Sequences of Integrals in Experimental Mathematics

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Abstract: *Some sequences of integrals have nice patterns as can be found in CAS-assisted experiments. In the present paper, we present some interesting examples of this kind and explain useful tips about the use of Maple. We also give a proof of a formula in Fourier analysis. It is based on the study of a sequence of integrals and is well suited for experimental mathematics. It can replace a conventional, tricky proof based on a strange lemma.*

1 Introduction

It is a fun, to be sure, to learn established facts, but it gives more joy to discover mathematical facts by oneself. Such a thrilling experience used to be a privilege of few talented people. Now, however, ordinary people can feel the joy of discovery with the help of a computer algebra system. One can perform a significant amount of compution in mathematical experiments. Some experiments may produce only messy results, while others yield really good, beautiful patterns. When the latter is the case, one is tempted to formulate and prove a general statement.

In the present paper, we give some examples of sequences of integrals with good patterns. First we calculate the first one hundred or so terms of a sequence and try to find a rule. We formulate a hypothesis based on such a CAS-assisted experiment. The next step is to prove it in full generality and it is done by pencil and paper.

We will conclude this paper by the discussion about a formula in Fourier analysis. The formula itself is not new and an ingenious proof is given in the famous book [4]. Indeed, it is so ingenious that it is not clear how one can hit upon it. Only a mathematician with exceptional intuition can do it. To teach this kind of proof has the risk of discouraging students by making them feel inferior to talented people. Here we give an alternative, natural proof based on a CAS-assisited experiment and well-known methods in analysis. Such an approach can be taken by an ordinary person and does not require an inspiration of a genius. It can boost students' self-confidence and encourages them to find something new.

2 Sequences of integrals

Problem 1 *Find the value of* $S_n = \int_0^\pi \sin^n x\,dx$ *for* $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

The following Maple commands[1] generate the sequence $\{S_n\}_{n=1}^{100}$.

```
sinint:=n->int(sin(x)^n, x=0..Pi);
sinintseq:=seq(sinint(n), n=1..100);
```

We start the sequence `sinintseq` from $n = 1$ because the index of the first element of a list in Maple is not 0 but 1. Here we give only the first sixteen terms for lack of space. They are:

$$2, \frac{1}{2}\pi, \frac{4}{3}, \frac{3}{8}\pi, \frac{16}{15}, \frac{5}{16}\pi, \frac{32}{35}, \frac{35}{128}\pi, \frac{256}{315}, \frac{63}{256}\pi, \frac{512}{693}, \frac{231}{1024}\pi, \frac{2048}{3003}, \frac{429}{2048}\pi, \frac{4096}{6435}, \frac{6435}{32768}\pi.$$

Odd-numbered terms are rationals and even-numbered terms are rationals multiplied by π. That is the case for $n \geq 11$, too. To see the pattern more clearly, it helps to calculate the ratio S_n/S_{n-2}, not S_n/S_{n-1}.

```
evenodd:=seq(sinintseq[n]/sinintseq[n-2], n=3..100);
```

The result is the simple *recurrence formula*

$$\frac{S_n}{S_{n-2}} = \frac{n-1}{n} \tag{1}$$

for $2 \leq n \leq 100$. Here we have included $n = 2$ since $S_0 = \pi, S_1 = 2, S_2 = \pi/2, S_3 = 4/3$. It follows immediately that we have $S_n = 2(n-1)!!/n!!$ if n is odd and $S_n = \pi(n-1)!!/n!!$ if n is even at least for $n \leq 100$. Recall that $n!! = n(n-2)\cdots 3 \cdot 1$ if n is odd and that $n!! = n(n-2)\cdots 4 \cdot 2$ if n is even. By convention, $(-1)!! = 0!! = 1$.

We expect that our sequence satisfies a simple recurrence formula (1) for an arbitrary n. We rewrite S_n into $S_n = \int_0^\pi \sin^{n-1} x(-\cos x)'\,dx$ and integrate it by parts. Then we get $S_n = (n-1)(S_{n-2} - S_n)$, which implies (1). The explicit expression of S_n given above holds true for any n. *Integration by parts will be used repeatedly to derive recurrence formulas in this manuscript.*

Problem 2 *Calculate* $J_n(x) = \int_0^x (1+t^2)^{-n-1/2}\,dt$ $(n \geq 0)$ *and find the values* $J_n(\infty) = \lim_{x\to\infty} J_n(x) = \int_0^\infty (1+t^2)^{-n-1/2}\,dt$ *for* $n \geq 1$. *(The limit diverges for* $n = 0$.*)*

The change of variables $x = \tan\theta\,(0 \leq \theta \leq \pi/2)$ shows $J_n(\infty) = \int_0^{\pi/2} \cos^{2n-1}\theta\,d\theta$ and the method in the previous problem can be applied. Here, however, we study the integral $J_n(x)$ without changing variables. We employ the following Maple commands:

```
J:=n->int((1+x^2)^(-n-1/2), x);
for n from 0 to 50 do J(n); od;
```

[1] See [2] or [3] for basics of Maple.

They yield

$$J_0(x) = \operatorname{arcsinh}(x), \quad J_1(x) = \frac{x}{\sqrt{1+x^2}}, \quad J_2(x) = \frac{1}{3}\,\frac{x\,(3+2\,x^2)}{(1+x^2)^{3/2}},$$
$$J_3(x) = \frac{1}{15}\,\frac{x\,(15+20\,x^2+8\,x^4)}{(1+x^2)^{5/2}}, \quad J_4(x) = \frac{1}{35}\,\frac{x\,(35+70\,x^2+56\,x^4+16\,x^6)}{(1+x^2)^{7/2}},$$
$$J_5(x) = \frac{1}{315}\,\frac{x\,(315+840\,x^2+1008\,x^4+576\,x^6+128\,x^8)}{(1+x^2)^{9/2}}, \ldots$$

Obviously $n = 0$ is an exception: $\operatorname{arcsinh} x = \log\bigl(x + \sqrt{x^2+1}\,\bigr)$. From now on, we consider only positive values of n. The denominators are $(1+x^2)^{n-1/2}$. Set $K_n(x) = (1+x^2)^{n-1/2}J_n(x)$. It is a polynomial whose degree is obtained by

```
K:=n->(1+x^2)^(n-1/2)*int((1+x^2)^(-n-1/2), x);
s:=seq(K(n), n=1..50);
s2:=seq(degree(s[n]), n=1..50);
```

We see that the degree of $K_n(x)$ is $2n-1$. The coefficient of the leading term is complicated. We calculate the ratio of its values for $n+1$ and n by

```
s3:=seq(lcoeff(s[n+1])/lcoeff(s[n]), n=1..49);
```

Then we find that the ratio is $2n/(2n+1)$.

Now let us prove the observation given above for an arbitrary n. Recall that the logarithm is integrated by $\int \log x\,dx = \int x' \log x\,dx = x\log x - x + C$. This technique works for $J_n(x) = \int_0^\infty t'(1+t^2)^{-n-1/2}\,dt$ and we get

$$J_n(x) = x(1+x^2)^{-n-1/2} + (2n+1)\int_0^x t^2(1+t^2)^{-n-3/2}\,dx.$$

Since $t^2 = (1+t^2)-1$, we get $J_n(x) = x(1+x^2)^{-n-1/2} + (2n+1)[J_n(x) - J_{n+1}(x)]$. It implies

$$J_{n+1}(x) = \frac{2n}{2n+1}J_n(x) + \frac{1}{2n+1}x(1+x^2)^{-n-1/2}. \tag{2}$$

Usually a recurrence formula is accompanied by an initial condition, but (2) can do without one because $J_1(x) = x(1+x^2)^{-1/2} + 0\cdot J_0(x) = x(1+x^2)^{-1/2}$. We can calculate $J_1(x)$ even if we do not know $J_0(x)$. We have $K_1(x) = x$ and (2) implies

$$K_{n+1}(x) = \frac{2n}{2n+1}(1+x^2)K_n(x) + \frac{x}{2n+1}. \tag{3}$$

It follows that $K_n(x)$ is a polynomial of degree $2n-1$ consisting of odd powers of x. The coefficient of x^{2n-1}, which we denote by a_n, is given by $a_n = (2n-2)!!/(2n-1)!!$. Since $J_n(x) = (1+x^2)^{-n+1/2}K_n(x) = (1+x^2)^{-n+1/2}(a_n x^{2n-1} + \cdots)$, we have $J_n(\infty) = a_n$.

Problem 3 *Assume $a > 0$. Find the value of $L_n = \int_0^1 x^a(\log x)^n\,dx$ for $n \in \mathbb{N}$.*

Let us define the function L in n by the following Maple commands. We fix *a for the time being.*

```
assume(a>0);
L:=n->int(x^a*log(x)^n, x=0..1);
```

We find that $L_0 = 1/(a+1)$, $L_1 = -1/(a^2+2a+1)$, $L_2 = 2/(a^3+3a^2+3a+1)$, $L_3 = -6/(a^4+4a^3+6a^2+4a+1)$. The denominators are powers of $a+1$, so the expression for $L(n)$ would be simplified by factorization. We apply

```
for n from 0 to 50 do factor(L(n)); od;
```

and the denominators are certainly $(a+1)^{n+1}$ and the numerators seem to be $(-1)^n n!$ (although factorials of large numbers are difficult to identify). In other words, it seems that

$$L_n = \int_0^1 x^a (\log x)^n \, dx = \frac{(-1)^n n!}{(a+1)^{n+1}}. \tag{4}$$

It reminds us the well-known exercise in calculus "Find the n-th derivative of the function $y = 1/(1+x)$". Of course, the answer is $y^{(n)} = (-1)^n n!/(1+x)^{n+1}$. That is, (4) says

$$L_0 = L_0(a) = \frac{1}{a+1}, \quad L_n = L_n(a) = \frac{d^n}{da^n} L_0(a). \tag{5}$$

Why is that so? It is explained by *differentiation under the integral sign*. Indeed, since $d\alpha^t/dt = \alpha^t \log \alpha$, we have $\partial x^a/\partial a = x^a \log x$. It implies

$$\frac{d^n}{da^n} L_0(a) = \frac{d^n}{da^n} \int_0^1 x^a \, dx = \int_0^1 \frac{\partial^n}{\partial a^n} x^a \, dx = \int_0^1 x^a (\log x)^n \, dx = L_n(a).$$

Problem 4 *Find the value of*

$$I_n = \int_0^\pi \frac{\sin^n x}{(a + ip\cos x)^{n+1}} dx$$

for $a > 0$, $p \in \mathbb{R}$, $n \in \mathbb{N}$.

The evaluation of these integrals is difficult even for Maple. A simple observation greatly alleviates the burden on the computer. It is easy to see that I_n is homogeneous of degree $-n-1$ in (a,p) and we have

$$I_n = \frac{1}{a^{n+1}} J_n, \quad J_n = \int_0^\pi \frac{\sin^n x}{(1 + iq\cos x)^{n+1}} dx, \quad q = \frac{p}{a}.$$

It is enough to calculate J_n. It is done by

```
assume(q, real);
J:=n->factor(int(sin(x)^n/(1+I*q*cos(x))^(n+1), x=0..Pi));
for n from 1 to 20 do J(n); od;
```

The output is

$$J(1) = -2\frac{1}{(i-q)(i+q)}, \qquad J(2) = \frac{1}{2}\frac{\pi}{(q^2+1)^{3/2}},$$
$$J(3) = \frac{4}{3}\frac{1}{(i-q)^2(i+q)^2}, \qquad J(4) = \frac{3}{8}\frac{\pi}{(q^2+1)^{5/2}},$$
$$J(5) = -\frac{16}{15}\frac{1}{(i-q)^3(i+q)^3}, \quad J(6) = \frac{5}{16}\frac{\pi}{(q^2+1)^{7/2}}, \ldots$$

The odd-numbered and even-numbered terms show a little different patterns. It suggests that a recurrence formula relates I_n to I_{n-2}. If $q = 0$ and $a = 0$, then J_n is nothing but S_n in Problem 1. We can guess that $J_n = S_n/(q^2+1)^{(n+1)/2}$ and that

$$I_n = \frac{S_n}{(a^2+p^2)^{(n+1)/2}}. \tag{6}$$

Let us derive a recurrence formula. Integration by parts gives

$$\begin{aligned} I_n &= \int_0^{\pi} \frac{(-\cos x)' \sin^{n-1} x}{(a+ip\cos x)^{n+1}} \\ &= (n-1)\left\{\int_0^{\pi} \frac{\sin^{n-2} x}{(a+ip\cos x)^{n+1}}\, dx - I_n\right\} + \int_0^{\pi} \frac{i(n+1)p\sin^n \cos x}{(a+ip\cos x)^{n+2}}\, dx. \end{aligned}$$

The integrals are derivatives of I_{n-2} and I_n with respect to a and p up to some factors. We get

$$\left(p\frac{\partial}{\partial p} + n\right) I_n = \frac{1}{n}\frac{\partial^2}{\partial a^2} I_{n-2} \quad (n \geq 2). \tag{7}$$

It is easy to check that $I_n = S_n(a^2+p^2)^{-\frac{n+1}{2}}$ satisfies (7).

There remains the problem of uniqueness of I_n. Given I_{n-2}, isn't there another function i_n that satisfies (7)? If that is the case, we have $p(I_n)_p + nI_n = p(i_n)_p + ni_n$, where the subscript p denotes the differentiation in p. Set $f = I_n - i_n$, then $pf_p = -nf$ and $f = \text{const.}p^{-n}$ as is proved by separation of variables. Since f is bounded near $p = 0$, the constant factor must be zero and we have $I_n = i_n$.

3 Fourier analysis

In this section, we study the following formula in Fourier analysis[2].

Theorem 5 *For all $a > 0$ and $t \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} e^{-2\pi a|x|} e^{-2\pi i t\cdot x}\, dx = c_n \frac{a}{(a^2+|t|^2)^{(n+1)/2}}, \tag{8}$$

where $c_n = \Gamma[(n+1)/2]/(\pi^{(n+1)/2})^3$.

[2]To readers unfamiliar with this branch of mathematics, we recommend [1] as a good introduction.

[3]The value of the gamma function can be calculated by using $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$.

Its proof in [4], pp.6-7 is based on the unfamiliar formula

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u}\, du, \quad \beta > 0. \tag{9}$$

It appears suddenly out of nowhere and only gifted mathematicians can think of such a trick. We would like to find how a layperson can prove Theorem 5 with the help of a computer[4].

We employ the polar coordinates. Set $x = r\xi, \xi \in S^{n-1}$. Then $dx = r^{n-1}d\xi$, where $d\xi$ is the surface-are element of S^{n-1}. Let us denote the left-hand side of by (8) by I. Then a formula of Laplace transform gives

$$\begin{aligned} I &= \int_{S^{n-1}} d\xi \int_0^\infty r^{n-1} e^{-2\pi a r} e^{-2\pi i r t\cdot\xi} dr \\ &= \int_{S^{n-1}} \frac{(n-1)!}{\{2\pi(a + 2\pi i t\cdot\xi)\}^n} d\xi. \end{aligned}$$

By rotation, we can replace the vector t by $(|t|, 0, \ldots, 0)$ without changing the value of the integral. Indeed, let $\rho : S^{n-1} \to S^{n-1}$ be a rotation[5] which maps the north pole ${}^t(1, 0, ..., 0)$ to $t/|t|$. Set $\xi = \rho(\eta)$. Then $d\xi = d\eta$ and $t \cdot \xi = |t|\rho({}^t(1, 0, ..., 0)) \cdot \rho(\eta) = |t|{}^t(1, 0, ..., 0) \cdot \eta = |t|\eta_1$. Therefore we obtain (still using the letter ξ for the new variable)

$$I = \int_{S^{n-1}} \frac{(n-1)!}{\{2\pi(a + 2\pi i|t|\xi_1)\}^n} d\xi.$$

Let us introduce the angles $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$ and $\theta_{n-1} \in [0, 2\pi]$ by

$$\begin{aligned} \xi_1 &= \cos\theta_1,\ \xi_2 = \sin\theta_1\cos\theta_2,\ \xi_3 = \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \\ \xi_j &= \sin\theta_1\sin\theta_2\cdots\sin\theta_{j-1}\cos\theta_j, \ldots, \\ \xi_{n-1} &= \sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1}\cos\theta_{n-1}, \\ \xi_n &= \sin\theta_1\sin\theta_2\cdots\sin\theta_{n-1}\sin\theta_{n-1}. \end{aligned}$$

We have $d\xi = \prod_{j=1}^{n-1} \sin^{n-1-j}\theta_j d\theta_j$ and

$$I = \frac{(n-1)!}{(2\pi)^n} \int_0^{2\pi} d\theta_{n-1} \left(\prod_{j=2}^{n-2} \int_0^\pi \sin^{n-1-j}\theta_j\, d\theta_j \right) \int_0^\pi \frac{\sin^{n-2}\theta_1\, d\theta_1}{(a + i|t|\cos\theta_1)^{n-1}}. \tag{10}$$

We can complete the proof of Theorem 5 by using the results of Problems 1 and 4. .

References

[1] Körner, T. W. (1989). *Fourier Analysis.* Cambridge, UK: Cambridge University Press.

[2] Meade, D. B., May, S. J. M., Cheung, C-K. & Keough, G.E. (2009). *Getting Started with Maple, 3rd ed.* Hoboken, NJ: Wiley.

[4] When the present author used Theorem 5 in [5], he was not aware that it was written in [4]. He had to discover and prove it by himself. He used TI-92 plus and Maple.

[5] The orthogonal group $SO(n)$ acts on S^{n-1} transitively.

[3] Shingareva, I. & Lizárraga-Celaya, C. (2007). *Maple and Mathematica: A Problem Solving Approach for Mathematics*. Wien, Austria: Springer-Verlag.

[4] Stein, E. M. & Weiss, G. (1971). *Introduction to Fourier analysis on Euclidean spaces*. Princeton, NJ: Princeton University Press.

[5] Yamane, H. (2004). Fourier-Ehrenpreis integral formula for harmonic functions. *J. Math. Soc. Japan, 56* (3), 729-735. Tokyo, Japan: Mathematical Society of Japan.