Interactive Worksheets for Learning the Connection Between Graphic and Symbolic Object Representations

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Abstract: *Learning the close relation between graphic and symbolic object representations is a key to conceptual understanding of mathematical functions and vector equations. For learning such a relation, it is valuable that students manipulate the graphs of functions, and transform the graphic objects directly with observing the simultaneous change of related equations. Here is the need for tailored worksheets, preferably embedded into a www-based learning-support system.*
Interactive worksheets concerning linear algebraic concepts like vector operations, basis, linear functions and eigenspaces in the plane were tested in university courses during the last decade. Such worksheets allow direct interaction between the student and the dynamic figure containing geometric objects. Each figure is accompanied by problems to be solved when exploring relationships in the figure. The students' and teachers' positive feedback encourages extending the idea to three-dimensional linear algebra.
Next implementation of the worksheets shows graphic objects as the targets that students move and reshape their own objects to overlay. The simultaneous change of graphic and symbolic objects provides the students with opportunities to recognize their relations. This paper describes how the worksheets are designed, implemented into www-systems, and what reflections they received from students and teachers.

1. Introduction

It has been and still is difficult to improve students' conceptual understanding in vector equations of lines or planes in 3D space although the concept is foundation to mathematics, science and engineering. Some of the students could not even tell if an equation represented a plane or a line in 3D space. Those students tended to memorize a series of formulas without examining their features or understanding the relations of the formulas with the features of graphic objects. In another words, they cannot use visual reasoning in cooperation with analytic reasoning.

Even more fundamental are the abstract concepts of binary operations that occur in axiomatic linear algebra in university level mathematics. In addition to the ability of generalization the students should be guided to the idea of abstraction. Generalization means here going to higher dimensions of Euclidean spaces, but abstraction is needed for going beyond those concrete spaces. When going further to the concepts basis and linear transformation there is an obvious need for visualizations even in 1D and 2D Euclidean spaces. Most students do not see the geometric idea of basis even though they can solve the standard calculative exercises.

Both analytic and visual reasoning are important in mathematics. Analytic reasoning is based on the use of symbolic representations and construction of logical inference chains. Instead, visual reasoning is based on visual interpretations of mathematical concepts. Visualization makes it possible to perceive abstract mathematical objects through senses. Visual representations can be considered more concrete than analytic ones, because they are based on external objects. Analytic reasoning is often exact and detailed, but visual reasoning is needed to reveal wider trends of the whole problem solving process and holistic features of the problem situation. Often visualization acts as a map showing a direction for the reasoning process. [1]

Mathematicians usually have an ability to utilize visualization in an effective way in mathematical reasoning. For example, Stylianou [2] noticed that mathematicians used visualization in a very systematic way, so that in their reasoning the visual and analytic steps interact very closely with each other. However, this is not the case with students: According to Stylianou and Dubinsky [3], many students have difficulties in analyzing visual representations, and, therefore, they cannot utilize them in problem solving. Also Raman [4, 5] found that an essential difference between professional mathematicians and inexperienced students was that mathematicians considered visual and formal arguments closely connected so that the visual arguments in an essential way contributed to inventing ideas in constructing formal arguments. Instead, students could not recognize connections between visual and formal arguments. Stylianou and Silver [6] noticed that students consider visual representations useful mostly in geometric problems, whereas mathematicians see a wider variety of problems where visualization can be used. In addition, students often consider visual reasoning as a non-mathematical and unacceptable method, and, therefore, they may be reluctant to use it [7, 8]. Therefore, it is important that both analytic and visual reasoning and connections between them are emphasized in teaching of mathematics.

Traditionally visual representations have been drawn on a paper or on a blackboard etc. However, the development of information technology has made it possible to use interactive and dynamic visual representations. By using *dynamic geometry software* (DGS) students can themselves modify constructs based on visual or symbolic representations and see the effects of the modifications.

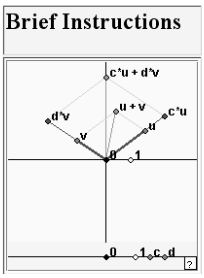
To utilize visual representations, some of the authors have redesigned their lesson plan at Toyota based on the MODEM theory: concept recognition, identification (in fact matching) and production [9]. The lesson plan is directed from concrete applications to abstract mathematical ideas, from handling graphic objects and identifying their characteristics toward defining symbolic expressions and rewriting them [10]. In the lessons, we used DGS, CAS, and the learning material created with these software programs to show simultaneous changes of graphic objects and the related symbolic expressions. But some students tended to lose the tracks among the software's rich but rather complicated functionalities. Teachers also wanted to know how much experience a student actually had with the learning material. Here is the need for tailored worksheets, preferably embedded into a www-based learning-support system.

In teaching binary operations, rather systematic www-worksheets were created and used in classroom sessions in Joensuu and Caen. The question types were slight modifications of the VSG problems of the MODEM theory. During the years the system was embedded in a course management system, and because of the simplicity of the questions and the structure it became a part of the linear algebra exam system, done in classroom or at home. This experiment showed for example that concept recognition questions provide us with an accurate tool for predicting the students' future success in abstract linear algebra course, see [11]. In teaching basis and linear transforms of the 2D space, the worksheets contain a more complicated structure. These are described in chapter 2.

In chapters 3 and 4, we explain how we designed interactive worksheets for learning the relation of single variable functions and their graphs using interactive graphic interface and built-in programming language of a CAS: Mathematica [12]. The same approach is used to design the worksheet for learning the relation of planes in 3D space and their vector equations. Attractive features and limitations of those interactive worksheets are to be discussed in this presentation.

2. Interactive worksheets for basis and linear functions

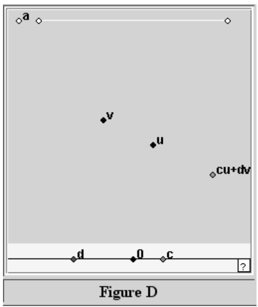
The worksheets Linear space and basis [13] and Linear functions in the plane [14] deal with linear algebraic concepts span, linear independence, basis and linear tranformations in 2D space. Both www-forms start with short descriptions of their purpose and content. A dynamic figure illustrating the vector space structure of the plane serves also as a short instruction on how to act with such figures (Figure 1).



(Text describing the dynamic figure idea and how to use them, e.g. what points can be dragged by the mouse, how to reset.
The scalar real line is at the bottom of the figure.)

Figure 1 Introductory dynamic figure in linear algebra worksheets

We only describe here the worksheet Linear space and basis in more details. It contains four chapters: I Span, II Linear independence, III Basis and IV Students feedback text input area. The chapters I-III start with recalling the concept definition in symbolic form. The span chapter has two kinds questions: 1) six for definition recognition, with a dynamic figure containing 1-3 vectors and their linear combinations with 1-3 scalars, 2) production questions asking what the span is. In the former the students should be able to recognize whether the span is the whole 2D space, in the latter they should express the span (when not the 2D plane) in verbal, geometric or symbolic form. To make the questions more tricky the figures only contain the vector heads, and in many figures the origin is hidden; finding it is crucial for understanding the situation, see Figure 2.



Objects in the left figure are in their initial positions, note the freely draggable white tools.
On the right the figure is manipulated so that one has found the origin and the span: c and d have been positioned at the real line 0 and a has been places at the plane origin. Vectors u and v are parallel and the white segment is placed on the span line.

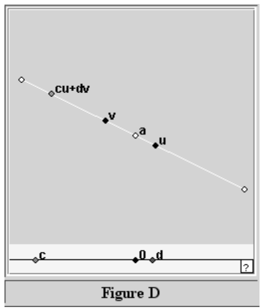
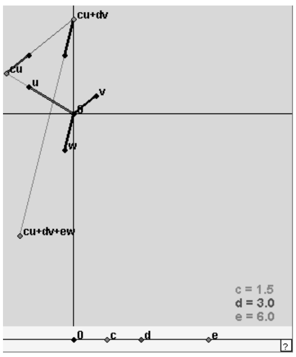


Figure 2 Finding the hidden origin and the span in one of the recognition tasks

This way one sees concretely the geometric idea of span. One can visit any point in the span with at least one choice of the multiplying scalars.

Chapter II has four dynamic figures, in each of which one has 1-3 vectors and scalars. The first problem is to find out whether the vectors are linearly independent or not. Being not linearly independent (i.e. being dependent) means geometrically that one can make an actual cruise using the linear combination with suitable multipliers (not all of them zero).



Objects in the left figure are in their initial positions. Choosing suitable c and d one can adjust e so that the linear combination cu+dv+ew reaches the origin, just increase e to be 2.2 in the right hand figure situation.

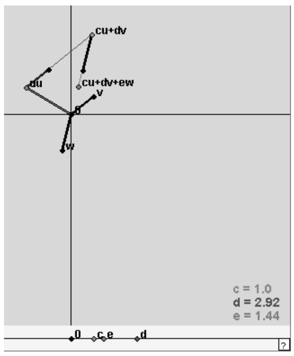


Figure 3 Finding suitable coefficients brings the linear combination to origin

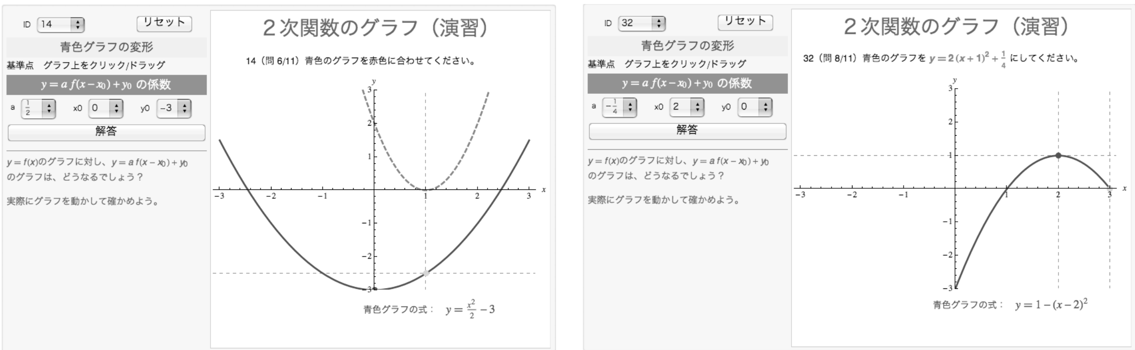
In each of the tasks with dependent vectors one is asked to input good scalar choices to the www form. Each student may find different (but proportional) values. Also a text area is available to write comments to the teacher.

Chapter III deals with basis, having three dynamic figure tasks about concept recognition. In addition, there is one dynamic figure task about determining vector coordinates with respect to a given basis, and one about finding the matrix for change of basis.

After a student has solved the tasks and sent the answers on the www-form to the teacher's email, the system shows them and also the teacher's answers and comments.

3. Interactive worksheets for learning functions

The relation of a quadratic function $f(x)$ and its graph $y=f(x)$ is a good introduction to design interactive worksheets because it is the first serious content where the relationship of symbolic and graphic forms becomes the key to understand the concept deeply. Fresh high school students, who have already learnt basics of quadratic function and the graph of $y=ax^2$ in junior high school, still need some learning to relate more general form $f(x)=a(x-x_0)^2+y_0$, where (x_0,y_0) shows the coordinates of the peak or bottom of the graph and a change the bending sharpness of the curve. In the worksheets shown in Figure 4, students are requested to do two kind of activities: the first activity is to manipulate the curve to fit to the target; and the second one is to find the parameters a, x_0, y_0 of the function which represents the curve.



Type A: Target is the curve　　　　　Type B: Target is the equation

Figure 4 Worksheets for learning graphic-to-symbolic relation of quadratic functions

During the first activity, the symbolic expression, which is shown under the graph, changes the parameters simultaneously when the student manipulates the solid curve by dragging one of the two points attached to the curve. If the student drag the point attached to the peak (or bottom), the curve changes its position without changing the shape. For changing the curve's shape, the student needs to drag the other point on the curve, which is initially located on the curve and one unit right to the peak. The vertical and horizontal lines crossing at this point also helps to measure the sharpness of the curve.

Matching to the target is simpler and could be completed without prior knowledge of the graphic-to-symbolic relations when it is shown as a curve (Figure 4, Type A). It becomes more difficult if the target is shown only as a symbolic expression (Figure 4, Type B). Students need to have or build some theory or assumptions about the graphic-to-symbolic relations during the activity. Anyway, it is in this activity that students are to learn the graphic-to-symbolic relations from simultaneous change of graphic and symbolic expressions.

In the second activity, students are to search the values of parameters a, x_0, y_0 of the graph. Because the symbolic expression of the curve has disappeared from the screen when the curve fits to the target in the first activity, the students have to analyze the graphical features of the curve and decide the values from them. They select the three values from the pop-up lists at the left side of the screen and click on the 'Answer' button. The second activity makes a quiz, and students are allowed to proceed to the next task only if all the parameters are answered correctly.

The worksheets hold the information such as if the student answered each quiz at the first attempt or not, time used to complete the tasks and the students' two-digits ID, and display them as a code on the ending screen. The code could be used to record the students' activities when they are used in a learning system.

The worksheets are *Computable Document Format* (CDF) files coded with Mathematica programming language (version 8) using its *Manipulate* function. They work on computers (Windwos PC, Mac, Linux) with Wolfram's CDF players installed on them, which are downloadable from Wolfram's website [12] freely although we still need Mathematica to create CDF files.

4. Interactive worksheets for learning equations of planes in 3D

Mathematica's built-in functions offer us a powerful interface, on which we could directly manipulate a graphic object and observe the simultaneous change of related symbolic expression. However, its directness is limited to the contents that handle 2D graphic objects and we have to use less direct interface for handling 3D graphic objects. *Sliders* are the alternative interface components affordable also to 3D, which is shown on the left side of the screen in Figure 5.

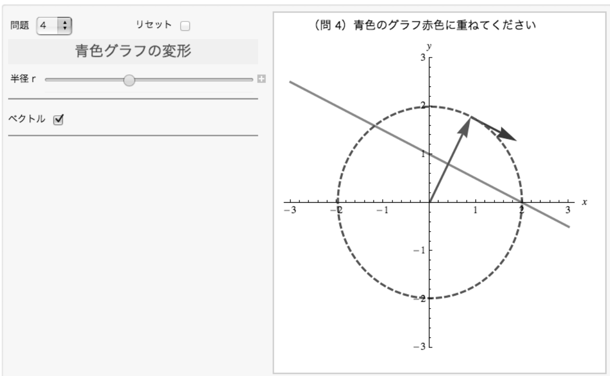


Figure 5 Example interface displaying normal and parallel vectors to a line in 2D

Figure 5 tries to show the same approach we employed to build interactive worksheets handling 3D graphic objects in a 2D example. It shows the relation of a line $ax + by = d^2$ in 2D and a circle $x^2 + y^2 = r^2$ centered at the origin. It also shows the parallel and normal vectors to the line. The normal vector is a position vector, whose starting point is fixed to the origin and end point is located on the circle. The starting point of the parallel vector is fixed to the end point of the normal vector.

If a students change the radius r of the circle with the *slider*, he can adjust the concentric circle to cross the line only at a point, where $r = d$ and the line becomes the tangent line of the circle. At this time, the normal vector with its direction and length, is the only information we need to know when we describe the tangent line symbolically, although the direction of the normal vector is fixed in this example.

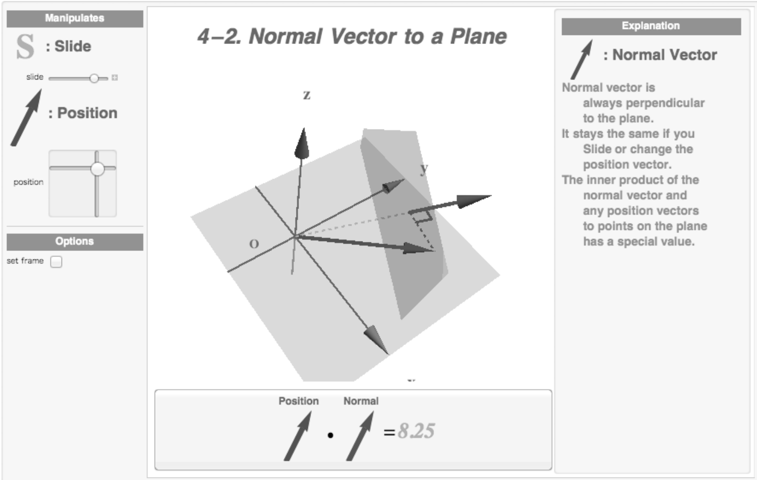


Figure 6 Example interface displaying normal vector to a plane in 3D

Figure 6 shows an extension of the approach to express a plane $ax + by + cz = d^2$ in 3D space [15]. Although the direction of the normal vector is fixed, its length d is changeable with the *slider*. The displayed plane also shifts the position and changes the distance to the origin in this case.

Figure 7 shows an example worksheet for learning graphic-to-symbolic relation of planes in 3D. The target is given as a plane; a graphic object. But it is not so easy to adjust the position and direction of the movable plane with four sliders on the left side of the screen, each of which is related to one of four parameters of the equation $ax + by + cz = D$ and the equation is displayed just under the graphic objects. Students with little prior knowledge of graphic-to-symbolic relation should need many trials and errors to fit their plane to the target. The special cases where the target plane is perpendicular or parallel to one the axes must to be included in initial tasks because they are relatively easier targets.

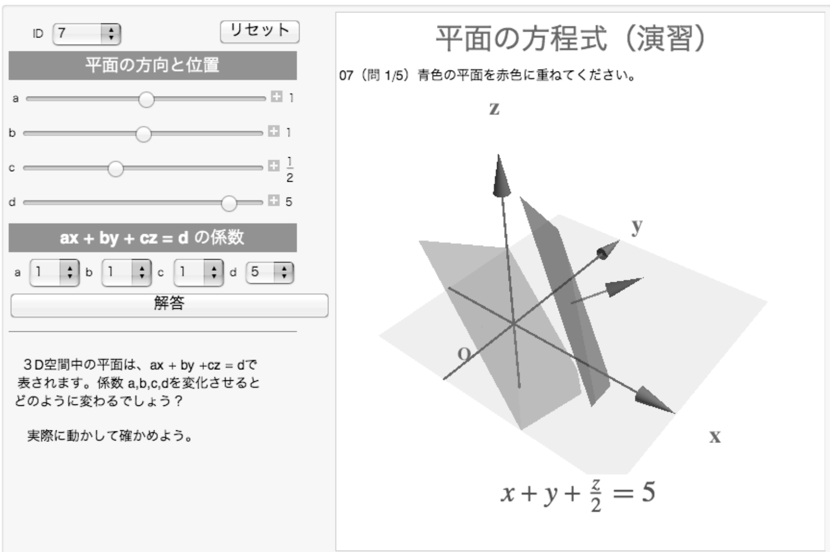


Figure 7 Worksheet for learning graphic-to-symbolic relation of planes in 3D

As in the worksheets for quadratic function (Figure 4), the worksheet has two activities; matching the plane to the target graphically and finding the values of parameters. However, current interface does not display enough graphical information to decide the parameters precisely, instead display the values just next to the sliders.

5. Working with interactive worksheets

Worksheets mentioned in chapter 2 have been used in University of Caen, France, during 2002-2012 with first year university students' linear algebra exercise sessions. Recently the www-form Linear space and basis has been used every year with about 60 students, in tutored 10-15 student groups, in 1.5-3 hour sessions. The Linear functions worksheet has been used when there has been more time than 1.5 hours.
The idea to use worksheets in English came from the French colleagues, they think it is a natural way to introduce English mathematical terminology. Because the French students' knowledge in English is often vague, we have always had 1-3 tutors, of which 1-2 local colleagues. During the last few years the French students' linguistic and technological abilities have improved much, many of them dare give their feedback in English although they are allowed to give written answers and comments also in French.
The French colleagues seem to appreciate the activities and the students are always very well occupied with the work, discussing lively together and with the tutors.
We have not done formal interviews, nor statistical measurements of the effects for learning, because as tutors we see immediately how the students catch the ideas, ask questions and help each other. However, here are some typical authentic feedback sentences:

"I think that it's a good way to work with maths contrarely than on papers. It helps the students to see maths differently."
"I think that is a good way to work. we can visualize most easily than on the paper specially for linear algebra."
"I think that it was good, i understood many thing. Thank you for your exercise."
"We think that it's a good exercise because it's illustrate our lessons graphically. Personally, we have more understand the Linear space and the basis, although it was in English."
"It's interesting to make Maths in English ! It' few difficult but with Professor , it's good !"
"I think that sort of worksheet were very interesting and pedagogic"

The interactive worksheets described in chapters 3 and 4 have been used as the supplemental exercises of the first and second year students at Toyota National College of Technology in Japan. We have prepared four 2D worksheets, which are designed to learn the graphic-to-symbolic relation of linear (absolute) function, quadratic function, rational and logarithmic functions. They are embedded into the web-pages of the learning system and work in students web browsers with the help of Wolfram's CDF players. The students' PCs in a computer laboratory have connections to the web server through the campus LAN.

49 students used the 2D worksheets in the spring semester of 2012. They were strongly recommended to use the worksheets because they had shown poor performance in the prior paper-and-pencil exercises. Each of the 2D worksheets contained from 11 to 14 tasks and the average completing time was 7 minutes per worksheet. If we define the

successful achievements as answering 80% or more correctly at the first attempts in the second activities of the tasks, there were 64 successes (79%) and 15 failures.

Six students used the 3D worksheet shown in Figure 7 during the same period also as the supplemental exercises. The worksheet contained 5 tasks and the average completing time was 5 minutes. All achievements were 100% successful because the second activities in the 3D worksheet are only the confirmation of parameters. There were some reports of malfunctioning related to the 3D worksheet presumably caused by slower response of the 3D worksheet.

6. Discussions

We are generally satisfied with the interactive worksheets handling 2D graphic objects because they are relatively easy to design and useful. They provide students with direct access to graphic objects and their responses are quick enough. Completing the tasks in the worksheets tells that the student can connect the graphs to right symbolic expressions, however, it is not still obvious if the student understand the connection deeply so that he can use the knowledge in his problem solving processes.

The weakness of 2D worksheets is their low impact to some students. The graphic-to-symbolic relations represented in the worksheets are too obvious to advanced students and sometimes also to average students. Pencil-and-paper is a more flexible tool for them. In Japan, we intend to use the worksheets mainly as a part of supplemental exercises for slow learners.

On the other hand, we still have much to improve 3D worksheets with its indirect interface, slower response, and more complicated programming source code. Basically we are dividing a big general-purpose worksheet into a series of smaller worksheets of similar activities in a united interface. 3D worksheets could have deeper impact to our students whose conceptual understandings in 3D vector equations and other concepts are still low.

Although for many linear algebraic phenomena 2D dynamic figures are a vital source for studying the dependencies between plane points, vectors and operations, and cannot overcome by static figures or paper-and-pencil experiments, much more fruitful – and challenging – is the 3D space case. The 3D work done in Toyota and the integration of appropriate questions with dynamic figures embedded within a pedagogically reasonable framework in Joensuu should improve both partners' educational achievements.

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