Determination Stability Ray of a Decision Making Unit in Definition of the Right and Left Returns to Scale

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Abstract: *This paper calculates stability ray of a decision making unit* (DMU_o)*, which the right and left its returns to scale* (RTS)*, is given, such that its RTS characteristic, in the RTS classification remains.*
With regarding to definable hyperplanes of the right and left RTS and classification of DMU, models for measuring stability ray, are proposed.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for evaluating decision making units (DMUs) with common input and output terms that its first introduction by Charnes, Cooper and Rhodes (1978) provided, as the CCR model. One of great importance provided investigations in DEA, is the concept of returns to scale. some methods are obtained for determination this concept in DEA (Banker et al., 2004; Zarepisheh et al., 2006). With determination the right and left RTS, behavior of a DMU_o, in direction of increasing and decreasing of inputs, is determined. Hadjicostas and Soteriou (2006), proposed a method for determination of these two notions in RTS, that needs solving two linear programming models.
Zarepisheh and Soleimani-damaneh (2009), a new method based on the dual-simplex algorithm provided to do this.
In this paper we provide models for determination alteration level of inputs and outputs of a DMU_o, that the right and left its RTS is given, such that its RTS characteristic, in the RTS classification remains.

2. Preliminaries and basic definitions

Consider n $DMUs$ where each DMU_j $(j \in J = \{1,2,...n\})$ produced s outputs $y_{rj}(r = 1,...s)$, using m inputs $x_{ij}(i = 1,...m)$. Define $x_j = (x_{1j}, x_{2j},..., x_{mj})^T$ and $y_j = (y_{1j}, y_{2j},..., y_{sj})^T$ as input and output vectors of DMU_j, respectively. Also $X = [x_1, x_2,..., x_n]$ and $Y = [y_1, y_2,..., y_n]$ are $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively. The production possibility set T is represented as:
$T = \{(x, y) \in R_+^{m+s} \mid y \text{ can beproduced from } x\}$.
Banker et al (1984) have deduced the following production possibility set, considering some postulates. This set is denoted by T_v, which signifies the prevalence returns to scale
$T_v = \{(x.y) \in R_+^{m+s} \mid \exists \lambda \in R^n : X\lambda \le x, Y\lambda \ge y, e\lambda = 1, \lambda \ge 0\}$

Where e and 0 are two vectors with all component equal to one and zero, respectively.
The efficiency of a specific DMU_o $(o \in \{1,...,n\})$ can be evaluated by the BCC (Banker, Charnes and Cooper) model of DEA, which has been introduced by banker et al. (1984) under the T_v set, and its output-oriented envelopment form is as follows:

$$\max\{\alpha | X\lambda \le x_o, Y\lambda \ge y_o, e\lambda = 1, \lambda \ge 0\}. \tag{2.1}$$

The dual of this model, called multiplier form, is

$$\min\{vx_o + u_o | vX - uY + u_o \ge 0, uy_o = 1, u \ge 0, v \ge 0\}. \tag{2.2}$$

Definition 1. DMU_o is called an output-oriented technically efficient unit if $\alpha^* = 1$ (where '*' denotes optimality).

Definition 2. if (v^*, u^*, u_o^*) is a optimal solution of model (2.2), then $v^*X - u^*Y + u_o^* \ge 0$ is the supporting hyper plane of production possibility set.

Definition 3. if $v^*X_o - u^*Y_o + u_o^* = 0$, then DMU_o is a efficient decision making unit, and hyper plane of $v^*X - u^*Y + u_o^* \ge 0$ is a passing hyper plan of (x_o, y_o).

Definition4.if $v^*X_o - u^*Y_o + u_o^* = 0$ and $v_i^* \ne 0 (i = 1,...,m), u_r^* \ne 0 (r = 1,...,s)$,then $v^*X - u^*Y + u_o^* \ge 0$ is strong efficient hyper plane.

3. Determination stability ray in definition of the right and left returns to scale

Assume that DMU_o is an output-oriented technically efficient unit, and the right its RTS (ρ_o^+) and the left its RTS (ρ_o^-) is determined (see Hadjicostas and Soteriou, 2006; Zarepisheh and Soleimani-damaneh, 2009).
Note: With regard to, RTS is a property of the frontier at a specific point, not a property of the DMU that sites at that point, therefore right RTS of DMU_o, is passing hyper plane characteristic of DMU_o, that in direction increasing of alteration level, is defined. And left RTS of DMU_o, is passing hyper plane characteristic of DMU_o, that in direction decreasing of alteration level, is defined.
For determination alteration level of inputs and outputs of the DMU_o, we classify two different classes.
First class: Assume the right RTS of DMU_o equal to the left RTS of DMU_o, means $\rho_o^+ = \rho_o^-$.
So the following model calculates the increasing level in the input and output vectors of DMU_o

$$
\begin{aligned}
&\max \quad \theta_o \\
&\max \quad \alpha_o \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} + \theta_{io} \ \ i = 1,\ldots,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} + \alpha_{ro} \ \ r = 1,\ldots,s \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \sum_{i=1}^{m} v_i (x_{io} + \theta_{io}) - \sum_{r=1}^{s} u_r (y_{ro} + \alpha_{ro}) + u_o = 0 \\
&\qquad \sum_{i=1}^{m} v_i (x_{ij}) - \sum_{r=1}^{s} u_r (y_{rj}) + u_o \ge 0 \ \ j = 1,\ldots,n \\
&\qquad \sum_{i=1}^{m} v_i (x_{io}) + u_o = 1 \\
&\qquad \sum_{r=1}^{s} u_r (y_{ro}) = 1 \\
&\qquad \lambda_j \ge 0,\ v_i \ge 0,\ \theta_{io} \ge 0,\ u_r \ge 0,\ \alpha_{ro} \ge 0,\ \forall j,i,r \\
&\qquad u_o \ \ \textit{free in sign}
\end{aligned}
\tag{3.1}
$$

Where $\theta_o = (\theta_{1o},\ldots,\theta_{mo})$ and $\alpha_o = (\alpha_{1o},\ldots,\alpha_{so})$.
If (θ_o^*, α_o^*) is the optimal solution of model (3.1), then (θ_o^*, α_o^*) is stability ray measure of DMU_o, such that on the definable hyper plan of the right RTS remains.
If model (3.1) has been infinite optimal solution, then DMU_o is on the weak hyper plane in input-oriented.
And the following model calculates the decreasing level in the input and output vectors of DMU_o

$$
\begin{aligned}
&\max \quad \theta_o \\
&\max \quad \alpha_o \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} - \theta_{io} \ \ i = 1,\ldots,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} - \alpha_{ro} \ \ r = 1,\ldots,s \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \sum_{i=1}^{m} v_i (x_{io} - \theta_{io}) - \sum_{r=1}^{s} u_r (y_{ro} - \alpha_{ro}) + u_o = 0 \\
&\qquad \sum_{i=1}^{m} v_i (x_{ij}) - \sum_{r=1}^{s} u_r (y_{rj}) + u_o \ge 0 \ \ j = 1,\ldots,n \\
&\qquad \sum_{i=1}^{m} v_i (x_{io}) + u_o = 1 \\
&\qquad \sum_{r=1}^{s} u_r (y_{ro}) = 1 \\
&\qquad \lambda_j \ge 0,\ v_i \ge 0,\ \theta_{io} \ge 0,\ u_r \ge 0,\ \alpha_{ro} \ge 0,\ \forall j,i,r \\
&\qquad u_o \ \ \textit{free in sign}
\end{aligned}
\tag{3.2}
$$

If $\left(\theta_o^*, \alpha_o^*\right)$ is the optimal solution of model (3.2), then $\left(\theta_o^*, \alpha_o^*\right)$ is stability ray measure of DMU_o, such that on the definable hyper plane of the left RTS remains.
Model (3.2) always has finite optimal solution.

We replace objective function in models of (3.1) and (3.2) with $\sum_{i=1}^{m} \rho_i \theta_{io} + \sum_{r=1}^{s} \mu_r \alpha_{ro}$, by assuming $\rho_i > 0$ and $\mu_r > 0$ as weight of ith input for $i = 1,...m$ and weight of rth output for $r = 1,...s$, respectively. Therefore a single-objective non-linear programming problem, is obtained that by mathematical softwaires (Lingo, Gams), is solved.

Second class: All the DMU_o that are not in the first class. Then passing hyper plane of these is not unique. For to find definable hyper plane of the right RTS and definable hyper plane of the left RTS two models is defined as follows:

$$\begin{aligned} \max \quad & u_o \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{ro} + u_o \geq 0 \quad j = 1,...,n \\ & \sum_{i=1}^{m} v_i x_{io} + u_o = 1 \\ & \sum_{r=1}^{s} u_r y_{ro} = 1 \\ & v_i \geq 0,, u_r \geq 0, \forall i, r \end{aligned} \tag{3.3}$$

If $\left(v^*, u^*, u_o^*\right)$ is a optimal solution of model (3.3), then $v^* X - u^* Y + u_o^* \geq 0$ is the definable hyper-plane of the right RTS of DMU_o.

$$\begin{aligned} \min \quad & u_o \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{ro} + u_o \geq 0 \quad j = 1,...,n \\ & \sum_{i=1}^{m} v_i x_{io} + u_o = 1 \\ & \sum_{r=1}^{s} u_r y_{ro} = 1 \\ & v_i \geq 0, u_r \geq 0, \forall i, r \end{aligned} \tag{3.4}$$

If $\left(v^*, u^*, u_o^*\right)$ is a optimal solution of model (3.4), then $v^* X - u^* Y + u_o^* \geq 0$ is the definable hyper-plane of the left RTS of DMU_o.
Therefore, with to find definable hyperplanes of DMU_o, following models for determination stability ray of DMU_o, is proposed:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \rho_i \theta_{io} + \sum_{r=1}^{s} \mu_r \alpha_{ro} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} + \theta_{io} \ \ i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} + \alpha_{ro} \ \ r = 1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{i=1}^{m} v_i^{*}\left(x_{io} + \theta_{io}\right) - \sum_{r=1}^{s} u_r^{*}\left(y_{ro} + \alpha_{ro}\right) + u_o^{*} = 0 \\
& \lambda_j \ge 0, \theta_{io} \ge 0, \alpha_{ro} \ge 0, \forall j,i,r
\end{aligned}
\tag{3.5}
$$

Where $\left(v_i^*, i = 1,\dots,m,\ u_r^*, r = 1,\dots,s,\ u_o^*\right)$ is the optimal solution of model (3.3). And $\left(\theta_o^*, \alpha_o^*\right)$ is stability ray measure of DMU_o, such that on the definable hyper plan of the right RTS remains.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \rho_i \theta_{io} + \sum_{r=1}^{s} \mu_r \alpha_{ro} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} - \theta_{io} \ \ i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} - \alpha_{ro} \ \ r = 1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \sum_{i=1}^{m} v_i^{*}\left(x_{io} - \theta_{io}\right) - \sum_{r=1}^{s} u_r^{*}\left(y_{ro} - \alpha_{ro}\right) + u_o^{*} = 0 \\
& \lambda_j \ge 0, \theta_{io} \ge 0, \alpha_{ro} \ge 0, \forall j,i,r
\end{aligned}
\tag{3.6}
$$

Where $\left(v_i^*, i = 1,\dots,m,\ u_r^*, r = 1,\dots,s,\ u_o^*\right)$ is the optimal solution of model (3.4). And $\left(\theta_o^*, \alpha_o^*\right)$ is stability ray measure of DMU_o, such that on the definable hyperplan of the left RTS remains.

4. A Numerical Example

In this section, to establish the validity of proposed method, we apply it to the numerical example which was provided by Hadjicostas and Soteriou (2006), and also is solved by Zarepisheh and Soleimani-damaneh (2009).

Suppose we have one input and one output (i.e., assume $m = s = 1$) and we observe $DMUs\ \ B - F$, that are output-oriented technically efficient ; see Fig. 1. Table 1gives the right and left RTS for $DMUs\ \ B - F$.

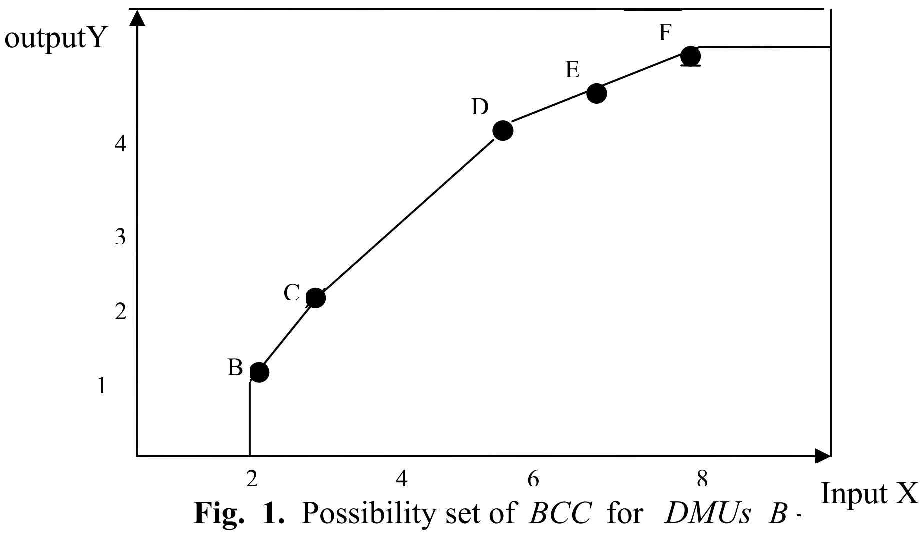

Fig. 1. Possibility set of BCC for $DMUs\ B$ -

Table 1
The right and left RTS of DMUs in fig. 1

	$B=(2,1)$	$C=(3,2)$	$D=(6,4)$	$E=\left(7,4\frac{1}{4}\right)$	$F=\left(8,4\frac{1}{2}\right)$
ρ_o^+,ρ_o^-	$2,\infty$	$1,\frac{3}{2}$	$\frac{3}{8},1$	$\frac{7}{17},\frac{7}{17}$	$0,\frac{4}{9}$

DMU_E is the under assessment, that for this unit $\rho_o^+=\rho_o^-$.
Therefore situates in the first class. Model (3.1) corresponding DMU_E, for the determination stability ray in the direction of the right RTS , is as follows:

$$\begin{aligned}
\max\quad & \theta+\alpha \\
s.t.\quad & 2\lambda_1+3\lambda_2+6\lambda_3+7\lambda_4+8\lambda_5\le 7+\theta \\
& \lambda_1+2\lambda_2+4\lambda_3+\frac{17}{4}\lambda_4+\frac{9}{2}\lambda_5\ge\frac{17}{4}+\alpha \\
& \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=1 \\
& v(7+\theta)-u\left(\frac{17}{4}+\alpha\right)+u_o=0 \\
& 2v-u+u_o\ge 0 \\
& 3v-2u+u_o\ge 0 \\
& 6v-4u+u_o\ge 0 \\
& 7v-\frac{17}{4}u+u_o\ge 0 \\
& 8v-\frac{9}{2}u+u_o\ge 0 \\
& 7v+u_o=1 \\
& \frac{17}{4}u=1 \\
& \lambda_j\ge 0,\ j=1,\ldots 5,\ v,u,\theta,\alpha\ge 0,
\end{aligned}$$

Optimal solution of the above model equal to:
$\left(\theta^*, \alpha^*, \lambda_5{}^*, v^*, u^*, u_o{}^*\right) = (1, 0/25, 1, 0/5882353E-01, 0/2352941, /5882353)$
$\left(X_o + \theta_o^*, Y_o + \alpha_o^*\right) = \left(7+1, \frac{17}{4} + 0/25\right) = \left(8, \frac{9}{2}\right) = DMU_F$
$\left(\theta^*, \alpha^*\right) = (1, 0/25)$ is the stability ray in definition of the right RTS.
Model (3.2) corresponding DMU_E, is as follows:

$$
\begin{aligned}
\max \quad & \theta + \alpha \\
s.t. \quad & 2\lambda_1 + 3\lambda_2 + 6\lambda_3 + 7\lambda_4 + 8\lambda_5 \le 7 - \theta \\
& \lambda_1 + 2\lambda_2 + 4\lambda_3 + \frac{17}{4}\lambda_4 + \frac{9}{2}\lambda_5 \ge \frac{17}{4} - \alpha \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\
& v(7-\theta) - u\left(\frac{17}{4} - \alpha\right) + u_o = 0 \\
& 2v - u + u_o \ge 0 \\
& 3v - 2u + u_o \ge 0 \\
& 6v - 4u + u_o \ge 0 \\
& 7v - \frac{17}{4}u + u_o \ge 0 \\
& 8v - \frac{9}{2}u + u_o \ge 0 \\
& 7v + u_o = 1 \\
& \frac{17}{4}u = 1 \\
& \lambda_j \ge 0,\ j = 1,\ldots,5,\ v, u, \theta, \alpha \ge 0,
\end{aligned}
$$

Optimal solution of the above model equal to:
$\left(\theta^*, \alpha^*, \lambda_3{}^*, v^*, u^*, u_o{}^*\right) = (1, 0/25, 1, 0/5882353E-01, 0/2352941, /5882353)$
$\left(X_o + \theta_o^*, Y_o + \alpha_o^*\right) = \left(7-1, \frac{17}{4} - 0/25\right) = (6,4) = DMU_D$
$\left(\theta^*, \alpha^*\right) = (1, 0/25)$, is the stability ray in definition of the left RTS.
DMU_C as the under assessment, with constant right RTS and increasing left RTS, is in the second class. Model (3.3) corresponding this unit, for to find definable hyperplan of the right RTS is as follows:

$$\begin{aligned} \max \quad & u_o \\ s.t. \quad & 2v - u + u_o \geq 0 \\ & 3v - 2u + u_o \geq 0 \\ & 6v - 4u + u_o \geq 0 \\ & 7v - \frac{17}{4}u + u_o \geq 0 \\ & 8v - \frac{9}{2}u + u_o \geq 0 \\ & 3v + u_o = 1 \\ & 2u = 1 \\ & v \geq 0, u \geq 0, u_o \ \textit{free in sign} \end{aligned}$$

$\left(v^*,\ u^*,\ u_o^*\right) = \left(0/3333333,\ 0/5,\ 0\right)$, is the optimal solution of the above model.

Model (3.4) corresponding this unit, For to find definable hyperplan of the left RTS is as follows:

$$\begin{aligned} \min \quad & u_o \\ s.t. \quad & 2v - u + u_o \geq 0 \\ & 3v - 2u + u_o \geq 0 \\ & 6v - 4u + u_o \geq 0 \\ & 7v - \frac{17}{4}u + u_o \geq 0 \\ & 8v - \frac{9}{2}u + u_o \geq 0 \\ & 3v + u_o = 1 \\ & 2u = 1 \\ & v \geq 0, u \geq 0, u_o \ \textit{free in sign} \end{aligned}$$

$\left(v^*, u^*, u_o^*\right) = \left(0/5, 0/5, -0/5\right)$, is the optimal solution of the above model.

With to find hyper planes, stability ray of this unit in direction of the right and left RTS, is calculated.

Model (3.5) corresponding DMU_C:

$$\begin{aligned} \max \quad & \theta + \alpha \\ s.t. \quad & 2\lambda_1 + 3\lambda_2 + 6\lambda_3 + 7\lambda_4 + 8\lambda_5 \leq 3 + \theta \\ & \lambda_1 + 2\lambda_2 + 4\lambda_3 + \frac{17}{4}\lambda_4 + \frac{9}{2}\lambda_5 \geq 2 + \alpha \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\ & 0/3333333(3 + \theta) - 0/5(2 + \alpha) + 0 = 0 \\ & \theta \geq 0, \alpha \geq 0, \lambda_j \geq 0, j = 1,\ldots,5 \end{aligned}$$

Optimal solution of the above model equal to:

$\left(\theta^*, \alpha^*, \lambda_3^*\right) = (3, 2, 1)$

$\left(X_o + \theta_o^*, Y_o + \alpha_o^*\right) = (3 + 3, 2 + 2) = (6, 4) = DMU_D$

$\left(\theta^*, \alpha^*\right) = (3, 2)$ is the stability ray in definition of the right RTS.

Model (3.6) corresponding DMU_C:

$$
\begin{aligned}
&\max \quad \theta + \alpha \\
&s.t. \quad 2\lambda_1 + 3\lambda_2 + 6\lambda_3 + 7\lambda_4 + 8\lambda_5 \leq 3 - \theta \\
&\qquad \lambda_1 + 2\lambda_2 + 4\lambda_3 + \frac{17}{4}\lambda_4 + \frac{9}{2}\lambda_5 \geq 2 - \alpha \\
&\qquad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\
&\qquad 0/5(3-\theta) - 0/5(2-\alpha) - 0/5 = 0 \\
&\qquad \theta \geq 0, \alpha \geq 0, \lambda_j \geq 0, j = 1,\ldots,5
\end{aligned}
$$

Optimal solution of the above model equal to:

$\left(\theta^*, \alpha^*, \lambda_1^*\right) = (1, 1, 1)$

$\left(X_o + \theta_o^*, Y_o + \alpha_o^*\right) = (3-1, 2-1) = (2,1) = DMU_B$

$\left(\theta^*, \alpha^*\right) = (1, 1)$ is the stability ray in definition of the left RTS.

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