Exploring Pattern Generalization in the Logo-based Microworld

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Abstract: *This paper aimed to design learning activities for pattern generalization in the Logo-based JavaMAL microworld. We focused on figural pattern activities using polycube pattern, included in the elementary school mathematics textbooks in Korea. We designed web 2.0 based JavaMAL microworld so that students can create and explore pattern objects interactively, and provided students with virtual manipulative and expressive tools to support their thought process for pattern generalization. We analyzed students' algebraic thinking based on their symbolic pattern expressions and responses in the pre-test and post-test. The results suggested that students' pattern reasoning became more structured and sophisticated, and the JavaMAL microworld was very useful to support students' algebraic thinking for pattern generalization in the context of pattern manipulation and construction.*

1. Introduction and Background

As Zazkis & Liljedahl [21] stated, patterns are the heart and soul of mathematics, and many basic principles of school mathematics have emerged as generalization of patterns in numbers and shapes. A number of researchers (Stacey [19], Duval [3], Radford [14], Lannin [5], Samson [17], Rivera [16]) have studied pattern generalization activities for students to express those generalization in algebraic terms. Following constructionism perspectives, we designed a Logo-based microworld environment where students can explore polycube patterns through learning-by-designing, embodied simulations, and executable expression to construct them. In what follows, polycube is a solid figure formed by joining one or more unit cubes face to face, and polycube pattern is a pattern made from a sequence of polycubes that change from one term to the next term in a predictable way. The second grade and sixth grade mathematics textbooks in Korea use polycubes to teach students a sense of 3D shapes and pattern regularity, and the textbooks emphasize mathematical way of expressing and verbalizing 3D polycube shapes. In this paper we studied students' pattern generalization activities under the support of JavaMAL microworld.

1-1. Logo-based Embodied Expressions for Polycube Patterns

Papert [13] developed the idea of constructionism by providing the learners better opportunities to construct and share for effective learning, and he designed Logo microworld and 'turtle geometry' in which 2D figures can be constructed by two commands (rotate and forward). In this environment, learners can design various figural artifacts using a powerful idea grounded in 'playing turtle' metaphor, and learning by making takes place as they solve construction problems. Following this fundamental idea of Logo microworld, Cho et al. [1] proposed a 3D representation system by which learner can design executable expression to make 3D solid following 3D turtle metaphor. Cho et al. also designed JavaMAL microworld that can provide web 2.0 environment where people can make and share 3D polycube solids by connecting the unit cubes.

Figure 1 explains the five basic embodied symbols used in JavaMAL, and also gives an executable expression that can make the polycube shown in Figure 1. In this system, letter 's' is an embodied symbol for a turtle to make a cube as moving one step forward from its position, and 'R' and 'L' are symbols for a turtle to change its direction to right and left respectively. In addition, 'u' and ‘d’ are polycube symbols for a turtle to make a cube as moving up and down as if it took the elevator. The left-side image in Figure 1 is a polycube solid constructed by the executable expression ‘ssRsusLssd’.

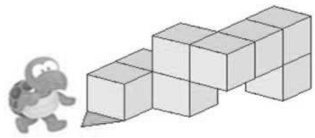
 **ssRsusLssd**	s : one step forward R : turn Right L : turn Left	Horizontal movement
	u : move up d : move down	Vertical Movement

Figure 1 3D representation system and executable expression by Cho et al. [1]

1-2. Web 2.0 based Learning Environment for Polycube Patterns

JavaMAL microworld may provide a web 2.0 learning environment where students can edit symbolic expressions to construct polycubes, and can get immediate feedback to reflect their thinking through possible error analysis. That is, JavaMAL microworld provides a learning environment where learners can construct, analyze, and manipulate their own polycube artifact with embodied symbols and explore it with others. Figure 2 is the JavaMAL microworld environment for polycube pattern activities.

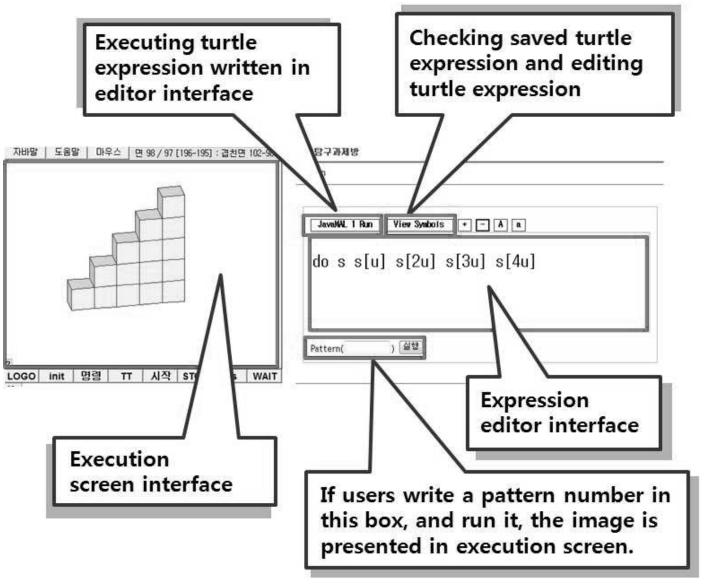


Figure 2. Web 2.0 based JavaMAL microworld environment for polycube pattern

In this microworld environment, the typed expression appears on the editor screen, and executing the expression makes polycube appear on the execution screen. The typed expression can be saved and shared with others. The executable expressions for polycubes made it possible to construct web 2.0 learning environment built on the idea of distributed cognition and collaborate knowledge building. JavaMAL microworld’s expressive power open windows for education Noss et al. [9] since we might get a glimpse of these meanings ourselves from the learner’s executable expressions and learner can refine those expressions through interactive reflection and error analysis. Note that Hwang et al. [4] designed virtual manipulatives and whiteboard system for polycubes, and

Mavrikis et al. [8] and Noss et al. [9] designed eXpresser microworld to support exploratory learning of algebraic generalization.

1-3. Pattern Generalizations in Mathematics Education

Types of patterns are various including number patterns, pictorial/geometric patterns, patterns in computational procedures, linear and quadratic patterns and repeating patterns [21]. In one-dimensional repetitive patterns research, Threlfall [20] suggested using repetitive patterns for symbol introduction, a conceptual stepping stone towards algebra and context providing for generalization. In addition, Stacey [19] researched linear patterns, using patterns in ladders and Christmas trees. In her linear patterns research, Stacey categorized the ways of pattern generalization into the following four types: Counting method, Difference method, Whole-object method, and linear method and analyzed how research participants generalized linear patterns. As a stage of pattern generalization and a principle of creating a pattern task, Stacey presented three linear patterns-related generalization problems. They are warm-up exercise, near generalization and far generalization. Extended from linear patterns, Orton and Orton [11] studied quadratic patterns and Fibonacci's sequence. Based on many previous researches, Lannin [5] later categorized learners' strategies shown in the process of pattern generalization in the various teaching experiments into the following five: Counting, Repetitive, Whole-object, Guess-and-check, and Contextual.

On the other hand, Samson [18] specifically mentioned dot patterns, match patterns, tile patterns and 2D & 3D cube block patterns as figural patterns. He argued sequence expressed with figural patterns potentially enables learners to understand the fundamental structure of patterns in more depth and adds more depth and width in pattern interpretation. Like this, figural patterns have an advantage in that students can find the regularity depending on geometric structure. This advantage may not only enrich formula's meaning but also make connection between algebra and geometry. Furthermore, Orton & Roper [12] commented that figural patterns are clearer and simpler to students than rows of numbers or tables, realistic, creative and more fundamental than symbols. Therefore, this paper focuses specifically on pattern generalization through figural patterns. Especially, we choose cube blocks as a pattern material and design activities to explore pattern generalization with cube blocks as many children have played Lego blocks since they were little and students have experience wood cube blocks when learning functions and regularity in elementary school education. In addition, cube blocks have a merit in that they can be used for various pattern activities from 2D to 3D environment.

The idea that pattern generalization is a basis and fundamental structure of algebraic thinking has been firmly accepted to many researchers. Mason [7] mentioned generality expression is a basis of algebra and also one of the good ways to introduce algebra, and Lee [6] stated everything about mathematics is about pattern generalization. Furthermore, Radford [15] commented pattern generalization is considered as one of the important method to introduce algebra to students and defined pattern generalization as follows:

> Generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S [15].

2. Pattern Generalization Activities in JavaMAL Microworld

In the study introduced below, we focus on the process of exploring pattern generalization through manipulation and construction activities that can be possible with JavaMAL microworld. The two subject groups are the ordinary students in the sixth grade and the gifted students in the eighth grade. The latter have taken lessons where they learned JavaMAL microworld and its related mathematics. After both groups of students performed pattern manipulation and construction activities, we observed their vignettes in the interaction with the environment. Here, manipulation means creating simulations by making changes to the created artifacts, and construction means creating artifacts by using a symbolic executable expression in the JavaMAL microworld.

2-1. Exploring pattern generalization through embodied simulation

The first activity to explore polycube pattern generalization in JavaMAL microworld is an embodied manipulation activity. It is simply done with the keys and mouse-dragging as directed in the left column in Figure 3, and this manipulation activities provides embodied simulations related to re-enactments of our sensory-motor experiences with polycube objects. de Koning et al. [2] claims that applying an embodied perspective to the design of animations will facilitate understanding of dynamic system, and we design JavaMAl microworld so that executable expressions based on body metaphor, manipulating and interacting with pattern objects, and web 2.0 environments are possible.

Basic Manipulation	Description
(a) Mouse right-dragging (Rotation)	This rotates the image of polycube shown in the screen.
(b) Shift key+Mouse left-dragging (Translation)	This moves the image of polycube shown in the screen.
(c) Ctrl key+Mouse right-dragging	This shows gnomon[1] in each term. With mouse dragging, this makes gnomon appear or disappear repetitively.
(d) Shift key+Mouse right-dragging	This dismantles and rotates gnomon or rotates to combine two polycubes.

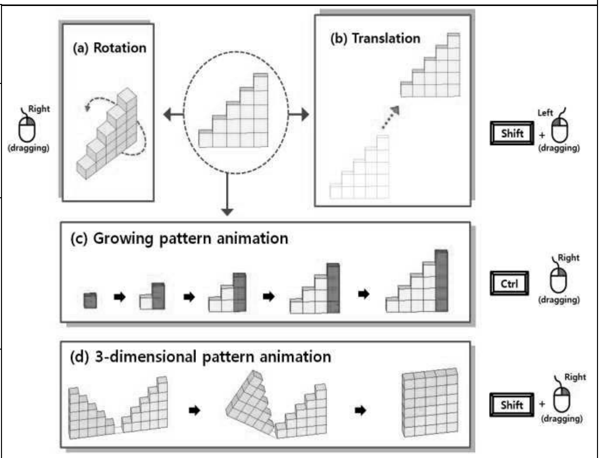


Figure 3. Basic manipulation activities for polycube pattern exploration

Based on the above manipulation tools, we designed the pattern exploration activity into two parts, depending on polycube pattern task problems developed: pattern exploration for near generalization and pattern exploration for far generalization.

2-2. Pattern exploration for near generalization and far generalization

As in Figure 3-(c), learners can do exploration activities as representing the given patterns according to the instruction and observing gnomon in each term with ctrl key and mouse right-dragging. In this exploration for near generalization, we recommend teachers control the speed of mouse-dragging so that students can pay attention to the number of gnomon and the change of figures. This is designed for learners to observe how the number of gnomon increases in each term and have them predict a tendency of pattern and its generalization. This design is related to the

1 This is an important mathematical concept to analyze the difference between the two neighboring terms in figurate number pattern, and gnomon means a piece to be added to make the next figurate number (i.e. as the term number is increased by one).

repetitive strategy among Lannin's generalization strategies [5], which focuses on local difference between neighboring terms in the pattern.

As in Figure 3-(c) and Figure 3-(d), learners can do exploration activities as representing the given pattern according to the instruction and gain a perspective to recognize the images of polycube with shift and mouse right-dragging. In this exploration for far generalization, we limited teacher's role to guide learners how to explore basically, but depending on learners' status, teachers are allowed to provide learners with additional guides so that learners can pay more attention to the number of gnomon with mouse-dragging. This design is related to contextual recognition strategy among Lannin's generalization strategies, which is relevant to a counting skill that finds regularity in patterns. That is, we aim learners to use contextual strategy through this activity.

2-3. Student vignettes through embodied simulation

Observing twenty gifted students in the 8th grade taking lessons at Seoul National University, we developed a pattern task based on the concept of figurate number like triangular, square and tetrahedral number as a mathematical content of the polycube pattern activities in the JavaMAL microworld environment. While the students were performing pattern task, they were asked to conjecture the number of the cubes in the 50th triangular number as in Figure 3-(d). Jaemin answered in the pre-test that the number of the cubes equals the amount of '1+2+3+ ... +50'. He used counting strategy as one of the generalization strategies proposed by Laninn [5].

The post-test, however, shows that Jaemin used contextual strategy in addition to counting strategy. He got the answer by calculating the area of 50+51 rectangular and dividing it by 2. This student not only checked sequential changes of triangular number with mouse-dragging but also combined two polycubes to find the generalized number through embodied simulation.

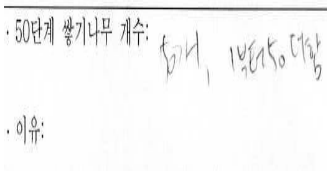
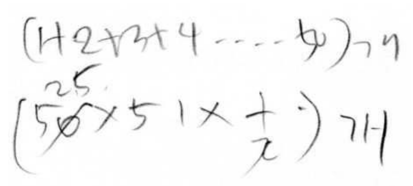
	The number of the cubes in the 50th term : the amount of '1+2+3+...+50'	
Jaemin's answer in the pre-test and its translation into English		Jaemin's answer in the post-test

Figure 4. Comparison of the answers in the pre- and post-test of the 50th triangular number

In addition, Yujin did a pattern activity that observes the changes of gnomon with ctrl key and mouse right-dragging for near generalization pattern exploration in the JavaMAL microworld environment. We saw that she expressed a bit dimly in the pre-test, but articulated clearly with a gnomon expression in the post-test as in Figure 5. She checked gnomon of the given polycube patterns through manipulation and activated expression with gnomon as a mediate. We believe this shows the possibility to raise generalization level up to the contextual generalization level which Radford [15] stated.

In this exploration, it was observed many students including Yujin used the terminology 'term number' or expressed a bit dimly or wrongly before the manipulation activity, but stated their answer correctly and clearly with gnomon in the post-test. Thus, we may interpret this that gnomon played an important role to mediate the level between factual generalization and symbolic generalization. From the perspective of pattern generalization level that Radford [15] proposed, on the other hand, we were able to confirm manipulative function in pattern activities in the JavaMAL microworld environment has possibility to increase the level in various terms like from naive induction to arithmetic generalization and from arithmetic generalization to factual generalization.

· 이유: 단계가 올라가면 모든 방향의 개수가 올라감	Reason: with 'term number' increasing, the number of cubes in all directions increases.	이유: 1단계 늘어날수록 (단계수+단계수-1)개의 그노몬이 늘어난다.	Reason: with one term increasing, (term number + term number-1) number of gnomon increases.
Yujin's answer in the pre-test and its translation into English		Yujin's answer in the post-test and its translation into English	

Figure 5. Answer to the task to describe the regularity in the triangular number polycube pattern

2-4. Exploring pattern generalization though executable expression

This part of the paper aims to provide students with the environment where they can explore and construct polycube patterns and to study how the polycube construction activities with embodied symbols affect students when they generalize and express polycube patterns. This study was conducted to twenty five ordinary students in the sixth grade in Si-Heung City, and tasks were given through 6 times of online lessons. These students are already familiar with basic polycube symbols in JavaMAL microworld because they participated in the creative artifact-making project with polycube for three months. We provided video lectures to facilitate students' understanding and students were also able to ask questions via comments written in web blog. Criteria of evaluation were turtle expression that students constructed and students' description.

Table 1. Lesson plan for pattern generalization activity in JavaMAL microworld

	1st and 2nd lesson	**3rd and 4th lesson**	**5th and 6th lesson**
Lesson contents	Introducing substitution symbols and designing polycube expressions using substitution symbols	Introducing (numerical) variables and designing polycube expressions using variables	Introducing the concept of general expression and finding general expression using 'term number'

In the first and the second lesson, we introduced a concept of substitution symbol to represent repetitive structure and commonality in polycube pattern. A concept of substitution symbol is introduced as a way to express the repetitive structure shown in the images using such a letter symbol as X.

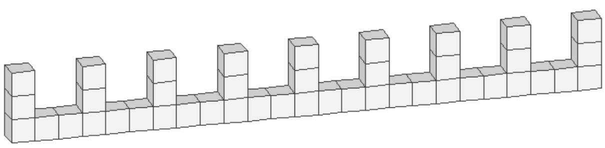


Figure 6. Task given for designing polycube expression with substitution symbols

Figure 6 is a task to design a polycube expression using substitution symbol. Out of 25 students, 21 students substituted a repetitive ‘L’ shape with letter X and express the task as 8X (e.g. XXXXXXXX) and one column (as in (a) and (b) in Figure 7). Three students substituted a column with X and a row with Y and express the shape as XYXYXYXYXYXYXYXYX (as in (c) in Figure 7). Finally, one student seemed to misunderstand the concept of substitution and listed up all of symbols that consisted of the patterns and substituted this with X (as (d) in Figure 7).

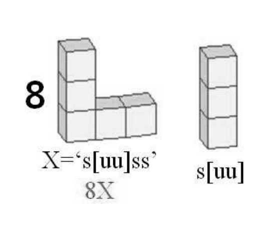
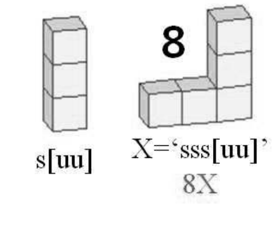
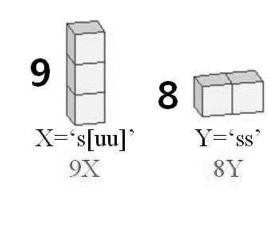
			X='s[2u]2ss[2u]2ss[2u]2ss[2u]2ss[2u]2ss[2u]2ss[2u]2ss[2u]sss2u'
(a) 15 students	(b) 6 students	(c) 3 students	(d) 1 student

Figure 7. Analysis of the students’ expressions for the task in Figure 6

Students didn’t have much difficulty in using substitution symbols and it is analyzed that substituting a new object with a new symbol was not that difficult to the students because they have already used ‘s, R, L, u and d’, corresponding to an object’s or turtle’s movement in JavaMAL microworld.

In the 3rd and the 4th lesson, pattern generalization tasks in Figure 8 were given to find 5th term, 20th term, and 100th term in the sequence of polycubes to design warm-up exercise task, near generalization task, and far generalization task applying Stacey’s pattern generalization stage and pattern task design principle [19]. Certainly the tasks aim to design general expression of the polycube pattern by finding general pattern rule, not to count the number of unit cube one by one. In addition, we introduced a ‘pattern-making box’ so that students can create *n*-th term in the pattern by typing the term number *n* (see Figure 2) and have students manipulate and explore created polycube. In this process, as a way for students to understand the concept of numerical variable, we introduced JavaMAL microworld’s variable ‘(n)’, which may correspond to an algebraic variable *n*. Students constructed not only *(n)* but also *(n+1)*, *(n+2)* and *(n-1)* on their own and challenged themselves to try to recognize the differences between variants and invariants through interaction with microworld. In the 5th lesson, we asked students to make and explain the general expressions that build polycube patterns in JavaMAL microworld. In the 6th lesson, we gave tasks to figure out the number of unit cubes in the *n*-th term in the polycube pattern. The pattern tasks presented in the 5th and the 6th lesson are as follows.

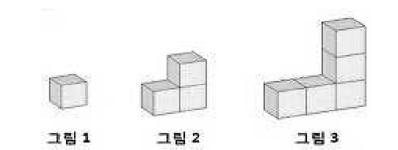
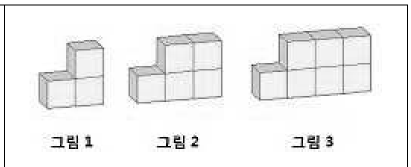
		
Task 1	Task 2	Task 3

Figure 8. Tasks given for designing general expression in the 5th and 6th lesson

2-5. Student vignettes through executable expression

The students who made correct general expressions that construct polycube patterns in the 5th lesson mostly reached the contextual generalization level in terms of Radford's generalization level [15], based on embodied strategy into such activities. Moreover, some students showed possibility that they reached the symbolic generalization level. Let's look at the task submitted by Ahn during the 3rd lesson. As in Figure 9, Ahn constructed the 3rd, 5th, 20th and 100th term in a pattern with the turtle symbols and gave explanations. Students expressed polycubes with repetitive structure simply with substitution symbols in the first and second lesson, and Ahn also successfully completed this task. However, as shown in figure 9, he used long repetitive symbols in the construction process of even 100th term as well as 3rd, 5th, and 20th term.

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<td>Task 1 given in Figure 8</td>
<td>Expression for the 20th term</td>
<td>Expression for the 100th term</td>
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Figure 9. Polycube expressions that Ahn wrotes to construct 20th and 100th term

Like the previous pattern studies, most students showed a tendency to construct their expression effectively by using substitution symbol in the 100th term even if they listed up the repetitive symbols till the 20th term. (Many students used '100X'.) However, Ahn wrote a long line of symbols even to the 100th term. We think repeating the same symbol over and over again was not so much challenging for him because he could easily copy and paste the symbols with the computer. (In fact, Ahn constructed the long symbols in the 100th term this way.) However, in the symbol construction process of creating n-th term, he was not able to use this way, so substituted the repetitive images with the symbol 'X' and expressed it with '(n)X'. In this process, Ahn described that he was able to feel the power of recognizing the commonality and using the symbol '(n)'. In the pattern generalization task, furthermore, we were able to see that he reached the symbolic generalization level finally as generalizing and expressing situations in a symbolic system in that a microworld activity is mediated (Figure 10).   .

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<td>X='s[2d]'
do (n)X</td>
<td>세로로는 3개씩 그리고 가로는 1번째에는
1, 2번째에는 2 이런씩으로 이어 가기
때문입니다. (n)곱하기3입니다.</td>
<td>In vertical direction, there are 3 cubes, and in horizontal direction, 1 cube for the 1st term, 2 cubes for the 2nd term, keeps on in this way. Thus (n) times 3</td>
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<tr>
<td>Ahn's expression to make the n-th term</td>
<td colspan="2">Ahn's explanation for the number of unit cubes in the n-th term and its translation into English</td>
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Figure 10. Ahn's expression and explanation in the pattern generalization task

3. Conclusion and Discussion

This paper examined if JavaMAL microworld could be an appropriate learning environment for polycube pattern generalization exploration. Based on the previous pattern generalization researches, we realized systematic and creative design for pattern activities in the environment and described the participants' response shown in the process of pattern generalization by applying the design to a few experimental groups. This study provided the participants with a final and clear objective, which is creating 'a pattern-making box', and designed manipulation and construction activities to achieve the objective. This objective corresponds to pattern generalization's ultimate goal and naturally induces the concept and proper understanding of symbols in the design. Through this design, in addition, we tried to discuss if this is an appropriate environment for pattern generalization exploration by analyzing and interpreting participants' response based on Lannin's generalization strategy [5] and Radford's generalization level [15]. As a result, we were able to find the appropriateness in many cases. However, we cannot argue that this design is the most optimized environment for understanding of symbols and pattern generalization. Some students constructed the general expression that produces patterns, using '(n)' variable perfectly. However, it was observed that they didn't understand the meaning of variables out of their description in the next task and it did not work as a tool for pattern generalization. Therefore, we would like to provide students with more elaborate design through follow-up studies.

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