Visualization of the Cross Ratio and its Geometric Application

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Abstract: *Cross ratio is a special number associated with an ordered quadruple of points. This number can be visualized in the three-dimensional hyperbolic space as a configuration of two geodesics. Using this visualization, we can show that the angle between two geodesics in the hyperbolic space is a simple function of the cross ratio. Furthermore, we will see that this angle has a relation with the triangle inequality in the Euclidean geometry.*

1. Introduction

Let $\hat{C}$ be the extended complex plane, that is, the complex plane augmented by the point at infinity (denoted by ∞). For four points a, b, c and d in $\hat{C}$, the cross ratio is defined as

$$[a, b; c, d] = \frac{a-c}{c-b} \cdot \frac{b-d}{d-a}.$$

How can we understand this number? This question is the starting point of this study. The upper-half space model of the three-dimensional hyperbolic space $\mathbf{H}^3$ enables us to visualize the cross ratio as a configuration of two geodesics as in Figure 1.1.

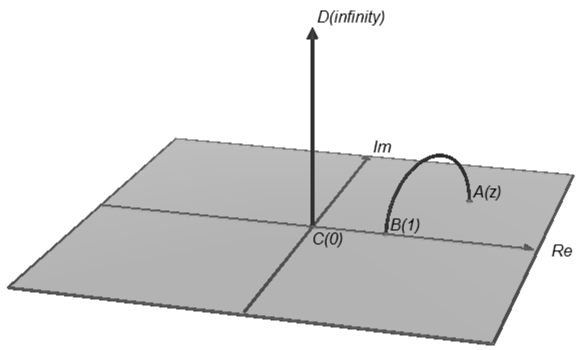

Figure 1.1 Visualization of the Cross Ratio.

We will see this visualization in Section 2. For a given cross ratio, what is the relation between two geodesics in $\mathbf{H}^3$? This question is the second topic of this research. The angle between two geodesics is given by a simple function of the cross ratio. To measure the angle, the common perpendicular of these two geodesics plays an important role. In Section 3, we will review the construction of the common perpendicular. In Section 4, we will derive the formula of the angle between two geodesics as a function of the cross ratio from Ptolemy's theorem. In addition, we will see that the angle of two geodesics in $\mathbf{H}^3$ has a close relation with the triangle inequality in the Euclidean geometry in Section 5. All pictures in this paper are created by dynamic geometry software *Cabri II plus* and *Cabri 3D.*

2. Visualization of the Cross Ratio

2.1. Two-dimensional visualization

Cross ratios are invariant under Mobius transformations. The general form of a Mobius transformation is given by

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

where α, β, γ and δ are complex numbers satisfying $\alpha\delta - \beta\gamma \neq 0$. A Mobius transformation is equivalent to a sequence of simpler transformations. Let:

$f_1(z) = z + \delta/\gamma$ (translation)
$f_2(z) = 1/z$ (inversion and reflection with respect to the real axis)
$f_3(z) = (-(\alpha\delta - \beta\gamma)/\gamma^2) \cdot z$ (dilation and rotation)
$f_4(z) = z + \alpha/\gamma$ (translation),

then these functions can be composed, giving

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Note that circle inversion itself is not a Mobius transformation, for example, the inversion in the unit circle given by

$$i(z) = \frac{1}{\bar{z}}$$

changes the value of the cross ratio:

$$[i(a), i(b); i(c), i(d)] = [1/\bar{a}, 1/\bar{b}; 1/\bar{c}, 1/\bar{d}] = \overline{[a, b; c, d]}.$$

Combining the fact above with the following equation,

$$[z, 1; 0, \infty] = \frac{z-0}{0-1} \cdot \frac{1-\infty}{\infty - z} = z,$$

we can visualize the cross ratio by dynamic geometry software with the following construction as in Figure 2.1.

Construction 2.1 (two-dimensional visualization of the cross ratio)

0. (Input) Four points $A(a), B(b), C(c)$ and $D(d)$ in a complex plane.
1. Circle $C1$ centered at D through C.
2. Point A' (resp. B'), inversion of A (resp. B) with respect to $C1$. ($C' = C, D' = \infty$)
3. Real axis through C as 0 and B' as 1.
4. Imaginary axis, clockwise rotation of the real axis with 90 degrees.
5. (Output) The value of the cross ratio [*a*,*b*;*c*,*d*], the coordinates of A' under the new axes.

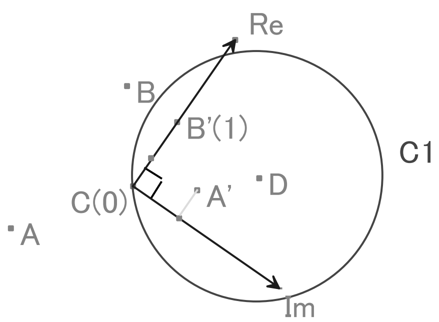

Figure 2.1 Two-dimensional Visualization of the Cross Ratio.

2.2. Three-dimensional visualization

We can apply the idea of two-dimensional visualization of the cross ratio to the upper-half space model of the three-dimensional hyperbolic space $\mathbf{H}^3$. In this model, geodesics are of two types: semi-circles and Euclidean rays perpendicular to the extended complex plane $\hat{C}$ which is the set of points at infinity ([1, page 343]). A Mobius transformation corresponds to an orientation-preserving isometry in $\mathbf{H}^3$. With this model, we can visualize the cross ratio in the similar way as in Figure 2.2.

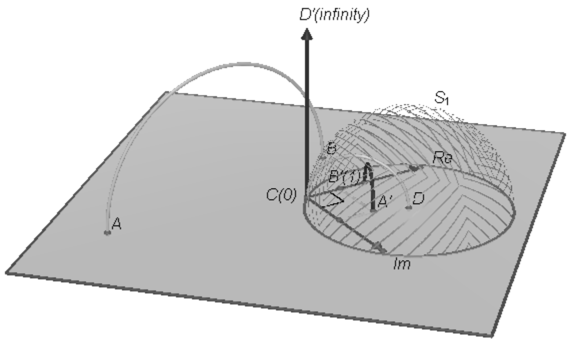

Figure 2.2 Three-dimensional Visualization of the Cross Ratio.

Construction 2.2 (three-dimensional visualization of the cross ratio)

0. (Input) Four points $A(a), B(b), C(c)$ and $D(d)$ in the extended complex plane $\hat{C}$.
1. Sphere $S1$ centered at D through C.
2. Point A' (resp. B'), inversion of A (resp. B) with respect to $S1$. ($C' = C, D' = \infty$)
3. Real axis through C as 0 and B' as 1.
4. Imaginary axis, clockwise rotation of the real axis with 90 degrees in the plane.
5. (Output) The value of the cross ratio [a,b;c,d], the coordinates of A' under the new axes.

Figure 2.2 suggests us that we can easily measure the cross ratio in the case that one of geodesics is a Euclidean ray. In addition, Figure 2.2 gives us more information than Figure 2.1, since A and B are connected by an arc and C and D' are connected by a line which indicates D' is a point at infinity.

Now let us go back to Figure 1.1. There are two geodesics AB and CD in the upper half space model in $\mathbf{H}^3$. D is the point at infinity, so CD is a Euclidean ray. For this configuration of four points A, B, C and D, we can give a complex number as follows:

1. Set a complex plane (real and imaginary axes) such that C is 0 and B is 1.
2. Measure the coordinate of A in this complex plane.

This value $A(z)$ is the very cross ratio for four points $A(z), B(1), C(0)$ and $D(\infty)$. In this way, we can identify a configuration of a semicircle AB and a ray CD as in Figure 1.1 with the value of the cross ratio $[A, B, C, D]$.

Under this identification, if z is a negative number, two geodesics AB and CD have an intersection, especially, if $z = -1$, the angle between AB and CD is equal to 90 degrees as in Figure 2.3. In the following sections, let us focus on the angle between two geodesics AB and CD in general.

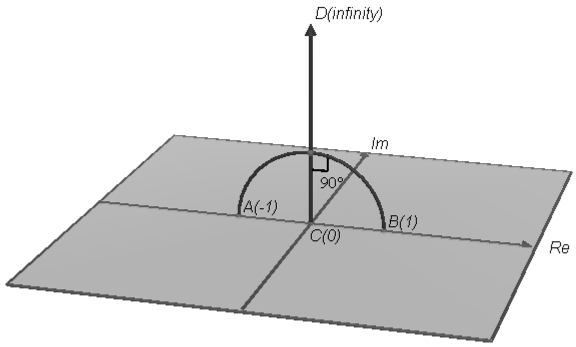

Figure 2.3 The Angle between Two Geodesics is a Right Angle in the Case $z = -1$.

3. The Common Perpendicular in the Hyperbolic Space

The common perpendicular plays an important role to measure the angle between two geodesics in twisted position. In Euclidean geometry as in Figure 3.1, the angle between two geodesics AB and CD is the dihedral angle between two planes ABH_2 and CDH_1, where segment H_1H_2 is the common perpendicular of AB and CD.

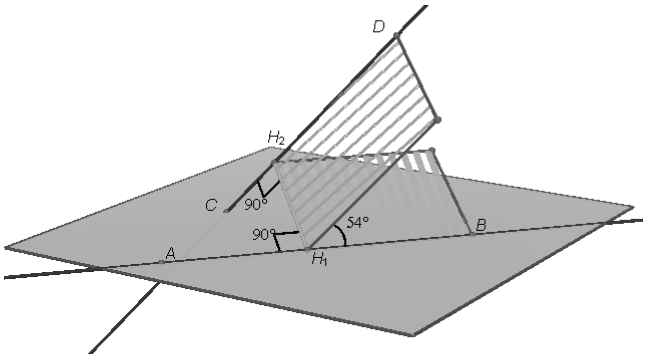

Figure 3.1 The Angle between Two Geodesics in the Euclidean Space.

In the same way, to measure the angle between two geodesics in $\mathbf{H}^3$, we have to construct the common perpendicular. Now, let us review the construction of the common perpendicular in $\mathbf{H}^3$ as in Figure 3.2 (see [2]).

Construction 3.1 (common perpendicular in the three-dimensional hyperbolic space)

0. (Input) Two geodesics, semi-circle AB and ray CD perpendicular to the base plane.
1. Bisector plane $P1$ of $\angle ACB$.
2. Point H_1, intersection of plane $P1$ and semi-circle AB.
3. (Output) Semi-circle on $P1$ centered at C through H_1.

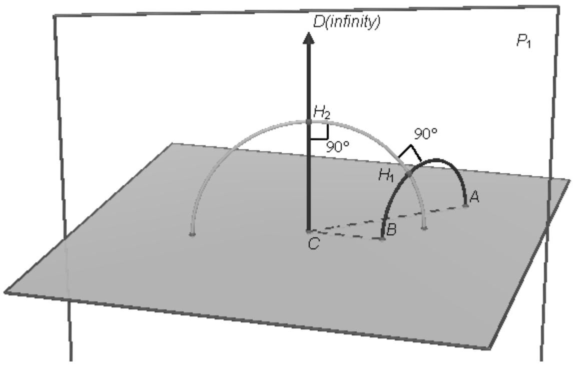

Figure 3.2 The Construction of the Common Perpendicular.

4. The Angle between Two Geodesics

Now we are ready to measure the angle between two geodesics in $\mathbf{H}^3$. In fact, the angle is given as an angle between a line and a circle on the base plane. To do this angle, let us prepare two geodesic planes: one is a Euclidean plane $P1$ including geodesic CD and common perpendicular H_1H_2, and another one is a Euclidean hemisphere including geodesic AB and common perpendicular H_1H_2. Figure 4.1 shows the construction of this hemisphere.

Construction 4.1 (geodesic plane including geodesic AB and common perpendicular H_1H_2)

0. (Input) Two geodesics, semi-circle AB and ray CD perpendicular to the base plane.
1. Circle $C1$ through A, B and C.
2. Point P, intersection of arc ACB and bisector plane between A and B.
3. (Output) Hemisphere $S1$ centered at P through A.

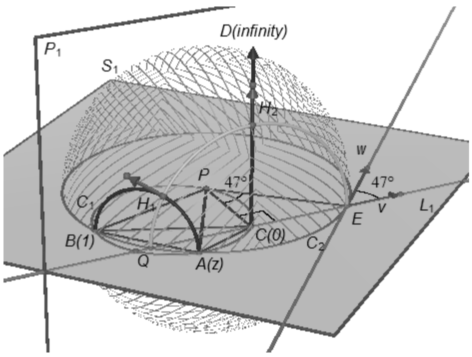

Figure 4.1 The Construction of the Geodesic Plane.

In Construction 4.1, it is trivial that semi-circle AB is included in hemisphere $S1$, since P is on the bisector plane between A and B. However, it is not trivial that common perpendicular H_1H_2 is included in $S1$. Let Q be another intersection point between circle ABC and bisector plane between A and B. Then, note that segment PQ is a diameter of circle $C1$, therefore $\angle PCQ = 90°$. Also, $AQ = BQ$ implies that $\angle ACQ = \angle BCQ$, that is, Q is on angle bisector plane $P1$ in Construction 3.1. Therefore, segment PC is perpendicular to plane $P1$, and the intersection between plane $P1$ and hemisphere $S1$ is a circle centered at C through H_1, that is, the common perpendicular H_1H_2.

In this way, the angle between geodesics AB and CD is the angle between plane $P1$ and hemisphere $S1$, and this angle is measured as the angle between two vectors $\vec{v}$ and $\vec{w}$ at an intersection of $P1$ and $S1$ as in Figure 4.1. Therefore, we can measure the angle on the base plane. The following theorem shows that the angle is a simple function of the cross ratio.

Theorem 4.1 For four points A, B, C and D on the base plane $\hat{C}$ in $\mathbf{H}^3$, let z be the cross ratio $[A, B, C, D]$. Then, the angle θ between geodesics AB and CD is given as

$$\cos\theta = \frac{1-|z|}{|1-z|}.$$

Proof. With a certain Mobius transformation, we can map four points A, B, C and D to z, 1,0 and ∞ as in Figure 4.1. The angle θ subtended by vectors $\vec{v}$ and $\vec{w}$ is equal to $\angle EPC$, hence,

$$\cos\theta = \frac{PC}{PE}.$$

On the other hand, by Ptolemy's theorem,

$$PC \cdot AB + PB \cdot AC = PA \cdot BC.$$

$PA = PB = PE$ implies that

$$PC \cdot AB + PE \cdot AC = PE \cdot BC,$$
$$PC \cdot AB = PE \cdot (BC - AC).$$

Therefore,

$$\cos\theta = \frac{PC}{PE} = \frac{BC - AC}{AB} = \frac{1-|z|}{|1-z|}. \tag{4.1}$$

■

In this way, we see that the angle between two geodesics is a function of the cross ratio. If $A = B$, i.e., $z = 1$, the angle is indefinite. In the case that $A \neq B$, the angle is 90° if and only if $|z| = 1$ as in Figure 4.2 and also in Figure 2.3.

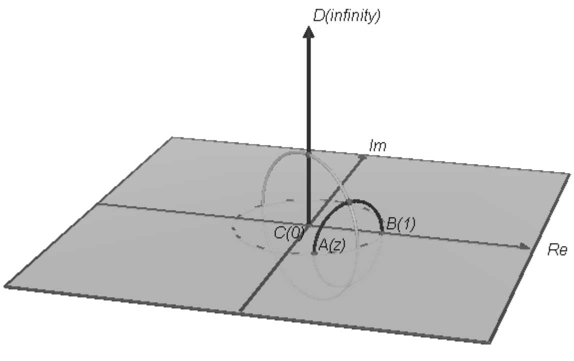

Figure 4.2 The Angle between Two Geodesics is a Right Angle in the Case $|z| = 1$.

5. Euclidean Geometry Revisited

Equation (4.1) indicates a relation with the triangle inequality in the Euclidean geometry. For a planar triangle ABC, the well-known triangle inequalities are

$$AC < AB + BC,$$
$$BC < AB + AC.$$

These inequalities imply that

$$\left|\frac{BC - AC}{AB}\right| < 1.$$

The left term is the value of cosine of the angle between two geodesics AB and CD in Theorem 4.1. In this sense, we can say that hyperbolic geometry is deeply connected with Euclidean geometry.

References

[1] Berger, M. (1987). *Geometry II.* Berlin Heidelberg, Germany: Springer-Verlag.

[2] Maeda, Y. (2010). *Construction of common perpendicular in hyperbolic space.* Proceedings of the Fifteenth Asian Technology Conference in Mathematics, pp. 210-218.