Applications of CAS to analyze the step response of a system with parameters

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Abstract

Recently, Computer Algebra System (CAS) such as Maple and Mathematica increases its popularity in the community of education, mathematics, sciences and engineering, because they can treat symbols, which conventional numerical software packages can not. In this paper, we apply CAS to the control engineering where CAS are quite useful, since symbols can represent unknown values such as unknown dynamics, design parameters and modeling error in a natural way.

We focus on a system with linear differential equation $x'(t) = Ax(t)+Bu(t), y = Cx(t)$ where A, B and C are matrices whose entries are polynomials in parameter k. We examine the behavior of the step response of the system, which is expressed by $y(t) = C(e^{At} - E)A^{-1}B$, where e^{At}, E and A^{-1} denote the matrix exponent of matrix At, the identity matrix and the inverse of matrix A, respectively. To analyze the behavior of $y(t)$, we approximate e^{At} with matrix Pade approximation, and compute a rational function approximation of $y(t)$. This enables us to examine various properties of $y(t)$. For example, we can compute approximations of the peaktime and the peakvalue explicitly as a rational function of k, which makes clear the relations between those values and parameter k. We present some analysis and design examples of the system, utilizing these computations.

1 Introduction

Recently, computer algebra system (CAS) such as Maple and *Mathematica* increases its popularity in various research fields. Control engineering is one of such research field where symbolic computation, which is the one of the key features of CAS, is quite useful, because symbols can represent unknown parameters such as unknown dynamics, design parameters and modeling error in a natural way.

For example, [1]-[3] apply Quantifier elimination (QE), which is a new technique of computer algebra, to the design of control systems. References [4]-[5] present methods to compute H_∞ norm of a given parametric system (system with a parameter) as a root of polynomial, using a similar technique to QE. In [6], the same author presents a method to compute H_2 norm of a given parametric system.

In this paper, we focus on a system with

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

where A, B and C are $n \times n$, $n \times 1$, $1 \times n$ matrices whose entries are polynomials in parameter k, $x(t)$ is a vector, $u(t)$ and $y(t)$ are scalars. We define $u(t)$ by

$$u(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0) \end{cases} \tag{2}$$

and examine the behavior of the step response of the system. As is well-known, the solution of the above differential solution (the step response of the system) is given by

$$y(t) = C(e^{At} - E)A^{-1}B, \tag{3}$$

where e^{At}, E and A^{-1} denote the matrix exponent of matrix At, the identity matrix and the inverse of matrix A, respectively.

Since $y(t)$ contains non-trivial function e^{At}, the relation between $y(t)$ and parameters in matrices A, B, C is not simple, especially when matrices A, B, C contain parameters. In this paper, we approximate matrix exponential e^{At} using matrix Pade approximation in [7], which enables us to approximate $y(t)$ by a rational function of t and parameter k. With the approximation of $y(t)$, we can examine various properties such as the peaktime and the peakvalue. We give some illustrative examples of control systems. Although we assume that given system contains only one parameter k, the method in the paper can be easily extended to multi-parameter case.

2 Problem formulation

Fig. 1 shows a typical step response where $y(\infty) = 1$. The maximum value of $y(t)$ is called "peakvalue" and the time when $y(t)$ takes the peakvalue is called "peaktime" (see Fig. 1). We denote the peaktime and peakvalue by t_{peak} and y_{peak}, respectively. The difference between y_{peak} and $y(\infty)$ ($y(t)$ at $t = \infty$), i.e., $|y_{\text{peak}} - y(\infty)|$ is called "overshoot". Similarly, we define undershoot as the difference between $y(0)$ and $\min_t (y(t))$ (see Fig. 1). The minimum time t satisfying $y(t) = y(\infty)$ is called "rise time" and is denoted by t_{rise}. A system is called stable if and only if all of eigenvalues of matrix A in (1) have negative real parts. It is known that if the system is stable, then $y(\infty)$ is finite and $y(t)$ converges to $y(\infty)$ as time t goes to infinity as in Fig 1.

In this paper, we consider a system defined by linear differential equations (1) and (2). We assume that the system contains parameter k and compute the following properties of the system:

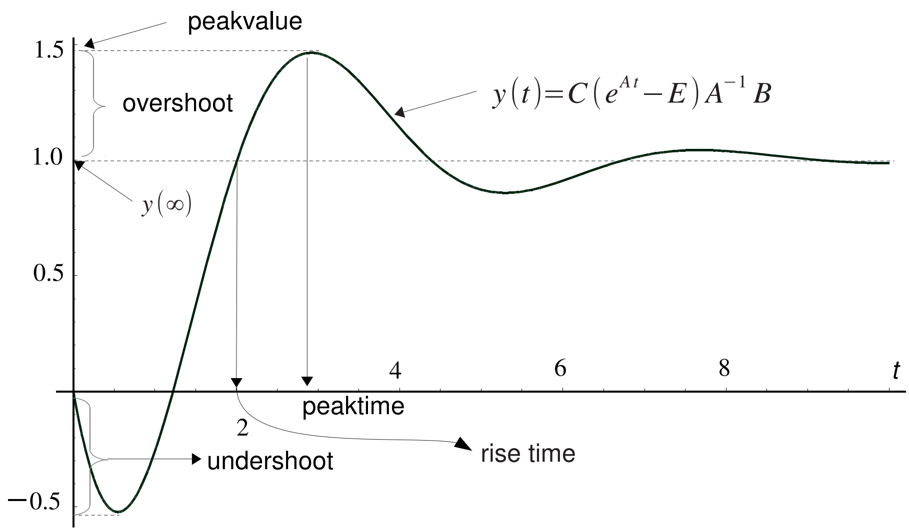

Figure 1: Step response

(P1) The range Ω of parameter k such that the system is stable, i.e., set Ω such that

$$\text{the system is stable} \Leftrightarrow k \in \Omega \tag{4}$$

(P2) A rational function approximation of rise time t_{rise}

(P3) Rational function approximations of the peaktime t_{peak} and the peakvalue y_{peak}

3 Algorithms

3.1 Matrix Pade approximation

The solution $y(t)$ of linear differential equation (1) and (2) is given by (3). Matrix exponential function e^{At} in (3) is formally defined by

$$e^{At} = E + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots , \tag{5}$$

which is difficult to compute, especially when matrix A contains parameter k. We apply matrix Pade approximation in [7] to approximate e^{At} by a matrix whose entries are rational functions of k and t. More concretely, we approximate e^{At} by

$$\left(E + \alpha_1 At + \alpha_2 A^2t^2 + \cdots + \alpha_p A^pt^p\right)^{-1}\left(E + \beta_1 At + \beta_2 A^2t^2 + \cdots + \beta_l A^lt^l\right), \tag{6}$$

where α_i, β_j $(i = 1, \cdots, p,\ j = 1, \cdots, l)$ are real numbers such that Taylor series expansion of $(1 + \beta_1 x + \cdots + \beta_l x^l)/(1 + \alpha_1 x + \cdots + \alpha_p x^p)$ agrees to that of e^x up to $(p + l)$th degree of x, i.e.,

$$e^x \equiv \frac{1 + \beta_1 x + \cdots + \beta_l x^l}{1 + \alpha_1 x + \cdots + \alpha_p x^p} \pmod{x^{p+l+1}}. \tag{7}$$

When matrix A contains parameter k, (6) is a matrix whose entries are rational function of k and t. Hence, with the above approximation of e^{At}, $y(t)$ in (3) can be computed in the form of a rational function of k and t. Thus, we obtain the following algorithm:

Algorithm 1:

Input : Matrices A, B, C in (1) ;
Output : A rational function approximation $q(k,t)/r(k,t)$ of $y(t)$;

Step 1: Determine α_i, β_j $(i = 1, \cdots, p, j = 1, \cdots, l)$ so that (7) satisfied.

Step 2: Compute a rational function approximation of $y(t)$ by

$$y(t) \cong C\left\{(E + \alpha_1 At + \cdots + \alpha_p A^p t^p)^{-1}\left(E + \beta_1 At + \cdots + \beta_l A^l t^l\right) - E\right\} A^{-1}B. \tag{8}$$

For example, let matrices A, B, C in (3) be

$$A = \begin{bmatrix} 0 & k+1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2k+1 & 2k+1 \end{bmatrix}. \tag{9}$$

We will compute a rational function approximation of $y(t)$ with $p = l = 2$. Since, we have

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \equiv \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2} \pmod{x^5}, \tag{10}$$

we see $\alpha = -\frac{1}{2}$, $\alpha_2 = \frac{1}{12}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{12}$ in Step 1 of **Algorithm 1**. Hence, $y(t)$ in (8) is computed by

$$\begin{aligned} y(t) &= C(e^{At} - E)A^{-1}B \\ &\cong C\left\{\left(E - \frac{1}{2}At + \frac{1}{12}A^2t^2\right)^{-1}\left(E + \frac{1}{2}At + \frac{1}{12}A^2t^2\right) - E\right\} A^{-1}B \\ &= \frac{6(2k^2 + 5k + 3)t^3 + 36(k+1)t^2 - 72(k+1)t}{(k+1)^2t^4 + 3(k+1)t^3 + 3(2k+3)t^2 + 18t + 36}. \end{aligned} \tag{11}$$

In [8], matrix exponential e^{At} is approximated by a matrix whose entries are truncated power series. In general, a rational function approximation is more accurate than the one by power series expansion.

3.2 Computations of properties (P1) - (P3)

3.2.1 Property (P1)

We can apply well-known Routh stability criteria to the characteristic polynomial $|sE - A|$ of matrix A to compute the equivalent condition for the system stability.

3.2.2 Property (P2)

With matrix Pade approximation of e^{At} in Sect. 3.1, $y(t)$ in (3) is approximated by a rational function of $q(k,t)/r(k,t)$ (i.e., $y(t) \simeq q(k,t)/r(k,t)$), where $q(k,t)$ and $r(k,t)$ are polynomials in k and t. This and $y(t) = y(\infty)$ imply $q(k,t) \simeq y(\infty)r(k,t)$. Thus, t_{rise} can be approximated by the root of polynomial

$$w(k,t) = q(k,t) - y(\infty)r(k,t) \tag{12}$$

with respect t (the rise time is the minimum t satisfying equation $w(k,t) = 0$). We compute a rational function approximation of a root of $w(k,t)$ by the following algorithm:

Algorithm 2:

Input : Bivariate polynomial $w(k,t)$;
Output : A rational function approximation $f(k)/g(k)$ of a root of $w(k,t) = 0$ with respect to t ;

Step 1: Compute a power series approximation $\alpha_0 + \alpha_1 k + \cdots + \alpha_m k^m \quad (\alpha_i \in \mathbf{R})$ of a root (the computation can be performed by symbolic Newton's method. for details of the computation, refer to [9]-[11].)

Step 2: Apply Pade approximation to the root and compute its rational function approximation $f(k)/g(k)$.

Thus, we obtain the following algorithm to compute a rational function approximation $f(k)/g(k)$ of t_{rise}.

Algorithm 3:

Input : Matrices A, B, C in (1) ;
Output : A rational function approximation of t_{rise} ;

Step 1: Compute a rational function approximation $q(k,t)/r(k,t)$ of $y(t)$ by **Algorithm 1**.

Step 2: Let $w(k,t) = q(k,t) - y(\infty)r(k,t)$ and compute a rational function approximation of a root of $w(k,t) = 0$ by **Algorithm 2**.

3.2.3 Property (P3)

Fig. 1 tells us that t_{peak} is the time t satisfying $\frac{dy}{dt} = 0$, and from (3), we see that $\frac{dy}{dt} = Ce^{At}B$. Since e^{At} can be approximated by (6), $\frac{dy}{dt} = Ce^{At}B$ can be approximated by a rational function $q(k,t)/r(k,t)$ of k and t. This implies that an approximation of t_{peak} is given as a root of $q(k,t) = 0$ with respect to t. Since a root of polynomial $q(k,t)$ can be approximated by a rational function of k with **Algorithm 2**, we obtain the following algorithm to compute a rational function approximation of t_{peak}.

Algorithm 4:

Input : Matrices A, B, C in (1) ;
Output : A rational function approximation of t_{peak} ;

Step 1: Determine α_i, β_j $(i = 1, \cdots, p, j = 1, \cdots, l)$ so that (7) satisfied.

Step 2: Compute an approximation of $\frac{dy}{dt}$ by

$$\frac{dy}{dt} \cong C\left\{(E + \alpha_1 At + \cdots + \alpha_p A^p t^p)^{-1}\left(E + \beta_1 At + \cdots + \beta_l A^l t^l\right)\right\}B. \tag{13}$$

Step 3: Let the numerator of rational function (13) be $q(k,t)$. Then compute a rational function approximation of the root of $q(k,t) = 0$ with respect to t by **Algorithm 2**.

From the definition, y_{peak} is given by $y(t_{\text{peak}})$. Thus, given a rational function approximation $f(k)/g(k)$ of t_{peak}, a rational function approximation of y_{peak} is given by $q(k, f(k)/g(k))/r(k, f(k)/g(k))$ where $q(k,t)/r(k,t)$ is a rational function approximation of $y(t)$.

However, it is impractical to use $q(k, f(k)/g(k))/r(k, f(k)/g(k))$ directly as a rational function approximation. Although the function is a rational function, its denominator and numerator are, in general, of high degree in k. Hence, to be more practical, we need to approximate the function by a rational function of lower degree in k, applying Pade approximation to it.

Algorithm 5:

Input : Matrices A, B, C in (1) ;
Output : A rational function approximation of the peakvalue ;

Step 1: Compute a rational function approximation $f(k)/g(k)$ of y_{peak} by **Algorithm 4**.

Step 2: Compute function $q(k, f(k)/g(k))/r(k, f(k)/g(k))$ and apply Pade approximation to the function to obtain a rational function approximation of low degree in k.

4 Design examples

Let matrices A, B, C in (1) be as in (9). We will design a system, computing properties (P1) - (P3).

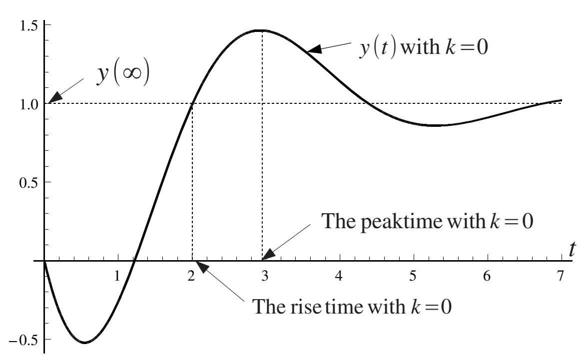

Figure 2: $y(t)$ with $k = 0$

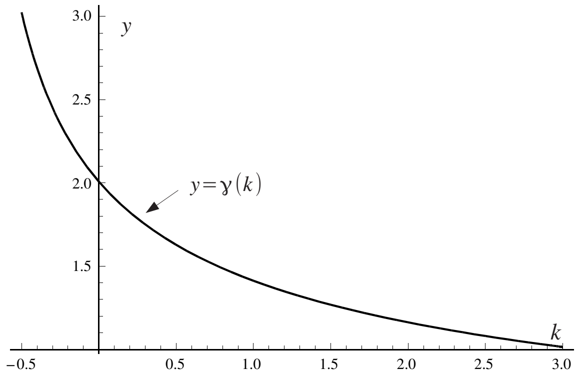

Figure 3: $y = \gamma(k)$

4.1 Property (P1)

The characteristic polynomial of matrix A is given by $|sE - A| = s^2 + s + 2k + 2$. Applying Routh stability criteria to the characteristic polynomial, we find that the system is stable if and only if $k + 1 > 0$. Thus, the region Ω is given by

$$\Omega = \{\ k \in \mathbf{R} \mid k > -1\ \}. \tag{14}$$

4.2 Property (P2)

We compute a rational function approximation of $t_{\rm rise}$ by **Algorithm 3**. First, we compute a rational function approximation of $y(t)$ with $p = l = 5$ by **Algorithm 1**. From Taylor series expansion of e^x, we obtain

$$e^x \equiv \frac{1 + \frac{1}{2}x + \frac{1}{9}x^2 + \frac{1}{72}x^3 + \frac{1}{1008}x^4 + \frac{1}{30240}x^5}{1 - \frac{1}{2}x + \frac{1}{9}x^2 - \frac{1}{72}x^3 + \frac{1}{1008}x^4 - \frac{1}{30240}x^5} \pmod{x^{11}}. \tag{15}$$

Thus, $y(t)$ is approximated as

$$\begin{aligned} y(t) &\simeq \frac{q(k,t)}{r(k,t)} \\ &= C\left\{\left(E + \cdots - \frac{1}{30240}A^5t^5\right)^{-1}\left(E + \cdots + \frac{1}{30240}A^5t^5\right) - E\right\}A^{-1}B \\ &= \frac{2(k+1)^5t^{10} - 60(k+1)^4t^9 - 210(k+1)^3(4k+3)t^8 + \cdots - 571153600t}{(k+1)^5t^{10} + 15(k+1)^4t^9 + 15(k+1)^3(2k+9)t^8 + \cdots + 28576800}. \end{aligned}$$

Since $y(\infty) = 1$, we define polynomial $w(k,t)$ by $w(k,t) = q(k,t) - y(\infty)r(k,t) = q(k,t) - r(k,t)$ and compute a rational function approximation of its root by **Algorithm 2**.

First, we will compute power series expansion of the root of $w(k,t)$ associated to $t_{\rm ise}$. Fig. 2 shows plot of $y(t)$ when $k = 0$. From the figure, we see that $t_{\rm rise}$ is equal to 2.0069 when $k = 0$, which implies that constant term of its power series expansion is 2.0069. Computing the power series expansion whose constant term is 2.0069, we obtain

$$2.0069 - 1.0754k + 0.93963k^2 - 0.924662k^3 + 0.9537k^4 - 1.0069k^5 + \cdots + 1.5129k^{10}, \quad (16)$$

whose Pade approximation $\gamma(k)$ is given by

$$\gamma(k) = \frac{2.0069 + 5.2005k + 4.6847k^2 + 1.7064k^3 + 0.21319k^4 + 0.0041291k^5}{1 + 3.1271k + 3.5418k^2 + 1.7447k^3 + 0.34846k^4 + 0.019395k^5}. \quad (17)$$

Fig. 3 shows plot of $\gamma(k)$, which indicates t_{rise} decreases as parameter k increases.

4.3 Property (P3)

First, we compute a rational function approximation of $\frac{dy}{dt}$ by **Algorithm 4**. Real numbers α_i, β_j $(i = 1, \cdots, 5,\ j = 1, \cdots, 5)$ in Step 1 of **Algorithm 4** are determined from (15). Thus, a rational function approximation of $\frac{dy}{dt}$ in Step 2 is given by

$$\begin{aligned} \frac{dy}{dt} &\cong C\left(E - \frac{1}{2}At + \cdots - \frac{1}{30240}A^5t^5\right)^{-1}\left(E + \frac{1}{2}At + \cdots + \frac{1}{30240}A^5t^5\right)B \\ &= \frac{2(k+1)^6t^{10} + 90(k+1)^5t^9 - 30(k+1)^4(58k+51)t^8 + \cdots - 57153600}{(k+1)^5t^10 + 15(k+1)^4t^9 + 15(k+1)^3(2k+9)t^8 + \cdots + 28576800}. \end{aligned}$$

In Step 3, we let the numerator of the above rational function be $q(k, t)$, and compute a rational function approximation of the root of $q(k, t) = 0$ corresponding to t_{peak} with **Algorithm 2**. Computing the power series approximation of the root of $q(k, t) = 0$ corresponding to t_{peak} (from Fig. 2, we see that constant term of the power series of t_{peak} is 2.9209), we obtain

$$2.9209 - 1.4551k + 1.1943k^2 - 1.1202k^3 + \cdots + 1.5945k^{10} \pmod{x^{11}}, \quad (18)$$

whose Pade approximation $\delta(k)$ is given by

$$\delta(k) = \frac{2.9209 + 7.4317k + 6.5432k^2 + 2.3168k^3 + 0.28039k^4 + 0.0052484k^5}{1 + 3.0425k + 3.3470k^2 + 1.6000k^3 + 0.31000k^4 + 0.016713k^5}. \quad (19)$$

Substituting $\delta(k)$ into t of $\frac{q(k,t)}{r(k,t)}$ in (16), we obtain a rational function approximation $\frac{q(k,\delta(k))}{r(k,\delta(k))}$ of y_{peak}. The denominator and the numerator of $\frac{q(k,\delta(k))}{r(k,\delta(k))}$ are of 55-th degree in k, and we apply Pade approximation to the rational function and obtain a new rational function approximation

$$\eta(k) = \frac{1.4642 + 4.2553k + 4.4800k^2 + 2.0591k^3 + 0.38919k^4 + 0.021239k^5}{1 + 2.5962k + 2.3121k^2 + 0.80967k^3 + 0.090334k^4 + 0.0014513k^5} \quad (20)$$

of lower degree. Thus, we have computed rational function approximations $\delta(k)$ of t_{peak} and $\eta(k)$ of y_{peak}. Fig. 4 shows plot of $\eta(k)$, which indicates y_{peak} increases as parameter k increases.

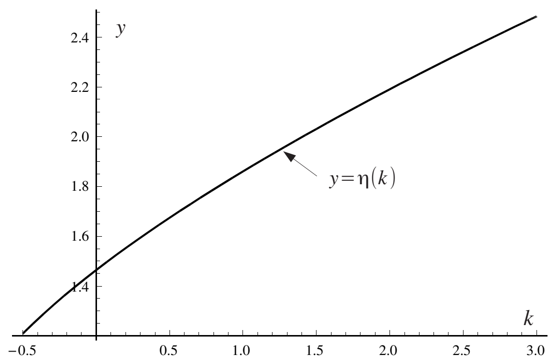

Figure 4: $y = \eta(k)$

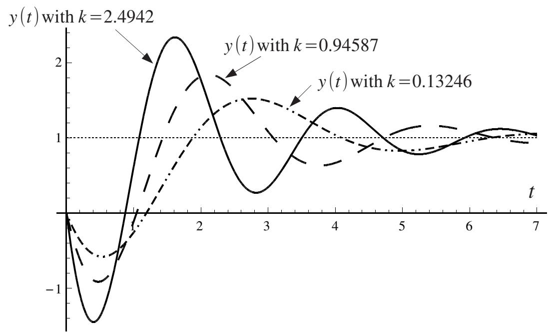

Figure 5: $y(t)$

4.4 Parameter optimization

Using properties (P1) - (P3) we have computed so far, we decide desirable parameter value of k. For quick response of the system, t_{rise} ($\simeq \gamma(k)$) is better to be small. Hence, parameter k should be large (see Fig. 3). On the other hand, for smooth behavior of the system, y_{peak} ($\simeq \eta(k)$) is better to be small, which implies parameter k should be small (see Fig. 4). Therefore, there is a tradeoff between quickness and smoothness of the system. To take the balance of quickness and smoothness, we consider the optimization problem that minimizes the cost function

$$\phi(k) = \gamma(k) + \lambda\eta(k), \tag{21}$$

where λ ($\in \mathbf{R},\ 0 < \lambda$) is a weight to determine the balance between quickness and smoothness of the system. When $\lambda > 1$, we put priority on smoothness, and otherwise, we put priority on quickness. Since $\phi(k)$ is a rational function, minimization of the function $\phi(k)$ can be easily performed by computing real roots of $\phi'(k) = 0$.

For $\lambda = \frac{1}{2}, 1, 2$, function $\phi(k)$ is minimized at $k = 2.4942, 0.94587, 0.13246$, respectively, and the system stable at all those values, because $k = 2.4942, 0.94587, 0.13246 \in \Omega$. Fig. 5 shows plots of $y(t)$ with $k = 2.4942,\ 0.94587,\ 0.13246$.

5 Conclusion

Given a system with linear differential equation $x'(t) = Ax(t) + Bu(t),\ y = Cx(t)$, the solution $y(t) = C(e^{At} - E)A^{-1}B$ contains matrix exponential function e^{At}, and is difficult to deal with, especially when the system contains parameters. This makes properties of the system such as t_{rise} and y_{peak} hard to handle. This paper presented a method to compute the following properties of the system:

(i) The region Ω of parameters where the system is stable.

(ii) A rational function approximation of t_{rise}.

(iii) Rational function approximations of t_{peak} and y_{peak}

Since above properties are expressed in rational functions, design methods such as optimizations can be easily performed. We showed a design example, utilizing $t_{\rm rise}$ and $y_{\rm peak}$.

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