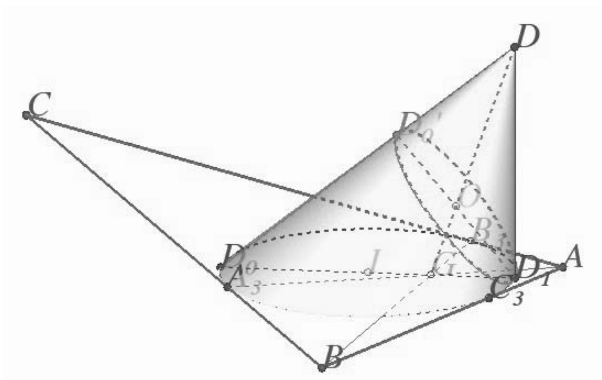


Figure 3 Construction of the center of projection

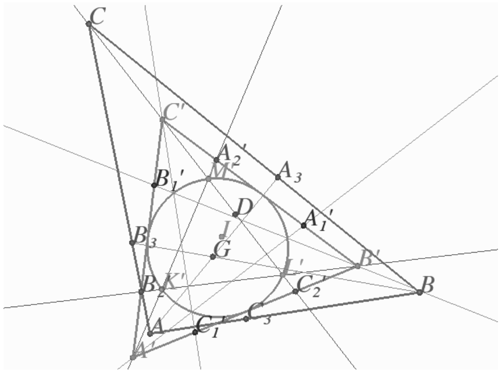


**Figure 4** Construction of the base Sejfriedian for equilateral triangle

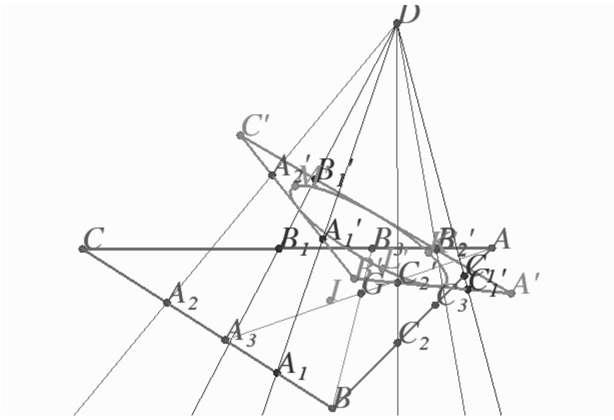


**Figure 5** Central projection to the *ABC* plane

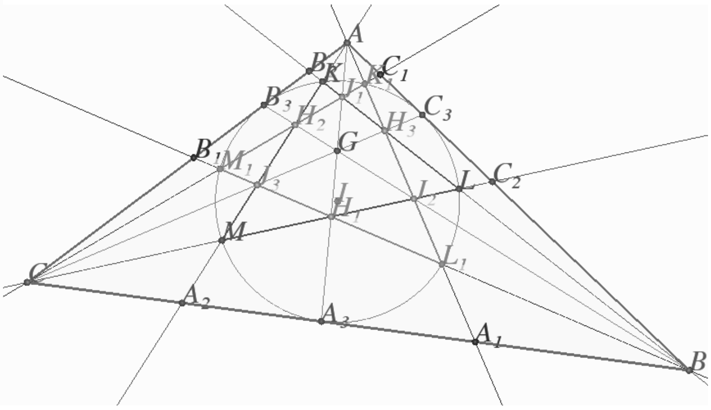


**Figure 6** The base Sejfriedian for given triangle

## 6. Construction of the Sejfriedian for any reference triangle

To construct the Sejfriedian for arbitrary $\lambda$ and given triangle, we may construct corresponding Sejfriedian for equilateral triangle and find the projection center *D*. The center is determined by three parameters. For example, these parameters are: the radius of the Sejfried circle, the angular position of the point $D_1$ (a base of the perpendicular from *D*) on the circle, and the distance from *D* to the plane of the circle. As a result of the projection, we obtaine the triangle. We may create three combined equations defining the ratio of the sides for resulting triangle. Solving this system, we build triangle similar for given triangle. We show the Sejfriedians with $\lambda=\frac{1}{2}$ and $\lambda=\frac{3}{2}$, constructed by this method. For the proofs in part 7 there is important the principle possibility of constructing the Sejfriedian for given triangle using central projection of the Sejfriedian for equilateral triangle.

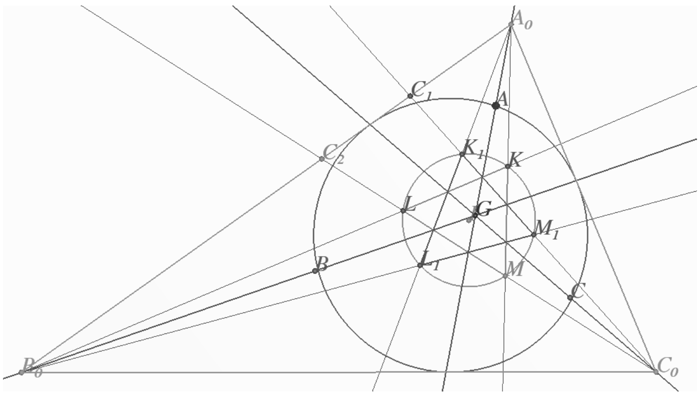


**Figure 7** Sejfriedian for $\lambda=\frac{1}{2}$.

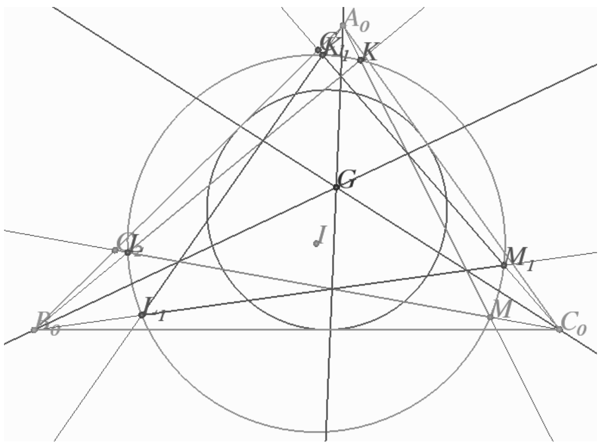


**Figure 8** Sejfriedian for $\lambda=\frac{3}{2}$.

## 7. Sejfriedian invariants

**Theorem 1** *Sejfriedian points for arbitrary* $\lambda$ *and triangle meet the condition:*

$$\frac{AB_1}{CB_1}\cdot\frac{AB_2}{CB_2}\cdot\frac{CA_1}{BA_1}\cdot\frac{CA_2}{BA_2}\cdot\frac{BC_1}{AC_1}\cdot\frac{BC_2}{AC_2}=1. \quad (16)$$

**Proof.** The Ceva conditions for the vertices of Sejfried triangles and point *G* may be written as follows:

for the vertex of $J_1$
$$\frac{BC_1\cdot CA_3\cdot AB_2}{AC_1\cdot BA_3\cdot CB_2}=1, \quad (17)$$

for the vertex of $H_2$
$$\frac{BC_1\cdot AB_3\cdot CA_2}{AC_1\cdot CB_3\cdot BA_2}=1, \quad (18)$$

for the vertex of $J_3$ $$\frac{AB_1 \cdot BC_3 \cdot CA_2}{CB_1 \cdot AC_3 \cdot BA_2} = 1, \tag{19}$$

for the vertex of $H_1$ $$\frac{AB_1 \cdot CA_3 \cdot BC_2}{CB_1 \cdot BA_3 \cdot AC_2} = 1, \tag{20}$$

for the vertex of $J_2$ $$\frac{CA_1 \cdot AB_3 \cdot BC_2}{BA_1 \cdot CB_3 \cdot AC_2} = 1, \tag{21}$$

for the vertex of $H_3$ $$\frac{CA_1 \cdot BC_3 \cdot AB_2}{BA_1 \cdot AC_3 \cdot CB_2} = 1, \tag{22}$$

for the point $G$ $$\frac{CB_3 \cdot BA_3 \cdot AC_3}{CA_3 \cdot AB_3 \cdot BC_3} = 1. \tag{23}$$

We multiply the left and right sides of (17) – (22) equations and the square of (23) equation. Then we extract the square root of the result and obtain the required equality.■

**Theorem 2** *Sejfriedian points for arbitrary* $\lambda$ *and given triangle meet the conditions:*

$$\frac{BA_2 \cdot A_1C}{BA_1 \cdot A_2C} = \frac{AC_2 \cdot BC_1}{AC_1 \cdot BC_2} = \frac{CB_2 \cdot AB_1}{CB_1 \cdot AB_2} = \left( \frac{\sqrt{16-\lambda^2} + \lambda\sqrt{3}}{\sqrt{16-\lambda^2} - \lambda\sqrt{3}} \right)^2.$$

**Proof.** We devide (12) on (13) equations and get

$$\frac{BA_2 \cdot A_1C}{BA_1 \cdot A_2C} = \frac{(B, A_1; A_2, C)}{(B, A_2; A_1, C)} = \frac{(1-\xi)^2}{\xi^2} = \left( \frac{\sqrt{16-\lambda^2} + \lambda\sqrt{3}}{\sqrt{16-\lambda^2} - \lambda\sqrt{3}} \right)^2. \tag{24}$$

We get similarly

$$\frac{AC_2 \cdot BC_1}{AC_1 \cdot BC_2} = \left( \frac{\sqrt{16-\lambda^2} + \lambda\sqrt{3}}{\sqrt{16-\lambda^2} - \lambda\sqrt{3}} \right)^2, \tag{25}$$

$$\frac{CB_2 \cdot AB_1}{CB_1 \cdot AB_2} = \left( \frac{\sqrt{16-\lambda^2} + \lambda\sqrt{3}}{\sqrt{16-\lambda^2} - \lambda\sqrt{3}} \right)^2. \tag{26}$$

In the case of the base Sejfriedian we get

$$\frac{BA_2 \cdot A_1C}{BA_1 \cdot A_2C} = \frac{AC_2 \cdot BC_1}{AC_1 \cdot BC_2} = \frac{CB_2 \cdot AB_1}{CB_1 \cdot AB_2} = \left( \frac{\sqrt{5}+1}{2} \right)^4 = \phi^4 \approx 6{,}854.$$ ■

**Theorem 3 (Sejfried theorem)** *Sejfriedian points for arbitrary* $\lambda$ *and given triangle meet the conditions:*

$$\frac{AB_1}{CB_1} \cdot \frac{CA_1}{BA_1} \cdot \frac{BC_1}{AC_1} = \left( \frac{\sqrt{16-\lambda^2} + \lambda\sqrt{3}}{\sqrt{16-\lambda^2} - \lambda\sqrt{3}} \right)^3, \tag{27}$$

$$\frac{AB_2}{CB_2} \cdot \frac{CA_2}{BA_2} \cdot \frac{BC_2}{AC_2} = \left( \frac{\sqrt{16-\lambda^2} - \lambda\sqrt{3}}{\sqrt{16-\lambda^2} + \lambda\sqrt{3}} \right)^3. \tag{28}$$

$$\frac{AB_1}{CB_1}\cdot\frac{CA_1}{BA_1}\cdot\frac{BC_1}{AC_1}+\frac{AB_2}{CB_2}\cdot\frac{CA_2}{BA_2}\cdot\frac{BC_2}{AC_2}=2+\frac{432}{(4-\lambda^2)^3}. \tag{29}$$

*In the case of the base Sejfriedian we get:*

$$\frac{AB_1}{CB_1}\cdot\frac{CA_1}{BA_1}\cdot\frac{BC_1}{AC_1}=\phi^6, \qquad \frac{AB_2}{CB_2}\cdot\frac{CA_2}{BA_2}\cdot\frac{BC_2}{AC_2}=\phi^{-6}. \tag{30}$$

$$\frac{AB_1}{CB_1}\cdot\frac{CA_1}{BA_1}\cdot\frac{BC_1}{AC_1}+\frac{AB_2}{CB_2}\cdot\frac{CA_2}{BA_2}\cdot\frac{BC_2}{AC_2}=18.\blacksquare \tag{31}$$

**Proof.** Let's multiply the equations (24), (25) and (26) and multiply the product by the square of the equation (16). Let's make square-rooting, and we get (27). We divide the equations (16) on (27), and get (28). We add the equations (27) and (28), and get (29).■

**Theorem 4** *Let the pair of Sejfried triangles* $KLM$ *and* $K_1L_1M_1$ *be given. Then*

$$\frac{1}{KL^2}+\frac{1}{KM^2}+\frac{1}{LM^2}=\frac{1}{K_1L_1^2}+\frac{1}{K_1M_1^2}+\frac{1}{L_1M_1^2}. \tag{32}$$

**Proof.** Let us consider stereographic projection of the Sejfriedian for equilateral triangle *A'B'C'* from point *D* into the plane of the original triangle *ABC*. We consider it as the inversion with the center *D* which transforms the points $K', L', M', K_1', L_1', M_1'$ of the Sejfriedian for equilateral triangle *A'B'C'* into the points $K, L, M, K_1, L_1, M_1$, respectively. According to the construction, the point $D_0'$ belongs to the circumcircle of regular triangles $K'L'M'$ and $K_1'L_1'M_1'$, $DD_0'$, is perpendicular to the plane $K'L'M'$. We use the property of inversion and obtain

$$\frac{K'L'}{KL}=\frac{K'D\cdot L'D}{R^2}, \tag{33}$$

where *R* is the inversion radius. We use the following form of (33) $\frac{K'L'^2\cdot R^4}{KL^2}=K'D^2\cdot L'D^2$.

We add up similar relations for all sides of Sejfried triangles. We take into account that $K'L'=K'M'=L'M'=K_1'L_1'=K_1'M_1'=L_1'M_1'$.

Then we obtain

$$K'L'^2\cdot R^4\left(\frac{1}{KL^2}+\frac{1}{KM^2}+\frac{1}{LM^2}\right)=K'D^2\cdot L'D^2+K'D^2\cdot M'D^2+L'D^2\cdot M'D^2, \tag{34}$$

$$K_1'L_1'^2\cdot R^4\left(\frac{1}{K_1L_1^2}+\frac{1}{K_1M_1^2}+\frac{1}{L_1M_1^2}\right)=K_1'D^2\cdot L_1'D^2+K_1'D^2\cdot M_1'D^2+L_1'D^2\cdot M_1'D^2. \tag{35}$$

We note, that $K'D^2=K'D_0'^2+D_0'D^2$. We use the equation (A3) in the form $K'D_0'^2+L'D_0'^2+M'D_0'^2=2a^2=K_1'D_0'^2+L_1'D_0'^2+M_1'D_0'^2$ and the equation (A5) in the form $K'D_0'^2\cdot L'D_0'^2+K'D_0'^2\cdot M'D_0'^2+L'D_0'^2\cdot M'D_0'^2=a^4$,

$K_1'D_0'^2\cdot L_1'D_0'^2+K_1'D_0'^2\cdot M_1'D_0'^2+L_1'D_0'^2\cdot M_1'D_0'^2=a^4$.

We add up the equations and obtain, that the right part of the equation (34) is equal to the right part of the equation (35). We know that the first factors in the left side of the equations (34) and (35) are equal. Hence the second factors are equal.■

**Theorem 5** *Let the pair of Sejfried triangles* $KLM$ *and* $K_1L_1M_1$ *be given. Then*

$$\frac{KL}{KM\cdot LM}+\frac{KM}{KL\cdot LM}+\frac{LM}{KL\cdot KM}=\frac{K_1L_1}{K_1M_1\cdot L_1M_1}+\frac{K_1M_1}{K_1L_1\cdot L_1M_1}+\frac{L_1M_1}{K_1L_1\cdot K_1M_1}, \tag{36}$$

$$\frac{KL^2+KM^2+LM^2}{S_{KLM}}=\frac{K_1L_1^2+K_1M_1^2+L_1M_1^2}{S_{K'L'M'}}, \tag{37}$$

where $S_{KLM}(S_{K_1L_1M_1})$ is the area of the Sejfried triangle $KLM\,(K_1L_1M_1)$.

$$ctg\,\alpha+ctg\,\beta+ctg\,\gamma=ctg\,\alpha_1+ctg\,\beta_1+ctg\,\gamma_1, \tag{38}$$

where $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1$ are the angles of Sejfried triangles.

**Proof.** We use the equation (33) and similar, and obtain that

$$\frac{KL}{KM\cdot LM}=\frac{K'L'\cdot R^2}{K'D\cdot L'D}\cdot\frac{K'D\cdot M'D}{K'M'\cdot R^2}\cdot\frac{L'D\cdot M'D}{L'M'\cdot R^2}=\frac{M'A_0'^2+A_0'D^2}{K'L'\cdot R^2}.$$

$$\frac{KM}{KL\cdot LM}=\frac{K'M'\cdot R^2}{K'D\cdot M'D}\cdot\frac{K'D\cdot L'D}{K'L'\cdot R^2}\cdot\frac{M'D\cdot L'D}{L'M'\cdot R^2}=\frac{L'A_0'^2+A_0'D^2}{K'M'\cdot R^2}.$$

$$\frac{LM}{KM\cdot KL}=\frac{L'M'\cdot R^2}{L'D\cdot M'D}\cdot\frac{K'D\cdot M'D}{K'M'\cdot R^2}\cdot\frac{K'D\cdot L'D}{K'L'\cdot R^2}=\frac{K'A_0'^2+A_0'D^2}{L'M'\cdot R^2}. \tag{39}$$

We add the expressions in the left and right parts of the equation (36) and use ($A$3). We obtain, that the equation (36) is correct.

If two triangles are inscribed in a circle, the ratio of their areas is equal to the ratio of the products of sides. This follows from the formula for the area of triangle $S=\frac{abc}{4R}$. Consequently, the equation (36) may be transformed into the equation (37).

It is known, that for any triangle we have $ctg\,\alpha+ctg\,\beta+ctg\,\gamma=\frac{a^2+b^2+c^2}{4S}$. Consequently, the equation (37) may be transformed into the equation (38).■

## 8. Appendixes

### Appendix 1. Properties of equilateral triangle

*In this section, we consider equilateral triangle ABC and point P belonging to the circumcircle ABC. Let the side of ABC is equal to a. We derive appropriate equations.*

Let us order the distances to the vertices. Let us take $AP \leq BP \leq CP$. It is known, that in this case we have $AP + BP = CP$. The angle between *AP* and *BP* is 120º, hence

$$AP^2 + BP^2 + AP \cdot BP = a^2, \tag{A1}$$

$$CP^2 - AP \cdot BP = a^2. \tag{A2}$$

We add up the equations (A1) and (A2) and get

$$AP^2 + BP^2 + CP^2 = 2a^2. \tag{A3}$$

Raising the equation (A2) in the square, we get

$$(CP \cdot (AP + BP) - AP \cdot BP)^2 = a^4, \tag{A4}$$

$$CP^2 \cdot AP^2 + CP^2 \cdot BP^2 + AP^2 \cdot BP^2 = a^4. \tag{A5}$$

The sines of the angles between *AP, BP,* and *CP* are equal to $\frac{\sqrt{3}}{2}$, hence the sum of the squares of the distances from *P* to the lines containing the sides of triangle $ABC$ is equal to $\frac{3}{4}a^2$.

## Appendix 2. Gergonne point coincides with the incenter

*Let the Gergonne point coincides with the center of the inscribed circle of the triangle. We prove that the triangle is equilateral.*

**Proof.** The radii of the inscribed circle are perpendicular to the sides at the points of contact, so they lie on the altitudes of the triangle. Therefore, the orthocenter is concur with the incenter. Consequently, the nine-point circle, passing through the bases of heights, is the same as the inscribed circle. Hence, the midpoints of the triangle sides are concur with the height bases. Therefore, the heights coincide with the medians. This is only possible for equilateral triangle.■

## Appendix 3. Projection of a circle into a circle

*Let right triangle* $ABD(AB \perp BD)$ *be given in space and it's height is* $BC \perp AD$. *The circle* ω *with the diameter* $AB$ *lies on a plane perpendicular to* $BD$, *the circle* ω' *with the diameter* $BC$ *lies on a plane perpendicular to* $AD$. *Let us prove that central projection from point* $D$ *transforms the circle* ω *into the circle* ω'. *That is, any point E of the circle* ω *corresponds to the point F of the circle* ω' *such that the points D, E and F are collinear.*

**Proof.** Let us construct the sphere with the diameter $AB$ (and with the center of *O'*) and the sphere with the diameter $BD$ (and with the center of *Q*). The plane of the circle ω is tangent to the second sphere at point *B* and is perpendicular to the diameter of the sphere *BD.* The intersection curve of the spheres is a circle belonging to the plane perpendicular to $O'Q$. Since $O'Q$ is the average line of the triangle $ABD$, the plane of this circle is perpendicular to $AD$. In accordance with the definition, the circle ω' transforms into the circle ω under stereographic projection from point $D$. In accordance with the properties of the stereographic projection, the circle ω transforms into ω' under central projection from point $D$.

Using the properties of crossing segments, we find that the beam $DG$ intersects the segment $BC$ at its mid-point *O* if and only if $BD = AB\sqrt{\frac{AG}{BG} - 1}$. ■

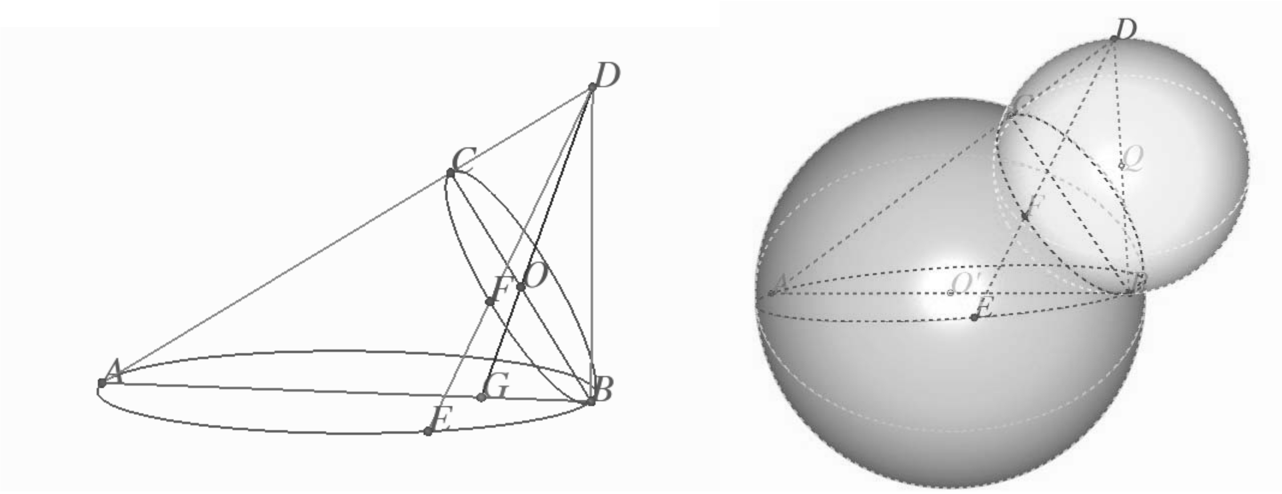


**Figure 9** Projection of a circle into a circle

## 8. Conclusion

The second author has just started to delve into the amazing properties of Sejfriedian and there are many properties that have not been proved. For example, $\frac{AK}{AK_1} \cdot \frac{BL}{BL_1} \cdot \frac{CM}{CM_1} = 1$. We have a lot of interesting research ahead.

## References


[1] M.Sejfried. *Amicable triangles and perfect circles*., at the Proceedings of the 15th International Conference on Geometry and Grafics (ICGG 2012), pp 682-687, 2012. ISBN 978-0-7717-0717-9.


## Software Packages


[2] GinMA Software, http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx