Special Pythagorean Triangles and Pentagonal Numbers

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Abstract: *The objective of this paper is to show that how research in Mathematics can be facilitated by the use of softwares. In this paper, Special Pythagorean Triangles, in terms of their perimeter to be Pentagonal numbers, are obtained with the help of Mathematica. Cases, when one leg and a hypotenuse are consecutive, are also discussed. A few interesting results are observed. 3D graph of corresponding Pythagorean triplets is plotted using software Mathematica.*

1. Introduction

Pythagorean Theorem, a seemingly simple theorem, has fascinated mathematicians all over the ages. The solutions of many problems related to it still continue to enrich Mathematics. Integral solutions of ternary quadratic equation are given by Gopalan, Somnath & Vanitha [1] and Gopalan & Kalinga Rani [2]. Special Pythagorean Triangles are generated by Gopalan & Vijyalakshmi [3] and Gopalan and Devibala [4]. In the Pythagorean Mathematics, Gopalan and Janaki [5] have given perimeter of Pythagorean Triangles as a pentagonal Number. These gave the motivation to explore the existence of Pythagorean triangles with their perimeters as pentagonal numbers with two consecutive sides.

2. Method of Analysis:

The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \tag{2.1}$$

is given by [6]

$$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \tag{2.2}$$

for some integers ***m, n*** of opposite parity such that ***m* > *n* > 0** and $(m, n) = 1$.

2.1 Perimeter is a Pentagonal number:

Definition 2.1: A natural number p is called a pentagonal number if it can be written in the form

$$\frac{\beta(3\beta-1)}{2}, \beta \in \mathrm{N}$$

If the perimeter of the Pythagorean Triangle (X, Y, Z) is pentagonal number p, then

$$X + Y + Z = \frac{\beta(3\beta-1)}{2} = p \tag{2.3}$$

By virtue of equation (2.2), equation (2.3) becomes

$$2m^2 + 2mn = \frac{\beta(3\beta-1)}{2}, \beta \in \mathrm{N}$$

$$\text{Or, } 2m(m+n) = \frac{\beta(3\beta-1)}{2} \tag{2.4}$$

$$\Rightarrow \beta = \frac{(1+\sqrt{1+48\text{m}^2+48mn})}{6} \text{ and}$$

$$\beta = \frac{(1-\sqrt{1+48\text{m}^2+48mn})}{6}$$

Discarding the second value of β as it is negative, we get

$$\beta = \frac{(1+\sqrt{1+48\text{m}^2+48mn})}{6} \tag{2.5}$$

Now, solving the equation (2.4) with *Mathematica by following command,*

```
FindInstance[2m^2 + 2mn − β * (3β − 1)/2 == 0&&n < m&&0 < m < 1000&&0 < n <
1000&&0 < β < 100000&&GCD[m,n] == 1,{m,n,β},Integers, 10000]
```

we get 999 values of m, n, β out of which 100 values are given in Table 2.1:

S.N.	m	n	β	S.N.	m	n	β	S.N.	m	n	β	S.N.	m	n	β
1	2	1	3	26	77	26	103	51	152	51	203	76	227	76	303
2	5	2	7	27	80	27	107	52	155	52	207	77	230	77	307
3	8	3	11	28	83	28	111	53	158	53	211	78	233	78	311
4	11	4	15	29	86	29	115	54	161	54	215	79	236	79	315
5	14	5	19	30	89	30	119	55	164	55	219	80	239	80	319
6	17	6	23	31	92	31	123	56	167	56	223	81	242	81	323
7	20	7	27	32	95	32	127	57	170	57	227	82	245	82	327
8	23	8	31	33	98	33	131	58	173	58	231	83	248	83	331
9	26	9	35	34	101	34	135	59	176	59	235	84	251	84	335
10	29	10	39	35	104	35	139	60	179	60	239	85	254	85	339
11	32	11	43	36	107	36	143	61	182	61	243	86	257	86	343
12	35	12	47	37	110	37	147	62	185	62	247	87	260	87	347
13	38	13	51	38	113	38	151	63	188	63	251	88	263	88	351
14	41	14	55	39	116	39	155	64	191	64	255	89	266	89	355
15	44	15	59	40	119	40	159	65	194	65	259	90	269	90	359
16	47	16	63	41	122	41	163	66	197	66	263	91	272	91	363
17	50	17	67	42	125	42	167	67	200	67	267	92	275	92	367
18	53	18	71	43	128	43	171	68	203	68	271	93	278	93	371
19	56	19	75	44	131	44	175	69	206	69	275	94	281	94	375
20	59	20	79	45	134	45	179	70	209	70	279	95	284	95	379
21	62	21	83	46	137	46	183	71	212	71	283	96	287	96	383
22	65	22	87	47	140	47	187	72	215	72	287	97	290	97	387
23	68	23	91	48	143	48	191	73	218	73	291	98	293	98	391
24	71	24	95	49	146	49	195	74	221	74	295	99	296	99	395
25	74	25	99	50	149	50	199	75	224	75	299	100	299	100	399

Table 2.1: Values of m, n, β

From the Table 2.1, we observe that

$$m = 3n - 1, \tag{2.6}$$

which gives

$$\begin{aligned} &1 + 48m^2 + 48mn = (24n - 7)^2 \\ &\Rightarrow \beta = 4n - 1 \end{aligned} \tag{2.7}$$

Table 2.1 confirms the value of β.
Equations (2.2), (2.6) & (2.7) generate solutions of equations (2.1) in correspondence with equations (2.3) and (2.4), i.e.,

$$\begin{aligned} X &= 8n^2 - 6n + 1, \\ Y &= 3n^2 - 3n, \\ Z &= 10n^2 - 6n + 1 \end{aligned} \tag{2.8}$$

The following tables give 10 Primitive Pythagorean Triangles with their perimeter as pentagonal numbers:

S.N.	**n**	m = 3n –1	β = 4n –1	**X**	**Y**	**Z**	$\mathbf{X^2}$	$\mathbf{Y^2}$	$\mathbf{X^2 + Y^2 = Z^2}$	X + Y + Z = β (3β - 1)/2
1	1	2	3	3	4	5	9	16	25	12
2	2	5	7	21	20	29	441	400	841	70
3	3	8	11	55	48	73	3025	2304	5329	176
4	4	11	15	105	88	137	11025	7744	18769	330
5	5	14	19	171	140	221	29241	19600	48841	532
6	6	17	23	253	204	325	64009	41616	105625	782
7	7	20	27	351	280	449	123201	78400	201601	1080
8	8	23	31	465	368	593	216225	135424	351649	1426
9	9	26	35	595	468	757	354025	219024	573049	1820
10	10	29	39	741	580	941	549081	336400	885481	2262

Table 2.2: Primitive Pythagorean Triangles (X, Y, Z) with X + Y + Z = β (3β - 1)/2

2.2 Hypotenuse and one leg are consecutive:

In such case either Z = X + 1or Z = Y + 1

Now, Z ≠ X + 1, for if Z = X + 1, then by equation (2.2) we get

$$m^2 + n^2 = m^2 - n^2 + 1$$

$$\Rightarrow 2n^2 = 1$$

which gives n as irrational number, which is a contradiction.

Assuming Z = Y + 1, then equation (2.2) gives

$$m^2 + n^2 = 2mn + 1$$

$$\Rightarrow (m-n)^2 = 1 \qquad (2.9)$$

$$\Rightarrow m = n + 1$$

This gives equation (2.4) as

$$(2n+1)(2n+2) = \frac{\beta(3\beta - 1)}{2}$$

$$\Rightarrow \beta = \frac{1 \pm \sqrt{96\mathrm{n}^2 + 144n + 49})}{6} \qquad (2.10)$$

By the following command, solving equation (2.10) by using the software mathematica for n < 10^{24} and for β < 10^{26}, we get only 12 solutions!

```
FindInstance[2(2n + 2)(2n + 1) − β ∗ (3β − 1) == 0&&0 < n
        < 1000000000000000000000000&&0 < β
        < 100000000000000000000000000,{n, β},Integers, 10000]
```

These special Pythagorean Triangles (X, Y, Z), with their perimeters as pentagonal numbers and each of which one leg and hypotenuse are consecutive, are given in the Table 2.3 below with values of n, β and p in Table 2.2. Their verifications are given in Table 2.4.

S.N.	**N**	**β**	***p***
1	1	3	12
2	170	279	116622
3	16731	27323	1119805832
4	1639540	2677359	10752375483642
5	160658261	262353843	103244308274126052
6	15742870110	25707999239	991351837295782869062

7	1542640612591	2519121571563	9518960238469798834608672
8	151163037163880	246848206013919	91401055218435171114129600882
9	14812435001447721	24188605067792483	877632922688454274568073593061692
10	1451467467104712850	2370236448437649399	8427031232253482725967471526448767102
11	142228999341260411651	232258983341821848603	80916353014465018446285387028887468653112
12	13936990467976415629020	22759010131050103513679	776958813217861874867749560283905947558415722

Table 2.2: Values of n, β and p

S.N.	**X**	**Y**	**Z**
1	3	4	5
2	341	58140	58141
3	33463	559886184	559886185
4	3279081	5376186102280	5376186102281
5	321316523	51622153976404764	51622153976404765
6	31485740221	495675918632148564420	495675918632148564421
7	3085281225183	4759480119233356776691744	4759480119233356776691745
8	302326074327761	45700527609217434394027636560	45700527609217434394027636561
9	29624870002895443	438816461344227122471601795083124	438816461344227122471601795083125
10	2902934934209425701	4213515616126741361532268296119670700	4213515616126741361532268296119670701
11	284457998682520823303	40458176507232509223000464515102473914904	40458176507232509223000464515102473914905
12	27873980935952831258041	388479406608930937433860843151484997363578840	388479406608930937433860843151484997363578841

Table 2.3: Special Pythagorean Triangles (X, Y, Z)

S.N.	β	$p = X + Y + Z = \beta(3\beta - 1)/2$
1	3	12
2	279	116622
3	27323	1119805832
4	2677359	10752375483642
5	262353843	103244308274126052
6	25707999239	991351837295782869062
7	2519121571563	9518960238469798834608672
8	246848206013919	91401055218435171114129600882
9	24188605067792483	877632922688454274568073593061692
10	2370236448437649399	8427031232253482725967471526448767102
11	232258983341821848603	80916353014465018446285387028887468653112
12	22759010131050103513679	776958813217861874867749560283905947558415722

Table 2.4: Verification of $X + Y + Z = \beta(3\beta - 1)/2$

2.2 3D Plot:

For n < 10000, equation (2.8) gives 9999 Pythagorean Triplets (X, Y, Z). Plotting these as ListPointPlot3D using software *Mathematica* by the following command,we get the following graph (Figure 2.1).

```
ListPointPlot3D[Data, ColorFunction → "DarkRainbow"]
```

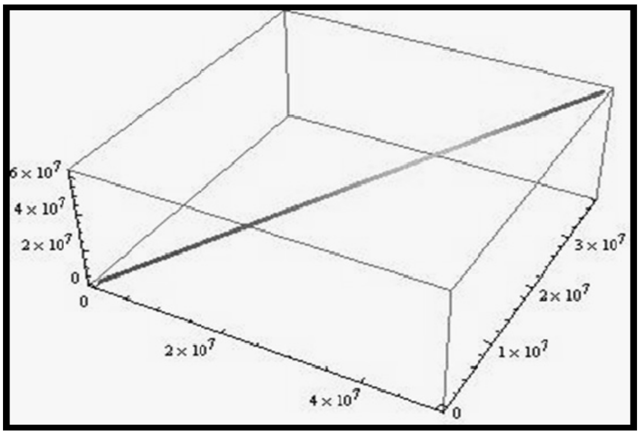

Figure 2.1:ListPlot3D of Special Pythagorean Triplets

4. Observations and conclusion:

We observe that

1. $X + Y + Z = 0 \pmod 2$.
2. $Y + Z - X = 0 \pmod 2$.
3. $(X + Y + Z)(X + Y - Z) = 0 \pmod 8$.
4. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.
5. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.

In conclusion, other patterns of Pythagorean Triangle can be found which satisfy the conditions discussed in the above problem.

References

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[3] Gopalan M.A., Vijyalakshmi P.; *Special pythagorean triangles generated through the integral solutions of the equation* $y^2 = (k^2 + 1)x^2 + 1$; Antarctica J. Math.; 7(5) (2010); 503-507.

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