A Didactical Transposition of the Perspective Theorem of Guidobaldo Del Monte with Cabri 3D

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Abstract: *The parallel perspectives used to represent 3D figures in 2D have the great advantage to respect the "parallelism" property, which is an enormous help for those who use these figures in 3D geometry problem solving. But these representations do not give a realistic impression. They do not display what our eyes see. At the beginning of the 15th century Guidobaldo (1545-1607) solved this problem mathematically in Perspectivae Libri Sex. He was the first (*[8]*) to give the proof of the positions of the vanishing point of a direction with respect to the position of the observer. Kirsti Andersen considers him as the father of the mathematical theory of perspective (*[1]*), We will see how Cabri 3D can help to understand what this theorem states. The principal aim of this paper is to show how I rediscovered this theorem using an experimental process (*[11]*), how I have discovered a more general form of it, especially that which includes the angle between the plane of the representation and the vertical plane parallel to the plane of the eyes (and the chin) of the observer. Vanishing points arose to help us understand the geometry of buildings, which we photograph with our cameras. This work started with a special task I had to achieve: help middle school students to understand the obtaining artistic results with their cameras before a competition focused on "the mathematics in the city". So, we can use the stages of this discovery as the base of an approach of the teaching of the conical representations. Back to the parallel perspective we will give some results never published about the coefficient of such perspectives with respect to the angles of their direction.*

1. Experimenting to discover three vanishing points

1.1. Alberti's approach

Alberti has shown how to draw on a plane exactly what the painter's eyes see (Figure 1 on the left). Jean-Marie Laborde has illustrated it dynamically with Cabri 3D (Figure 1 on the right). However, after displaying the representation of a cube, an important problem arose: parallel edges of the cube are not parallel in this representation. Guidobaldo del Monte focused his attention on the representation of parallel lines.

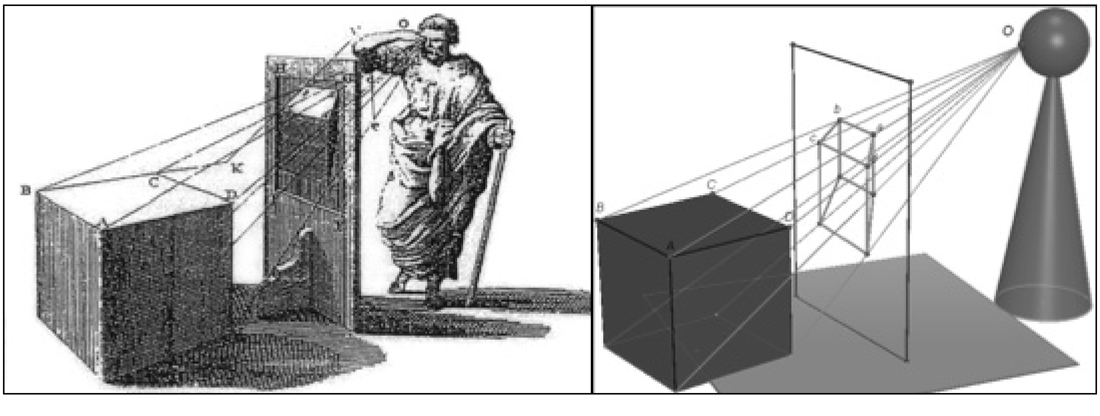

Figure 1

1.2. A camera + Cabri 2 Plus + an art deco building in Miami Beach = 3 vanishing points

Two teachers of a French middle school, one teaching "Geography" and the other "French", wanted me to show their students with dynamic geometry software what representation is obtained by a camera when taking pictures. It was a challenge because, if I have deep knowledge about parallel perspectives and especially on using Cabri 2 Plus, Cabri 3D and TI N'Spire with these representations at all levels ([3], [4], [6], [7], [13] and [14]), until now, I was not so familiar with the perspective of the painter, the conical one, even though I know that it is related to projective geometry.

So my **first reaction** was, as any researcher would have done: I began to read books about this perspective (especially [8] and [10]). But I found only books where only one thing was explained: the techniques of drawing usual objects using the vanishing point. That was very disappointing, because I wanted to understand the origin of these vanishing points (one, two, three… how many and why?) and their exact positions. [1] is an amazing work achieved by Kirsti Andersen about it.

Second reaction: experiment in order to visualize these vanishing points on a picture pasted in a Cabri 2Plus figure. It is what I did on a picture of an art deco building I have taken in Miami Beach.

Experiment 1: After pasting this picture (Figure 2 on the left), I have constructed 3 series of lines, the red ones superimposed on vertical lines of the building, the blue and the green lines superimposed on horizontal lines of this building knowing that the blue lines are perpendicular to the green lines. I noticed that the blue lines seem to cross at a same point; same conjecture for the green lines. Going upside down on this figure, we can do the same conjecture for the red lines (Figure 2 on the right). We have conducted a generative experiment ([5]), which had been done with middle school students (such experiments lead to conjectures).

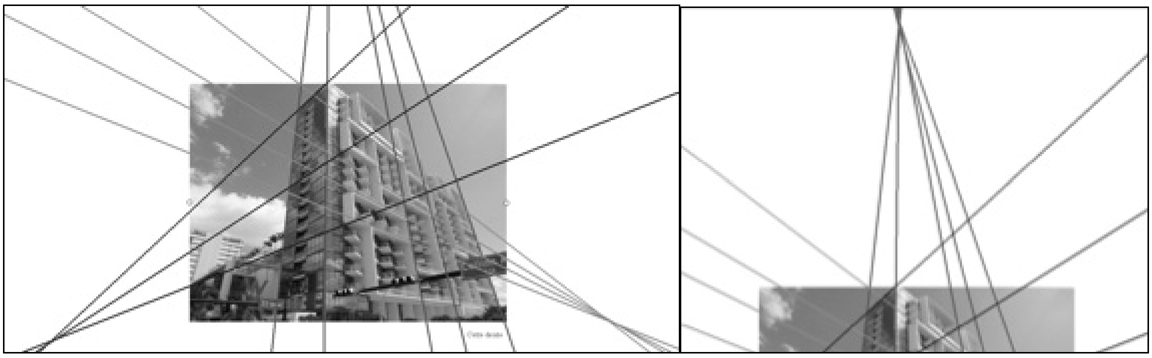

Figure 2

The question was: can we validate this conjecture (i.e. corroborate it, in the meaning of Popper, [11]) with other pictures?

Experiment 2: I then presented to the same students some other pictures taken in Chicago, in order to see what would be their reactions and their propositions (Figure 3 and Figure 4). Their first reaction was to paste these pictures in Cabri 2 Plus files: we did it. After that, they decided to draw lines superimposed to verticals and horizontals of these pictures: they told me that they wanted to check that these lines are crossing at the same points like in the first experiment. Here is briefly what happened for these four figures:

Figure 5 on the left: we have constructed the vertical lines and we obtained approximately the confirmation we expected.

Figure 5 on the right: we did the same constructions and obtained a common point for our vertical lines, but the common point was located under the picture.

Figure 6 on the left: we obtained only two vanishing points, the one relating to perpendicular lines belonging to planes parallel to the horizontal one.
Figure 6 on the right: we obtained one vanishing blue point corresponding to the direction perpendicular to the front plane of the experimenter. There were no vanishing points for the verticals.
We have undertaken validative experiments ([5]). Such experiments corroborate previous conjectures)

Figure 3

Figure 4

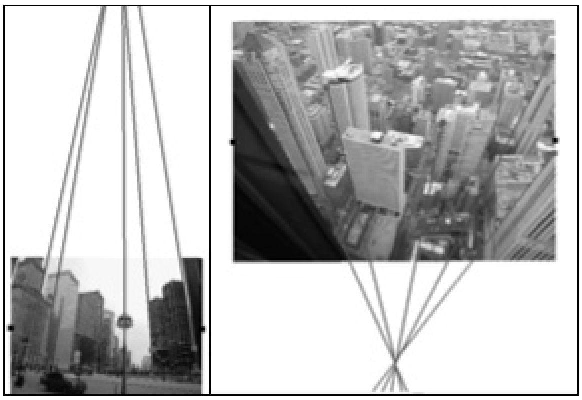

Figure 5

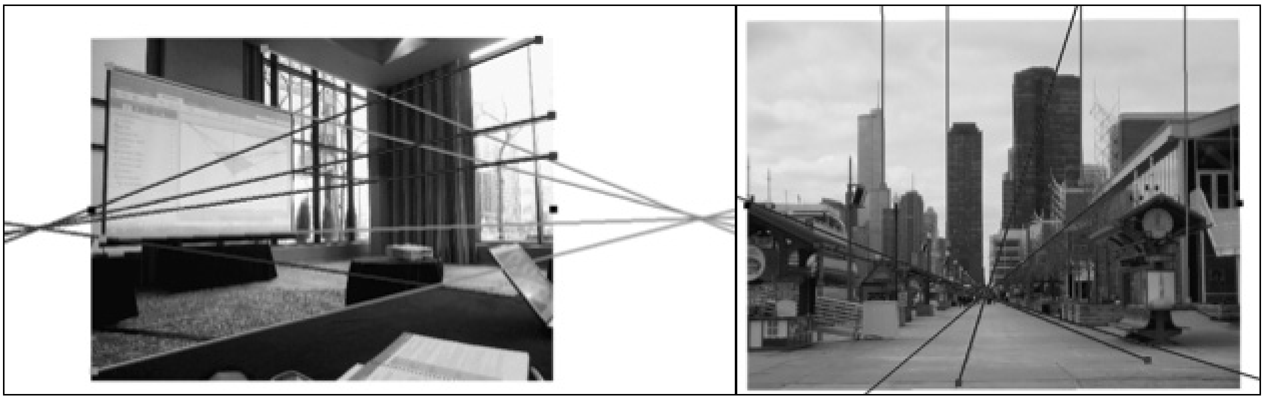

Figure 6

1.3. First conclusion

We did conjecture that an experimenter standing in front of a building at different levels can see lines having the same direction, crossing at a same vanishing point. We focused our observations on vertical lines and on pairs of horizontal lines perpendicular to each other. Sometimes a vanishing point exists, sometimes it doesn't. In the next paragraph, with two models created with Cabri 3D we will explore this phenomenon.

2. Two dynamic models to understand vanishing points

2.1. Model 1

This model is practically Alberti's one. It shows a box (whose base is a square) and its central projection on a vertical plane between this box and the eye of the experimenter. We have modelled this central projection for a spinning box that can turn around its vertical axis (knowing, according to Duval ([9]), that a dynamic approach of a figure is more heuristic than a static one).

In Figure 7 on the left, the vertical segment [FE] is representing the experimenter (F for feet and E for eye). He can see the projection of the box (projection centred in E onto a vertical plane) on the glass of a window included in this plane.

In Figure 7 (on the right), we have blackened the window glass to represent this projection and we have constructed the supporting lines of the horizontal edges of the represented box. This construction led us to a first conjecture: all the supporting lines of the same direction cross on a vanishing point. So we can discover the two vanishing points of the two directions of the horizontal edges of the box. One, blue and the other one, green. That is conjecture 1.

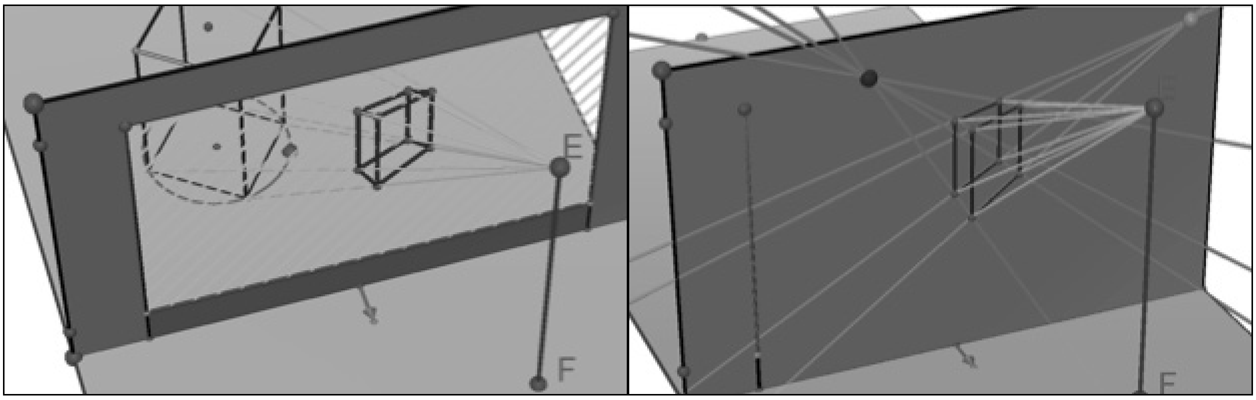

Figure 7

In Figures 8 on the left, we improved this conjecture: in changing the point of view, we can imagine that point E and the two vanishing points belong to the same horizontal plane. We validate this conjecture by constructing a plane parallel to the horizontal plane and passing through E: this plane cuts the plane of the representation on a red line containing the vanishing points. This property is kept when we change the height of the observer or when we rotate the box, even if when we change the position of F. That is conjecture 2.

In Figures 8 on the right, we obtain a more accurate conjecture about the positions of the two vanishing points V_1 and V_2. If we visualize the figure from above, as the box is turning around its axis, it is possible to conjecture that segments $[EV_1]$ and $[EV_2]$ stay parallel to the directions of the horizontal edges of the box. It is possible to verify it by superimposition of these segments with the parallel lines drawn from E to the directions of the horizontal edges of the box (praxeology G1 informatics, [5]). That is conjecture 3.

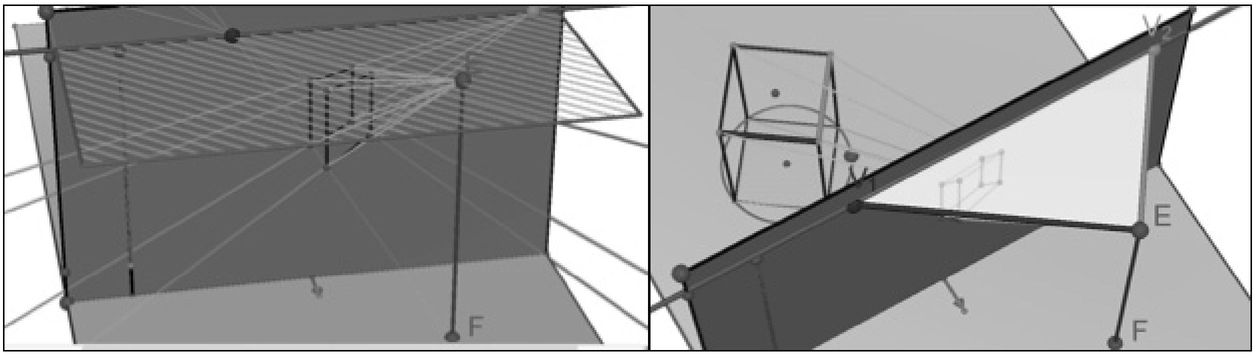

Figure 8

Eventually, we can group, these three conjectures onto the following one:

The two vanishing points of the directions of the horizontal edges of a vertical box are located at the intersection of the lines parallel to these directions drawn from the eye of the experimenter and the plane of representation. That is Guidobaldo del Monte’s theorem in its basic form. A proof will be given later for a more general form.

2.2. Model 2

What, neither Alberti nor Guidobaldo del Monte could imagine, is the invention of cameras and the automatic creation of what we see (really, still the result of a central perspective) on a very small

rectangle not necessarily vertical. To understand really what happened in the Miami and Chicago pictures, we have improved the previous model by allowing the plane of representation to be rotated around its intersection with the horizontal plane.
Remark: the experiments we have undertaken with Cabri principally allow us to understand the positions of the vanishing points when they do exist in relation to the location of the experimenter, the position of his eye(s) and the head angle of the camera. This second model will confirm in the most general case the first of Guidobaldo's conjectures we have retrieved.
Using this model (Figures 9), we have created the supporting lines of the vertical edges of the box. We can change the point of view of this figure and conjecture that the third vanishing point V_3 is located at the intersection of the plane of representation, and the vertical line passing through E (which is also a parallel line to the vertical edges of the box passing through E). Praxeology in this experiment is G2 informatics ([5]), because Cabri accepts to create this intersection point of 4 lines (deductive level of the software).

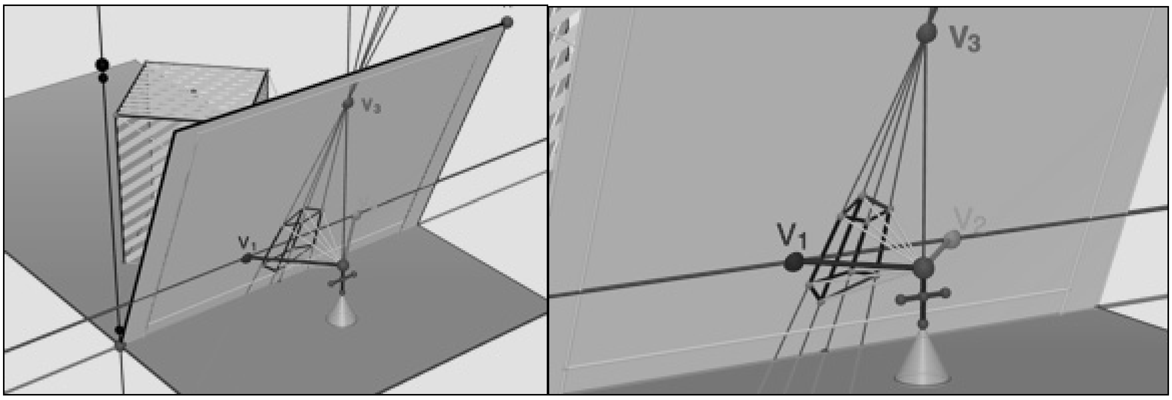

Figure 9

We can confirm this conjecture by rotating the box (V_3 does not change), by dragging the little observer (the three vanishing points change).

2.3. Second conclusion
2.3.1. Guidobaldo's general result: After experimenting with these two models, a more accurate conjecture arises: the vanishing point associated with a direction in a central perspective (conic projection on a plane) is the intersection point between the plane of representation (the plane of the conic projection) and the parallel line to the direction passing through the position of the eye of the observer.
2.3.2. Experimental validation in the general case: We represent a set of parallel lines and we construct their conic projection. We check that the projections have a common point and that this point is the intersection point between the plane of the perspective and a parallel line to these lines passing through the centre E of the perspective. (Figure 10 on the left). The confirmation is obtained in changing the direction of these lines, the position of the eye E.
2.3.3. Guidobaldo's theorem: It says that
The images of a set of parallel lines that cut the picture plane all meet in a point called their vanishing point. This point is the point of intersection of the picture plane and the line among the parallel lines that passes through the eye point.

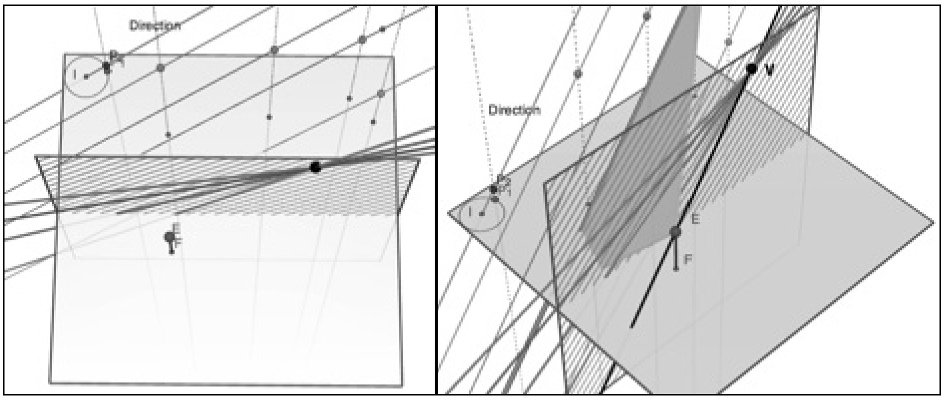

Figure 10

2.4. Proof of the theorem of Guidobaldo del Monte

We have represented in purple (Figure 10 on the right), the plane used to represent the purple projection (passing through V) of the purple line. This plane contains the parallel line to the purple line passing through E, and so this parallel line necessarily cuts the projection of the purple line in a point, which is necessarily the same for the other lines (because this parallel line is the same for all the lines). So Guidobaldo's theorem is proven in the more general case

3. About the coefficient of parallel perspectives

Parallel perspectives are parallel projections. They can be interpreted as the shadow of an object enlightened by the rays of the sun. When the shadow is obtained on a vertical plane, we have the usual perspective of textbooks. When the shadow is obtained on a horizontal plane, the perspective is the military one. What we will show, for each case, is a formula giving the coefficient of this perspective with respect to the two angles defining the direction of the rays of the sun. We recall that the coefficient of these perspectives is the length of the shadow of a unit vector perpendicular to the plane of projection.

3.1. Case of the usual parallel perspective ([13]) (Figure 11)

Direction (CP_2) of this perspective can be changed in moving point P_1 and P_2 that depend from the angles a and b. Vector $\overrightarrow{OI'}$ is the projection of the unit vector $\overrightarrow{OI}$ (perpendicular to the front green plane of projection). Its length is the coefficient of the perpective we want to find : it will be given by a formula depending on a and b. A vector of this direction is given by the coordinates :

$\begin{pmatrix} \cos(b).\cos(a) \\ \cos(b).\sin(a) \\ \sin(b) \end{pmatrix}$, knowing that $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $-\frac{\pi}{2} < b < \frac{\pi}{2}$. So a parametric equation of line D (parallel to (CP_2) and passing through I is given by : $\begin{pmatrix} x = 1 + \lambda.\cos(b).\cos(a) \\ y = 0 + \lambda.\cos(b).\sin(a) \\ z = 0 + \lambda.\sin(b) \end{pmatrix}, \lambda \in R$. I' is obtained when $x = 0$, so $\lambda = \frac{-1}{\cos(b).\cos(a)}$ and therefore :

$$I'\begin{pmatrix} 0 \\ -\dfrac{\cos(b).\sin(a)}{\cos(b).\cos(a)} \\ -\dfrac{\sin(b)}{\cos(b).\cos(a)} \end{pmatrix}$$ and then $$I'\begin{pmatrix} 0 \\ -\tan(a) \\ -\dfrac{\tan(b)}{\cos(a)} \end{pmatrix}$$. Finally the coefficient of the perspective is the norm of these coordinates : k $=\sqrt{\tan^2(a)+\dfrac{\tan^2(b)}{\cos^2(a)}}=\sqrt{\tan^2(a)+\tan^2(b)(1+\tan^2(a))}$. A symmetric form of this coefficient could be :

Coefficient of the usual parallel perspective : $k=\sqrt{\tan^2(a)+\tan^2(b)+\tan^2(a).\tan^2(b)}$.

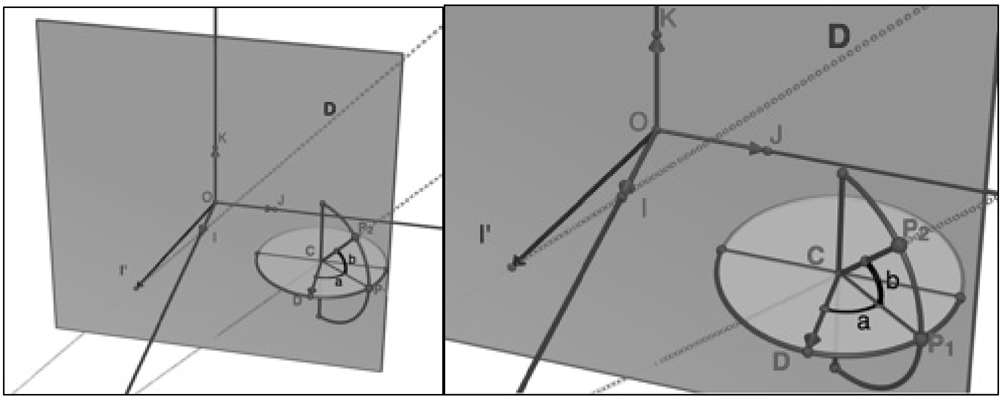

Figure 11

3.2. Case of the military perspective ([14]) (Figure 12)

Moving the front plane along the horizontal plane of projection can do similar work. The equation of line *D* is now:

$$\begin{pmatrix} x=0+\lambda.\cos(b).\cos(a) \\ y=0+\lambda.\cos(b).\sin(a) \\ z=1+\lambda.\sin(b) \end{pmatrix}$$, $\lambda \in R$, *I''* is obtained when $z=0$. We get easily : $\lambda=\dfrac{-1}{\sin(b)}$ and then :

$$I''\begin{pmatrix} -\dfrac{\cos(a)}{\tan(b)} \\ -\dfrac{\sin(a)}{\tan(b)} \\ 0 \end{pmatrix}$$ and the norm of the coordinates is : $\sqrt{\dfrac{1}{\tan^2(b)}}$ and finally :

Coefficient of the military perspective : $k = \dfrac{1}{|\tan(b)|}$.

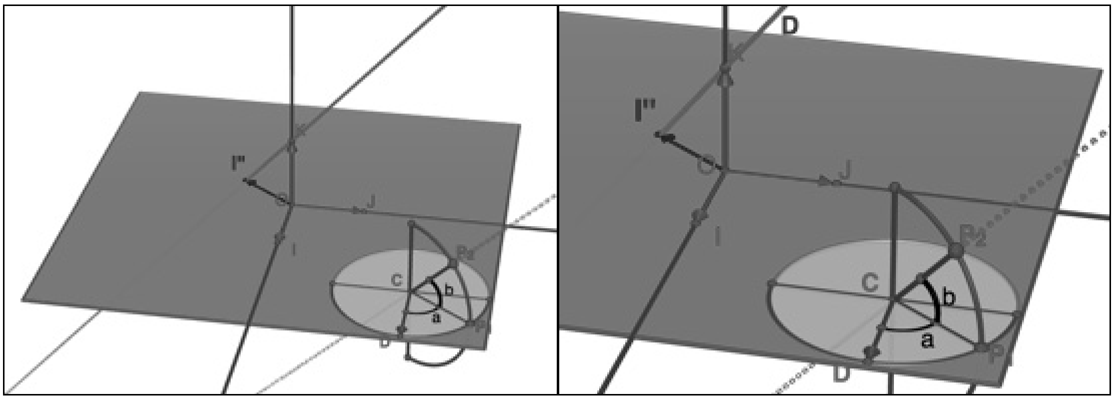

Figure 12

Remark : this coefficient does not depend of *a*.

3.3. A way to explore with TI N'Spire or Autograph

It is possible to use the 3D graphing tool of TI N'Spire to display the surface corresponding to the formula we got for the coefficient (see Figure 13 left) in the usual parallel perspective. The intersection of this red surface with the yellow plane $z = k$, is a closed curve with points giving the pairs (a,b) of angles in radians corresponding to this coefficient k. The intersection between this curve and the blue plane $x = a$ is a pair of points whose ordinates are the angles b, chosen in order to get coefficient k. This would be a way to find all the pairs of angles defining the direction of the projection giving the usual coefficients as 1 or ½ or to find angle b when k and a are given

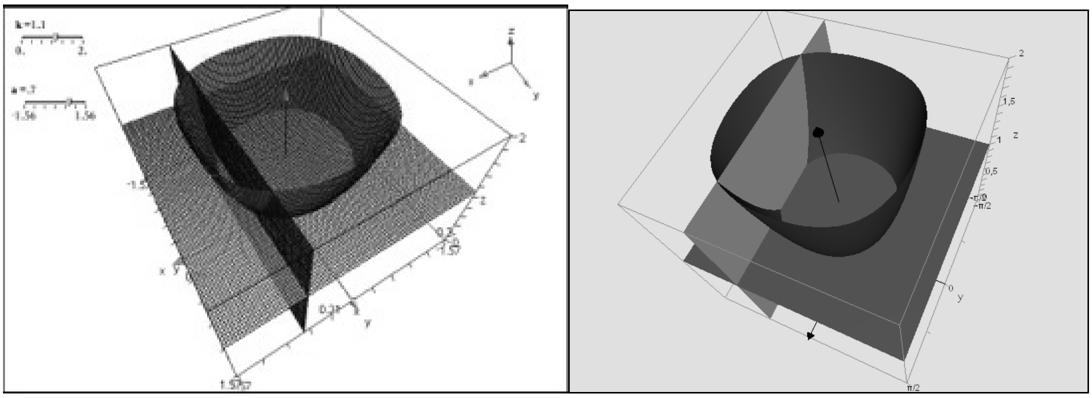

Figure 13

It is possible to do such a work with Autograph in using its Constant Controller instead of the sliders of TI N'Spire (Figure 13 on the right).

5. CONCLUSION

We have described research where the initial goal was not research work. Starting from the demand of a teacher, the expert thought that he had only to provide some nice DGS files (Cabri 2 Plus or Cabri 3D) to illustrate the central perspective for young students (a work of didactical transposition). But this work moves quickly onto a real research leading to the discovery of a theorem crucial to understanding the perspective obtained with our usual cameras (a same process which allowed the discovery of a theorem: see [1]). As, very often, this theorem, was, in its restricted form, already discovered and proven by somebody before, Guidobaldo del Monte at the beginning of the 15th century. At the end of this work, we realised that the Cabri 3D files created to understand a special phenomenon which led to this theorem, could be used for a very original presentation of the very difficult notion of central perspective for both students and teachers. By the way, we were improving our first didactical transposition with pictures pasted on Cabri and analysed by superimpositions with some lines constructed with this software. We can imagine that these files could also be used for teachers' training both to discover the software and to learn about the central perspective in a process of instrumental genesis ([13]).

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