Calculators and the Mathematics Curriculum

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Abstract: *Developers of mathematics curricula make choices regarding the kinds of technology that are to be used by students, which in turn influences the work of both students and teachers to learn and teach mathematics. This paper analyses the potential relationships between calculators and the mathematics curriculum, drawing implications for what can be learned through student access to different levels of calculators. Three different levels of calculators are considered in detail in the paper: scientific calculators, advanced scientific calculators and graphics calculators. Significant consequences of these choices are described and exemplified through a consideration of a number of mathematical topics that are commonly taught in many curricula in Asian countries.*

Introduction

Throughout the Asian region, mathematics curricula differ in terms of the technology that is approved and encouraged for use by students and teachers, as successive ATCM conferences have indicated over the lifetime of the conferences since 1995. The clearest way in which decisions regarding technology are communicated are the requirements and conditions imposed upon external examinations, which have an immediate effect on the everyday practices of mathematics classrooms, for understandable reasons. In this paper, we focus on the range of choices that are made regarding the use of calculators in mathematics curricula, in order to analyse and briefly illustrate their apparent consequences for the resultant experience of students in classrooms.

Curriculum

The term 'curriculum' is sometimes problematic and does not always have a consistent meaning from country to country or even from person to person within a country. Partly to remedy this problem, the term is sometimes used with an adjective, such as 'official' or 'intended' or 'implemented' or 'attained', recognising that what is written in government documents is not congruent with what happens in actual classrooms. In this paper, we mean the term 'curriculum' to be used in its widest sense: describing the totality of experiences associated with school classrooms, recognising that official statements differ from their interpretation by teachers, students or examiners, and recognising that, for individual students, many elements affect what they learn, think, believe and understand about mathematics. These include the resources of the classroom, both the teacher and the physical resources, the influences of external forces such as parents and examination authorities and the inevitable consequences of the ways of learning that are privileged in the classroom. Importantly, the curriculum comprises much more than a written statement of intent by a Ministry of Education or similar government agency with responsibility for education.

Ideally, curricula are carefully designed to accommodate all elements, including possible use of technology. However, as has been noted [4]:

> In the best of all possible worlds, mathematics curricula are designed, taking course objectives, learning experiences, teaching methodologies and assessment into account simultaneously. Unfortunately, such design

is rather rare in practice, and it is often the case that assessment is attended to after the rest of the course has been designed. Sound curriculum design is especially difficult in the presence of rapidly changing technologies, such as those of graphics calculators and other kinds of computers. It is more likely that curricula happen rather than they are designed in such a circumstance. (p.31)

Audiences

As is the case for countries in other parts of the world, mathematics is held in high esteem in Asian countries, so that the nature and scope of the mathematics curriculum is of wide interest. We recognise a range of audiences for this paper, especially those directly interested in school mathematics. As mathematics is a compulsory subject in all countries, this audience comprises not only professionals, but the wider audience of students, their parents and the wider community. The following three professional audiences are, however, of most interest to this paper.

Mathematics teachers in schools comprise the first audience. Teachers are obliged to implement the official curriculum, although they may not have any part in designing it. The extent to which teachers make individual choices regarding the use of calculators in their classrooms is understandably considerably constrained by official curricula and their associated publications. These include textbooks designed for a curriculum (both 'official' and otherwise) and external examination requirements. Teacher decisions are also affected by local constraints, including the costs of acquiring calculators for students to use and local school policies about technology use.

A second audience comprises curriculum developers and curriculum authorities, who are responsible for designing, monitoring and revising the curriculum, adjusting to external changes where necessary. The nature of such bodies varies from country to country, and their work varies considerably from country to country. Ideally, curriculum decisions are made by groups of people representative of relevant communities, such as teachers themselves, government officials with expertise in the area, university experts and broader community groups such as commerce and industry officials who are concerned with the employment of school graduates.

A third audience comprises university mathematics departments and their members, who are professional mathematicians and users of mathematics, and responsible for the mathematics curriculum immediately after secondary school. While some university mathematics staff in some countries have special responsibilities and interests in school mathematics, typically most staff teach the graduates of schools, and hence are interested in the school curriculum immediately before entering the university years.

Background

Over the past three decades, calculators have been introduced in many schools and school curricula around the world, and a considerable amount of research has been conducted, presented and published on the apparent consequences of their use. Calculators have been a significant part of the discussion at all ATCM conferences, and have frequently occupied a particular section of ther annual Proceedings. Attention has focused on graphics calculators and CAS calculators, with surprisingly little attention to scientific calculators, although these are more likely to be available to teachers in many countries. (e.g., see Xiao-Shan Gao *et al* for the situation in China [11]).

An excellent recent summary of these decades of research on calculators [8], involving meta-analyses, observed:

> In general, we found that the body of research consistently shows that the use of calculators in the teaching and learning of mathematics does not contribute to any negative outcomes for skill development or procedural proficiency, but instead enhances the understanding of mathematics concepts and student orientation toward mathematics. … In summary, a wide array of evidence of nearly four decades points to the usefulness of calculators for enhancing student achievement, learning concepts, orientation towards mathematics, and learning behaviors in mathematics.

For many countries today, it is assumed that mathematics curricula will involve students in regular use of technology, which includes calculators and computers. Consistent with the research findings, the overwhelming body of professional opinion in countries with experience in the use of calculators in schools suggest that the advantages of using them outweigh the disadvantages. Typical of recent statements is that by the US-based NCTM [7]:

> Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances student learning. … Students can learn more mathematics more deeply with the appropriate use of technology. … In mathematics-instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics. The existence, versatility and power of technology make it possible and necessary to re-examine what mathematics students should learn as well as how best they can learn it. (pp 24-25)

Similarly, in Australia, a professional conference observed many changes in the school curriculum that were associated with the increased use of graphics calculators. Among other things, the conference noted reports from many teachers that changes in teaching methods were observed, that classrooms acted as communities of inquiry, that challenging curriculum was developed and that there was an increased focus on understanding mathematics. [1]

As for many other aspects of the school curriculum, published research evidence is not the only factor that is taken into account in making decisions about the use of calculators. Indeed, it is not at all clear that research evidence is a major factor in many curriculum decisions in many countries (including Australia), which also seems to be the case for many curricula beyond schools. Rather, the opinions and beliefs of those responsible for making decisions seem to be of more importance than published research. The school mathematics curriculum seems to be no different in this respect from many other endeavours.

Rather than adding to empirical research, this paper aims to contribute to sound decision making about calculator use by providing an analysis and some examples of the effects of making curriculum choices regarding the use of calculators to be used at different levels of mathematics.

A hierarchy of calculators

Over the last decade, the range of calculator capabilities has increased sharply, so it is necessary first to recognise the range of calculators currently available. A number of levels and types of calculators can be recognised, as described in some detail in [5]. These tend to be hierarchical, in the sense that calculators at each level typically include and enhance the capabilities of those below it, at least for the same manufacturer.

Basic: These perform elementary arithmetic and are found in shops and market places throughout Asian countries, where they are used for basic arithmetic and to communicate prices.

Basic educational: These have been designed for school use with younger or middle-years children, and typically follow the rule of order of operations, handle fractions as well as decimals, and use scientific notation for large and small numbers.

Scientific: These allow more complicated computations, evaluate trigonometric, logarithmic, exponential and combinatoric expressions, recursion, generate pseudo-random numbers and handle elementary statistical calculations for univariate and bivariate data. Some are also programmable.

Advanced scientific: These provide for more sophisticated computations of many kinds, including those for complex numbers, numerical differentiation and integration, matrices, vectors, equations, probability distributions, bivariate regression and allow for some tabulation of functions.

Graphics: These offer a wide range of computations for equations, statistics, matrices, numerical calculus and finance, supported by a graphics screen which permits functions, geometric objects, conic sections and statistical graphs to be represented and manipulated in a range of ways.

Computer Algebra System (CAS): In addition to the suite of capabilities offered by graphics calculators, these provide for symbolic manipulations, especially those for elementary algebra and calculus (including factoring, expanding, solving, differentiating and integrating).

There are, of course, differences between calculators from different manufacturers, with different suites of capabilities and different operating systems. A very wide range of alternatives, ranging from low to high cabilities, at roughly corresponding prices, is now available.

Calculator choices

Most Asian countries do not permit the use of CAS calculators, and indeed many do not sanction the use of graphics calculators at this time. So, in this paper we restrict ourselves to three levels of calculators, reflecting the reality that curriculum practices in many Asian countries essentially permits and encourages calculator use in one of these three ways: scientific calculator, advanced scientific calculator or graphics calculator. We omit the case of no calculator use, since readers are no doubt already familiar with this.

The mechanism for specifying various kinds of calculator use in schools and universities can take several forms. The most common of these are explicit statements in the official curriculum documents referring to the use (or otherwise) of particular kinds of calculators, the inclusion of calculator information, like screens and activities, in official curriculum materials such as textbooks, and rules associated with high-stakes external examinations (such as those used for university entrance at the conclusion of secondary school).

In theory, mathematics teachers can use a range of materials in their classrooms (including calculators), but in practice they are very unlikely to use materials that are not officially approved for use by curriculum and examination authorities, for obvious reasons. For this reason, it is quite unusual for teachers in any countries to use calculators in their classroom teaching if the official curriculum and its associated examination do not sanction such use. In other words, the classroom use of technology tends to mirror the official position.

In considering the use of technology for a curriculum, both curriculum developers and examination authorities are generally very aware of the importance of making reasonable demands on schools across a country. If school financial resources are scarce, it would seem logical to consider the use of the least expensive technologies available, since these are likely to be the most affordable and thus have the greatest chance of being available to all students, rather than to only some students. In this regard Ruthven [9] noted many years ago, "In many settings, calculators offer the most realistic prospects of transforming classroom mathematics within the medium term so as to incorporate considered use of computational technology." (p. 464)

One of the reasons for Ruthven's optimism is that, unlike many other forms of technology, calculators in recent years have been essentially designed and improved for educational purposes, so that they are uniquely tailored to the preferences and needs of mathematics teachers. There has been considerable interest in other technologies for school mathematics, including computers, the Internet, smartphones, tablets and other portable devices, but all of these have been developed for essentially different purposes.

Modelling

For many years, mathematics teachers have tried to help students understand that mathematics is 'useful' for many practical purposes, as this seems to be a strong motivator for studying mathematics. Similarly, curricula have emphasised usefulness by explicit reference to mathematical modelling, applications of mathematics and particular applications of specific mathematical topics.

When curricula do not permit calculators to be used at all, however, it is difficult to provide an authentic and practical flavour to the mathematics curriculum. Real-world data is often complex and make computations difficult, so that, if they are done by hand, very few examples can be dealt with. An alternative is to use data that can be easily handled for computation purposes, but these are generally recognised by students as artificial and not realistic, thus reinforcing an impression that mathematics is not really relevant to the real world. Another alternative is to not use data, but to deal only with general relationships, but this is also unlikely to help promote the idea that mathematics is actually useful and will also have the effect of increasing the difficulty of the material for most students to a level beyond their comprehension.

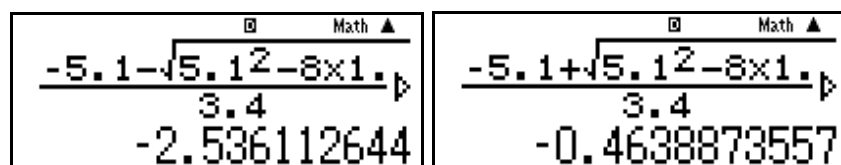## Exploring implications of calculators for curriculum

In this section, we outline some of the ways in which components of the mathematics curriculum may be affected by choices regarding calculator use. Two extended examples are provided, along with some briefer observations about several others.

The example of equations

The mathematical topic of equations is an important element in secondary mathematics curricula in all countries, and so is a good choice to illustrate our thinking. The essence of this example is that the curriculum available to students unavoidably depends on the technology that is assumed to be available. When no calculator is available to students, equation solving is essentially restricted to equations for which an exact solution method is available. This essentially restricts students to linear equations or systems of equations with integer coefficients or quadratic equations, or polynomial equations for which a solution is available through factoring a polynomial expression.

Scientific calculator

A scientific calculator, allows students to solve linear and quadratic equations for which the coefficients are not integers, since the standard algorithms can be readily carried out with the calculator. In particular, students can efficiently deal with some realistic equations for which the coefficients were obtained from data (such as those using a linear or quadratic regression function function). Figure 1 shows an example, involving the solution of the equation $1.7x^2 + 5.1x + 2 = 0$, using the quadratic formula. The capacity of the scientific calculator to handle the use of radicals, fractions, indices and an appropriate rule of order makes such calculators a great deal easier to use than would be the case for a basic caculator or by hand.

Figure 1. Solving a quadratic equation using the quadratic formula

Thus, more realistic examples can be dealt with, both in mathematics and science classrooms.

Advanced scientific calculator

An advanced scientific calculator offers students the additional opportunity to solve linear systems of equations efficiently and to solve cubic equations numerically. The screens below show that solutions of linear systems can be obtained through inputting the matrix of coefficients into the calculator, which is far less time-consuming than using a process of Gaussian elimination, even

with the calculator handling the arithmetic steps. Indeed, access to such a calculator allows for the idea of matrix representation of simultaneous linear equations to be included in the curriculum, and for matrix solutions to be part of the curriculum, extending the concept of solutions of systems of equations as well as providing access to a means for solving them.
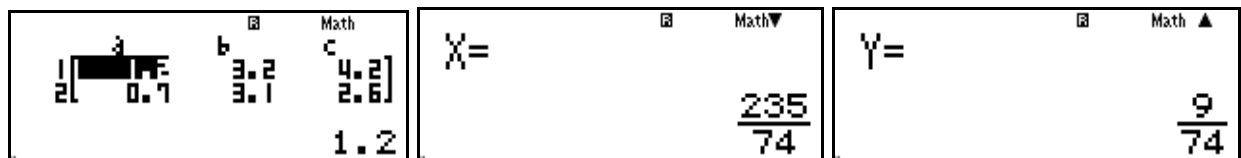

Figure 2. Solving a 2x2 linear system with non-integral coefficients

Similarly, the cubic equation $x^3 - x^2 = 2x - 1$ (for which the associated cubic expression has no rational factors) can be solved after inputting the coefficients. Students need to enter the coefficients of the equation according to the standard format, $ax^3 + bx^2 + cx + d = 0$. The solutions obtained are numerical approximations, as shown in Figure 3.
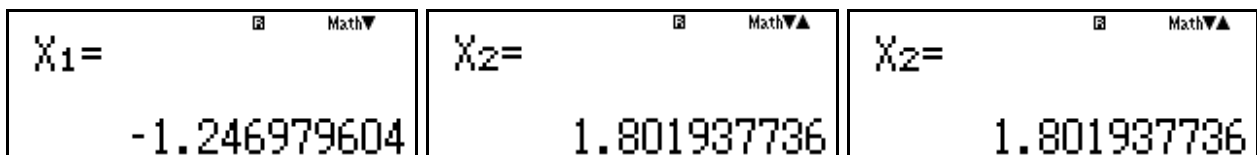

Figure 3. Numerical solutions of the cubic equation $x^3 - x^2 = 2x - 1$

In addition, this particular calculator also includes a numerical solver, shown in Figure 4, which removes the limitation of equation solution to those involving polynomial functions.
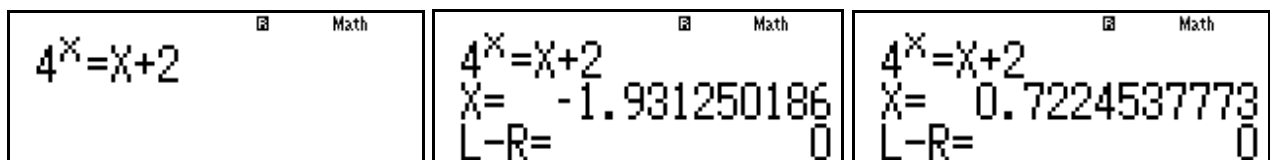

Figure 4. Solving the equation $4^x = x + 2$ approximately

Hence, a decision to permit advanced scientific calculators into the curriculum allows students to solve a wide range of equations and hence address a wide range of realistic practical problems, such as those involving heating or cooling or those involving regression. Not only can equations be solved, but the solutions obtained can be reasonably easily checked by substitution into the original equations, verifying that errors have not been made.

Graphics calculator
A graphics calculator includes all of the functionality of the advanced scientific calculator, as well as the additional capability to represent functions graphically and in tables.
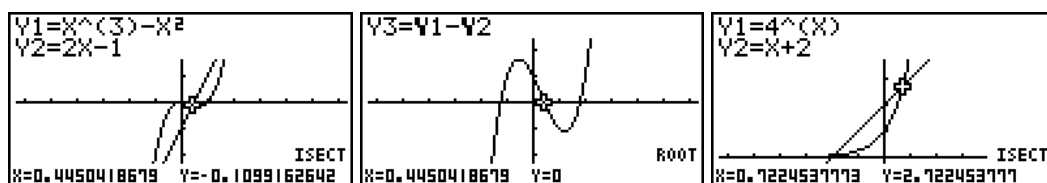

Figure 5. Examining graphical representations of solutions to equations

This allows students to understand the relationships between roots of functions and the solutions of equations by considering the associated graphs. Visualising relationships in this way offers an opportunity to understand the connections between elements of a problem and also to see that the number of solutions obtained is appropriate. To illustrate, Figure 5 shows graphs associated with each of the two equations addressed in Figures 3 and 4.

Graphical representations for equations offer more powerful learning opportunities, however, as well as efficient solutions of equations. In this case, the first screen in Figure 5 shows that the solution of a cubic equation can be regarded as finding the *x*-coordinates of the points of intersection of a pair of graphs, while the second screen shows that the solutions can also be regarded as the roots of an associated cubic function. By appropriate manipulations of the axes, students can see that each representation provides the same set of three solutions, and that further solutions are not possible. In a similar way, the third screen reveals that only two solutions are possible, provided that students understand the nature of linear and exponential functions. Having access to a range of ways of thinking about equations is intrinsically important, so that students are not restricted to a single, orthodox, method of addressing a particular mathematical situation.

This example illustrates the more general point: that a decision to permit the use of graphics calculators in the curriculum does more than merely extend the range of equations that can be solved. It also allows for a deeper consideration of the nature of equation solving, including the connections between equations and graphs of functions, than was possible without the graphs. In addition, the availability of more powerful calculators to students provides both an opportunity and an expectation that the curriculum will deal explicitly with both exact and numerical (i.e., approximate) solutions to equations, which is of considerable practical importance when realistic mathematical modelling is undertaken. [6]

The example of calculus
A second example of the relationships between the curriculum and calculators concerns introductory calculus. Again, as for the topic of equations, this is a common feature of many school mathematics courses and first year university courses across Asia, as well as other countries. If no calculator is used, the study of calculus will necessarily focus on general relationships. Many students in traditional calculus courses learn only the procedural techniques of differentiation and integration, without understanding their meaning or wider significance. [12]

Scientific calculator
Access to a scientific calculator allows students to evaluate numerically a derivative or an integral provided they have already obtained them symbolically, or to consider the nature of a limit by direct evaluation of functions. Figure 6 shows examples of calculations that become available to explore the idea of a limit at infinity, helping students to appreciate the idea of convergence for an important result in an engaging way.
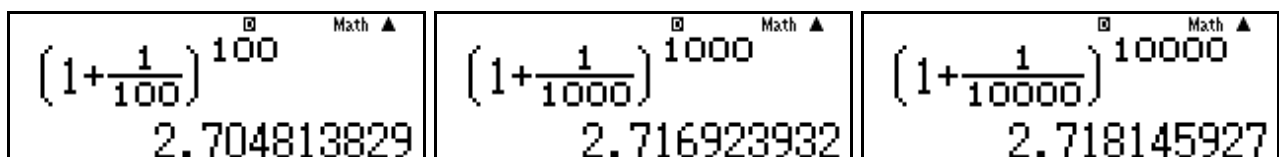

Figure 6. Calculations for exploring $e = \displaystyle\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

Advanced scientific calculator
Advanced scientific calculators have commands available for the numerical evaluation of derivatives and integrals, and so do not rely on students using formal algebraic rules in order to explore a mathematical situation. Figure 7 illustrates these three capabilities.
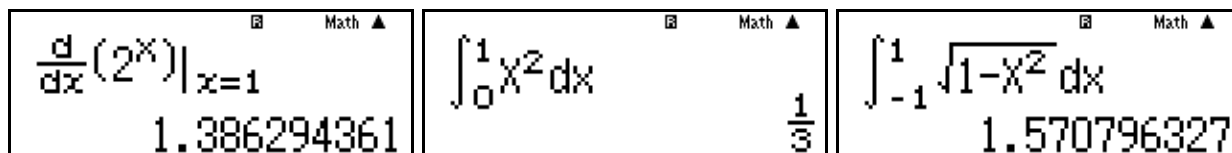

Figure 7. Numerical calculus on an advanced scientific calculator

Students can explore for themselves the nature of differentiation and integration, rather than relying entirely on formal methods of analysis. Thus, before they learn the (complicated) procedures needed for finding a derivative of the exponential function $f(x) = 2^x$, students can evaluate the derivative at various points and get a sense of the shape. They could also evaluate the derivative of points close to a critical point for a polynomial function to understand how the derivative changes. The second screen in Figure 7 shows an equivalent capability for (definite) integrals, while the third screen illustrates how a powerful result (the area of a circle) is related to the calculus. In these cases, the calculator enhances and enriches the more formal learning associated with the calculus.

Graphics calculator
As we have described in detail [4], access to a graphics calculator affords many new learning opportunities to students, and provides powerful ways of enhancing their understanding. Thus, students might use the calculator to engage with the idea of continuity and discontinuity, connect strongly the idea of a derivative with that of the rate of change of function, and distinguish the idea of a derivative at a point from a derivative function.
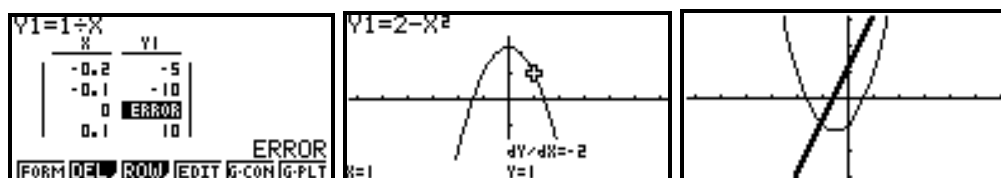

Figure 8. Exploring calculus with a graphics calculator

Figure 8 shows some examples of the possibilities provided by access to a graphics calculator. The first screen is concerned with probing the nature of a jump discontinuity for the reciprocal function, using a table of values. Further probing is accessible with changes to the step size of the table. The second screen shows that the rate of change of a function can be illustrated and explored before attention is focused on ways of finding derivatives symbolically, so that the shape of the curve and the derivative at a point can be strongly connected. The third screen illustrates one way is which the derivative function (graphed in bold) can be compared with a graph of the function, which provides many powerful opportunities to understand the nature of the relationship between the two functions.

With good teaching and use of the technology, the curriculum for students learning calculus with a graphics calculator available to them is more likely to be personally engaging and to focus on the ideas of the calculus than a traditional curriculum, steeped in formal analysis and a premature emphasis on proof for many students.

Other topics
Space does not permit an extended analysis of other topics in the same way as for equations and calculus. Instead Table 1 comprises a brief summary of some calculator capabilities and how they might be used for learning mathematics, further illustrating the main point of this paper.

Table 1. Calculators and some mathematical topics

Topic	Scientific	Advanced scientific	Graphics
Functions	Evaluate any function at any given point and study its properties	Tables of values, so graphs and intersections with other functions can be investigated	Tables and graphs of families of functions Solution of equations related to functions
Logarithmic and exponential functions	Explore logarithms and exponentials together to see connections	Table of values for trends in logarithmic function Different bases	Study of logarithmic functions and relationship to exponential functions Inverses
Trigonometry	Evaluation of trigonometric values to solve practical problems	Tables of values for trends Equation solver Exact values	Graphs, families of functions, transformations, unit circle, identities, trigonometric modelling
Probability and statistics	Mean, variance and standard deviation for univariate & bivariate data Combinations and permutations Random numbers	Choice of regression models allows for more sophisticated modelling Normal distribution Tables of random data available	Graphical analysis informs choice of regression models Data transformations Probability distributions Hypothesis tests Confidence intervals Simulations
Complex numbers	$\sqrt{-1}$ and other complex numbers give error	Square roots of negatives Calculations, rationalisation Argument, conjugate Polar form	Equations Powers and roots
Finance	Simple interest, compound interest (single values)	Table of values (e.g. annual compound interest)	Financial packages, annuities, bonds, spreadsheets

In each of the topics chosen for Table 1, a choice to use calculators of increasing sophistication opens a window for more powerful explorations by students and for more sophisticated understanding of the mathematics involved.

Discussion
Calculators have often been misunderstood as mere devices to provide answers to computational questions, although it is now clear after a great deal of research that their significance for mathematics education extends far beyond that. The evidence elsewhere is promising that calculators can be well used to improve the experience of learning and teaching mathematics, and it should not be too difficult for Asian countries to take advantage of what has been learned and adapt it to local conditions.

Effective educational use of calculators will require more than changes to official statements of curriculum, but also need to include changes to the ways in which mathematics is assessed. The use of calculators (or any other technologies) will not be maximally effective until there is a good measure of coherence between what is taught, how it is taught and learned and how it is evaluated. The need for this kind of coherence is especially important for tertiary entrance examinations.

There is a case for more careful attention to scientific calculators and advanced scientific calculators, if these are the most likely technologies to be available to teachers. A great deal can be achieved to help student learning with technologies less powerful than a graphics calculator.

Progress in these respects is unlikely to happen or be sustained without significant effort to support the work of teachers, who are ultimately responsible for the curriculum of their classroom. Professional development of existing teachers, as well as the development of good materials for teachers to use in their classrooms is a key factor in exploiting the advantages offered by the technology. [5], [10].

Conclusion

The examples chosen for this paper illustrate that the mathematics curriculum can be substantially improved by the integration of calculators; a large body of research has consistently established that such changes can be brought about. As noted in a meta-analysis regarding graphics calculators in [8], "There were no circumstances under which the students taught without calculators performed better than the students with access to calculators." While other technologies may attract more attention in our rapidly changing world, calculators remains a remarkably under-utilised resource in many countries, so that Ruthven's comments [9] nearly 20 years ago, suggesting that they are worth a closer look for education still seem to hold true.

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