From Ancient 'Moving Geometry' to Dynamic Geometry and Modern Technology

Jen-Chung Chuan,
NTHU, Taiwan
jcchuan@math.nthu.edu.tw

Mirek Majewski[1],
NYIT, UAE
mirek.majewski@yahoo.com

Abstract

In this paper we analyze the origins of dynamic geometry and show how some of the ancient Greek and medieval Islamic moving geometry constructions can be created with Dynamic Geometry software – Cabri, Sketchpad or any other program for geometry. We also show how the idea of moving geometry contributed to the development of modern technology.

Introduction

Ancient Greek and later medieval Muslim geometers highly valued geometric constructions that can be created using a straightedge, i.e. a ruler without marking, and compasses[2]. However, these traditional and noble, as they were considered, methods, were not good enough to produce solutions of a number of problems in geometry of this period of time. Such problems were, for example: trisection of an angle, construction of a regular nonagon or heptagon, squaring the circle or doubling the cube. Since these problems were quite important at this time, slightly less noble, but reasonably efficient methods to solve them were invented. One of these methods are so-called neusis constructions or constructions with compasses and a marked ruler. Ancient mathematicians created a number of constructions where, by moving a segment, a line or even a larger group of objects, a desired effect was achieved. For example, the famous construction of a heptagon by Archimedes, considered as the most unique and elegant construction from ancient times, was created using moving geometry. Expression *moving geometry* frequently occurs in manuscripts of medieval Muslim mathematicians and it is an alternative term for the Greek word *neusis* (νεῦσις). In modern literature, the Greek term is frequently replaced by the English expression *verging construction*. For the purpose of this paper we will frequently use term *moving geometry*, as it has a wider connotation.

Surprisingly, neusis and other constructions with moving elements resemble activities that are the essence of dynamic geometry software, e.g. create a geometric construction with a free element (a point, a segment or a line) and then move the free element to obtain a solution or to check if a hypothesis is valid.

---

[1] The work of this author on the project was partially supported by the National Science Council in Taiwan.

[2] The plural form compasses seems to be a bit strange in our times, but the ancient tool to draw a circle was a combination of two sharp pins. Compass is just another word for a pin.

Another group of geometric constructions based also on moving geometry are so-called linkages. Linkages were invented much later, but if we look at them closely, we may find the same or very similar concepts like those in geometry of ancient Greeks. We will discuss linkages and their applications in one of the last part of this paper.

While starting this project we had impression that writing this paper would be an easy task. The amount of literature on the history of mathematics as well as on problems related to moving geometry is enormous. Most of these resources present the main mathematical facts in a reasonably clear manner. However, proofs of these facts as well as historical information often raise various doubts, or simply are incorrect. Books and papers on the history of mathematics are often written by authors who are historians rather than mathematicians. Sometimes architects, or even people completely not related to mathematics attempt to write a publication related to mathematics or to the history of mathematics. These authors quite frequently are not able to check accuracy of proofs, and simply copy them from older publications adding some new errors along the way. On a few occasions, where a religious context was a part of the driving force for a publication, authors desperately tried to prove some ideas that simply do not have grounds or there are no reliable sources to support their thesis.

After examining a number of publications on the history of mathematics, both ancient Greek as well as medieval Muslim, we found that the most reliable sources are books by Sir Thomas Heath written at the end of the XIX and the beginning of XX century. His monographs (see [7][8][9]) on the history of Greek mathematics were major sources for our investigations. Most of Heath's publications can be obtained from Internet archives: Open Archive, Project Gutenberg, and Google Books. Another reliable source of facts in history of mathematics is a small monograph by Cajori (see [5]), perfectly restored by the Project Gutenberg team.

The linkages concept, although based on ancient geometry, have a few reasonably newer publications. For this paper, the main sources on linkages were two books: Yates's handbook on curves (see [13]) and Kempe's lecture on linkages (see [10]).

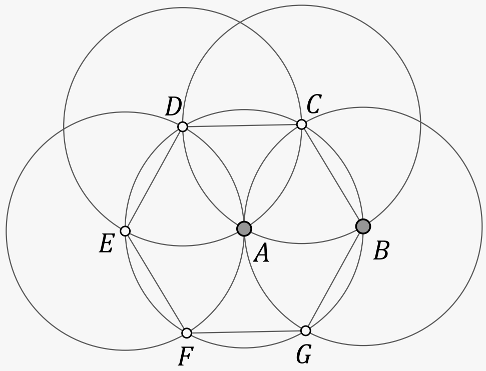
Geometric constructions

When we talk about a theorem or a complex object in elementary geometry we usually mean a set of elementary objects like points, lines, circles, etc., and some relations between them. We usually request that such object or a theorem should be constructed in some way. Constructability of an object is a test of its existence and belonging to a specific domain of geometric objects. For example, all objects that can be constructed using only a straightedge form a group of objects that are constructible with a straightedge only, and this group may differ from a group of objects that are constructible with different tools. Let us examine briefly the nature of geometric constructions and their origins.

A mathematician by a geometric construction will understand a group of objects that were obtained through specific geometric operations from some starting objects. For example, when we construct a regular hexagon we mean a construction shown in example 1.

EXAMPLE 1

Create points A and B
Create circle ○(*A*,*AB*)
Create circle ○(*B*,*AB*)
Create intersection point C
Create circle ○(*C*,*CB*)
Create intersection point D
Create circle ○(*D*,*DC*)
Create intersection point E
Create circle ○(*E*,*ED*)
Create points F and G
Create segments BC, CD, DE,EF, FG, GB

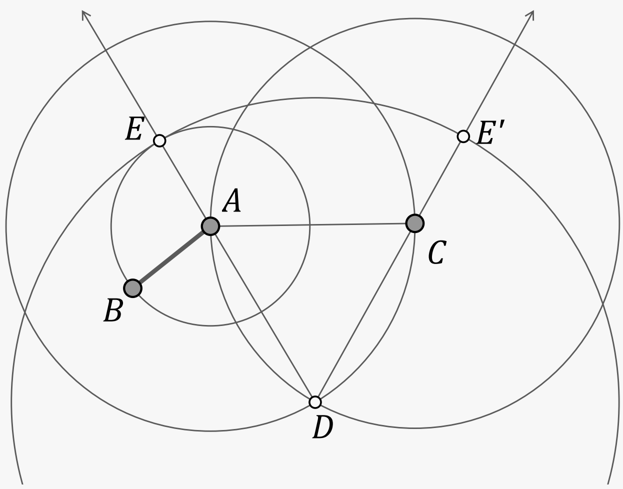


In this construction points *A* and *B* are the starting points and everything created later was obtained from these two points. Operations used here are the same operations that we use in high school geometry – draw a segment or a line connecting two points, draw a circle with a given center and a point on its circumference, create a point of intersection of lines and circles, etc. The only tools that we used here are a tool to draw a line and a tool to draw a circle. Of course we can think about geometric constructions with another set of tools but these two tools are frequently considered as the most appropriate tools for elementary geometry. Why?

The concept of solving problems in geometry using constructions was introduced by Plato, an ancient Greek mathematician and philosopher living around 427 – 347 B.C. Plato considered straight line as a model for the universe – a line can be extended to infinity and divided into an infinity number of parts. For Plato, the circle was a symbol of God or the Spirit. He considered these two objects to be somehow sacred. For this reason he, and many later mathematicians, insisted on using in geometric constructions exclusively the two tools: a straightedge and compasses. These two tools became a kind of sacred, or noble, tools in elementary geometry. It is important to note that a straightedge is a tool with one straight edge only. The other edge may not even exist or not be straight at all. In later times, geometers also used a straightedge with two parallel straight edges. For the purity of geometric constructions, some geometers, including Euclid, insisted that the compasses are collapsible and are not able to transfer distances. In other words the tools should be used exclusively to perform tasks mentioned in postulates 1, 2 and 3 of Euclid elements.

EXAMPLE 2

Assume points A, B, C are given and suppose that we wish to transfer distance AB to the point C.
CONSTRUCTION STEPS:
Create circle ○(*A*,*AB*)
Construct an equilateral triangle Δ*ACD and extend its sides DA and DC*
Create point E, the intersection point of line AD and circle ○(*A*,*AB*)
Create circle ○(*D*,*DE*)
Create point of intersection of the circle ○(*D*,*DE*) *with the line CD. This will be point E'. It can be easily demonstrated that*
E'C = AE = AB.



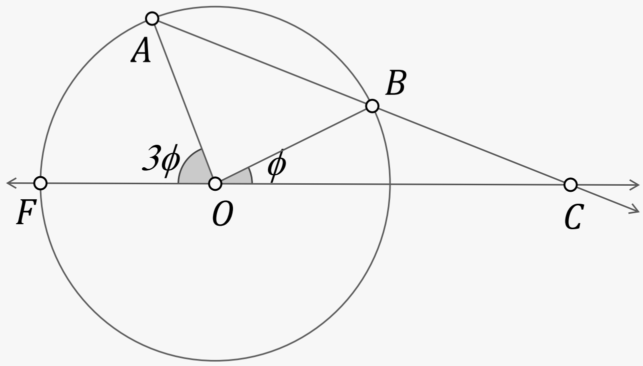
The requirement of not transferring distances by compasses is a bit redundant as we can easily transfer a distance between any two points to another place of the plane using geometric construction. This construction we will frequently use in this paper. Example 2 shows how it can be done.

As we know, Euclidean geometry was for many centuries the foundation of mathematics, art and architecture. However, ancient Greek mathematicians, as well as later Muslim geometers were trying for centuries to solve some problems, using constructions with straightedge and compasses only, without any special luck. One of such problems is hidden in the *Book of Lemmas*[3] written by Archimedes (see [7]). This problem is frequently used in modern geometry textbooks, and quite often without even mentioning its source.

EXAMPLE 3

High school geometry problem based on the eight proposition from Book of Lemmas by Archimedes

If a chord AB of a circle is extended to C, and BC is equal to the radius of the circle, and a line is drawn from C through the center of the circle, then $\angle AOF=3\angle BOC$



The proof of this fact is a simple high school exercise. The original proof given by Archimedes can be found in the Heath's monograph (see [7]). We can however, look at the problem from a different point of view. Suppose that the angle $\angle AOF$ inscribed in the circle $\circ(O,OA)$ was given. Can we construct, using straightedge and compasses only, a point B such that the angle $\angle BOC = \angle AOF/3$? One can easily recognize that at the very moment we are attempting to solve the famous problem of trisecting a given angle using a straightedge and compasses only. For many centuries, mathematicians as well amateurs of mathematical entertainments tried to solve this problem until finally in 1837 Pierre Laurent Wantzel, a French mathematician, proved that this problem cannot be solved using straightedge and compasses only (see [5]).

Meanwhile, ancient geometers invented a number of methods to solve the trisection of an angle problem. Each of these methods goes slightly beyond the straightedge and compasses limitations. One of them is so called *insertion principle* (see [12]), or *neusis* construction in Greek terms (see [8]), or *moving geometry* construction in medieval Islamic geometers notion (see [2]), or *verging constructions* in modern terms.

Verging constructions

Let us examine a few examples demonstrating how verging constructions work. Many interesting examples can be found in the literature. Therefore, after explaining the main concept of verging constructions, we will concentrate on a few examples only.

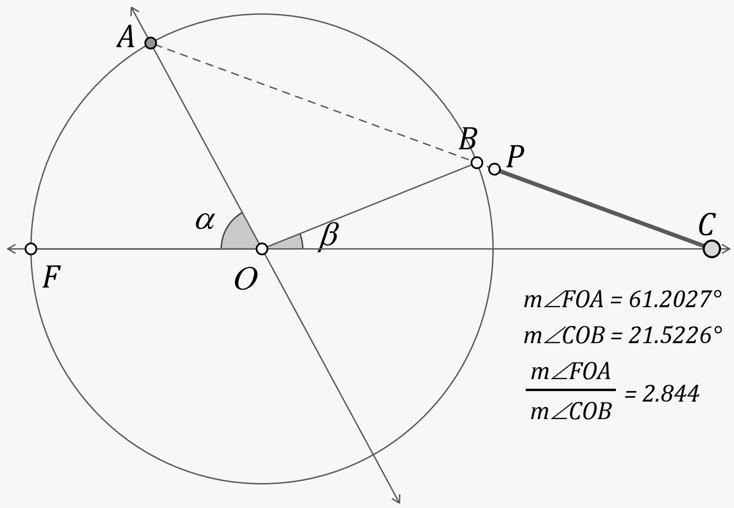
---

[3] The *Book of Lemmas* is a collection of fifteen geometrical propositions. The original book was not preserved to our times. However, we know it from Arabic medieval translations. A complete translation of Book of Lemmas can be found in the Heath monograph e.g. [7].

Let us go back to the concept presented in example 3. We will show how a typical verging construction looks and we will use Geometer's Sketchpad to experiment with this construction.

EXAMPLE 4

Suppose that angle α is given.
Draw a circle with the center in the vertex of the angle α. We will obtain points A and F.
Extend segment FO in both directions.
On the extension of FO mark a point C and draw line connecting A and C.
On the intersection of the line CA with the circle we will obtain point B.
Create segment CP on the line CA in such a way that C is one of its ends and its length is equal to the radius of the circle (see example 1 in this paper).



THE VERGING STEP: Now move the point C along the line FO until points B and P will join together. According to the eight proposition from the Book of Lemmas angle $\beta=\alpha/3$.

Now, after finishing this example we can make a summary of the common meaning of a verging construction.

A *verging construction* is a geometric construction where we have:

1. A line *l* with a fixed-length segment marked on it – the so-called *diastema,* and one fixed point *P* on it – the *pole* of the construction. Line *l* is sometimes called the *verging ruler.*
2. Two curves – one called *directrix* or *guiding line* and another one so-called *catch line.*
3. The fixed length segment on the line *l* (diastema) is located in such a way that one of its ends is located on the directrix and the other end, which we will call a *touch point*, after some manipulations of the line *l* should fit on the catch line.

In our example, point *A* is the pole, line *CA* is the verging ruler, segment *CP* is the diastema, *P* is the touch point, line *FO* is the directrix, and circle ○(*O*,*OA*) is the catch line. In this paper we will use the following convention: the verging ruler will be represented by a dashed line; the pole will be a red/dark point; the diastema will be shown as a thick segment, the end of diastema moving on the guiding line will be large light-blue point.

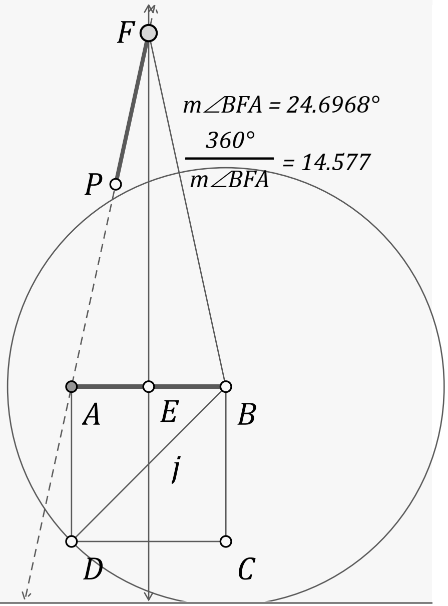
While reading historical or even modern geometry texts we find that people often misuse the verging construction term and sometimes use it for all geometric constructions where something is moving. The above definition, probably the most clear and accurate definition of verging constructions was adopted from a Wikipedia page.

The trisection of an angle is a problem that has had many different treatments and solutions. The one that was of interest to us while writing this paper was the verging construction approach. Other solutions can be found in the very rich literature for this problem.

Another very typical example of verging construction is the construction of a regular heptagon. This construction was probably created by Archimedes and enclosed in a manuscript that was partially reconstructed and translated to Arabic by Thabit ben Qurra. The original manuscript is lost.

EXAMPLE 5. CONSTRUCTION OF REGULAR HEPTAGON

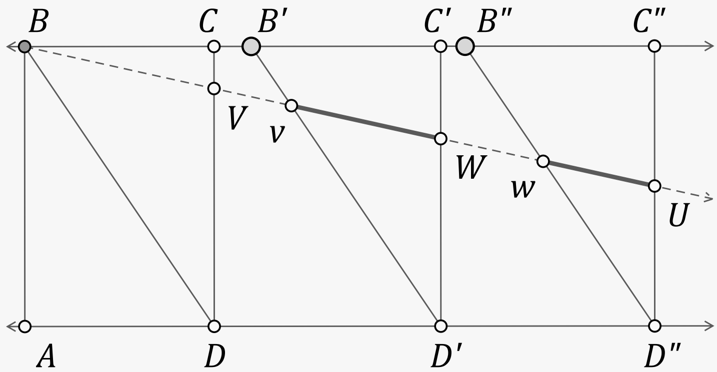
Start with a segment AB (this will be the side of the heptagon).
Construct square □ABCD with side AB, and draw its diagonal BD.
Draw circle ○(B,BD)
Find the midpoint of the segment AB and construct its bisector.
On the bisector of AB select a point F.
Draw the line AF.
On the line AF construct a segment with F as one its end and its length equal to the length of AB. The other end of the segment label as P.
THE VERGING STEP
Now, move point F until point P will touch the circle ○(B,BD).
The angle AFB is equal 360°/14. A reasonably easy proof of this fact can be done with trigonometry and solving an equation of order 3.



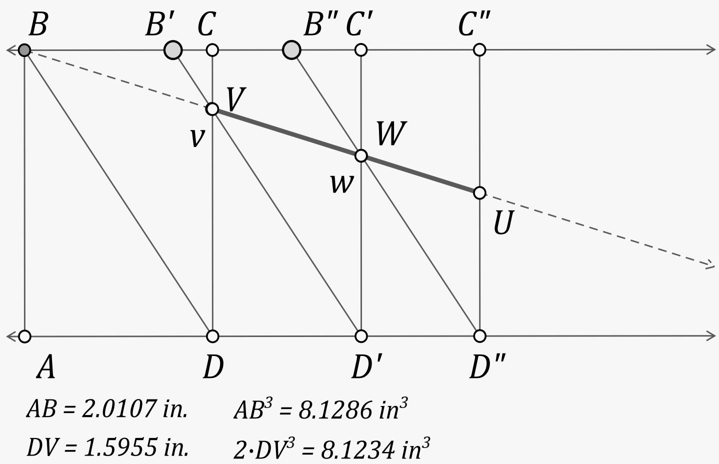
Now let us concentrate on another geometric problem from the same period of time – the doubling the cube problem. The construction presented here was invented by Eratosthenes, and it is a very unusual verging construction – we have to match two points, not one like in the trisection of an angle problem.

EXAMPLE 6

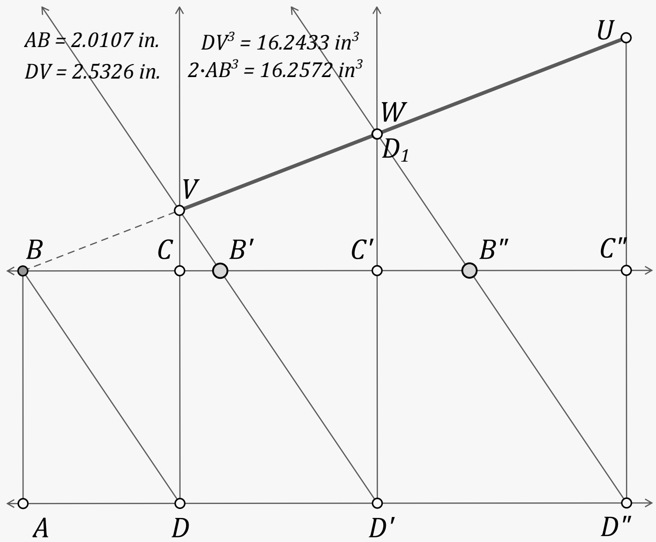
Construct segment AB
Construct two parallel lines perpendicular to AB and passing through its ends.
Construct three identical triangles ΔBCD, $\Delta B'C'D'$ and $\Delta B''C''D''$.
Construct the midpoint of the last vertical segment C″D″. This will be the point U.
Draw line passing through point B and U.
Mark points of intersection of the line BU with vertical segments as V and W, and with slant sides of triangles as v and w.



THE VERGING STEP
Move the point B′ along the top horizontal line until points V and v make one point.
Now, move the point B″ along the top horizontal line to get W = w. Note, after this step you may need to correct the locations of points V and v.
When you get V = v and W = w you will find by measuring appropriate distances that $AB^3 = 2DV^3$. A formal proof of this fact can be found in [8] and in a few other publications on the history of mathematics.



The construction presented above works the other way around than we needed. For a given segment AB it produces segment DV (smaller than AB) such that $AB^3 = 2DV^3$. However, we can easily modify this construction to get what we really want, i.e. DV such that $DV^3 = 2AB^3$.
A specific beauty of this problem is that it allows us to produce two cubes in any ratio by simply dividing the last right vertical segment into two parts of the ratio we need. In the above construction we had $UD'' = AB/2$. In the construction to the right, $UD'' = 2AB$. We can easily imagine the same construction using some other proportions.



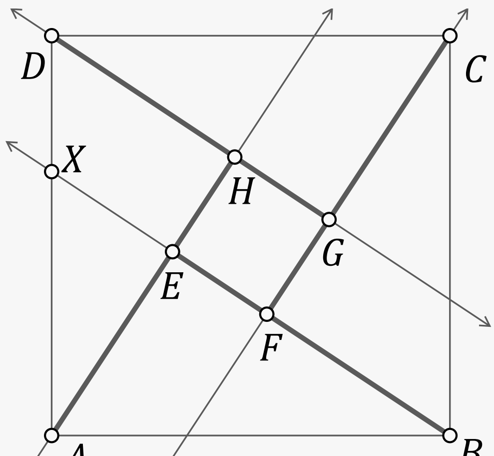
The problem of duplication of a cube has quite a rich history. The number of its solutions is incredible. Heath listed in his book more than twelve solutions by ancient Greek mathematicians only. While reading the Heath book, one may have a feeling that ancient Greek mathematicians held a kind of competition for solving this problem, and every day they were spending their siesta time discussing and solving geometry problems.

We will finish the chapter on verging constructions by showing one very unusual example from the Islamic world. It is closely related to the verging constructions, but it is not a verging construction in the sense of the definition we formulated at the beginning of this chapter.

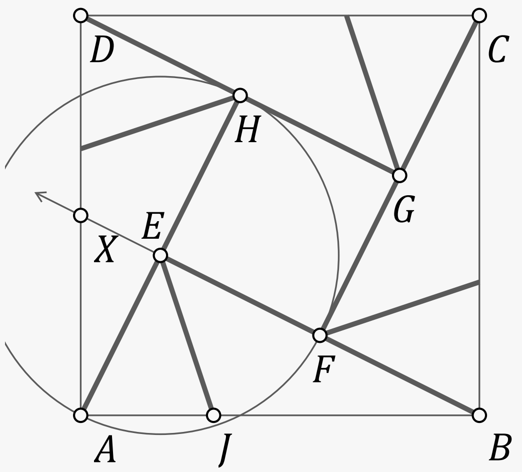
Islamic artists used a few techniques to develop their geometric ornaments. One of them was dissecting a square, or a rectangle, into smaller pieces, and these pieces dissecting again, and again. This way, design of the pattern would start from a very convenient shape that could be easily scaled and replicated to cover a specified plane region. One of the methods of creating such a pattern was based on the Pythagoras theorem (see [11]). The proof shown below in [11] is attributed to Abu I-Wafa (940-998). However, this particular proof of the Pythagoras theorem was also known to the Indian mathematicians of the same period of time.

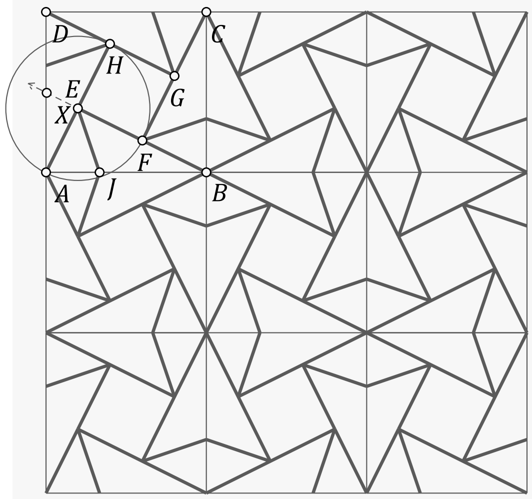
EXAMPLE 7 CONSTRUCTION OF ISLAMIC ORNAMENT BASED ON THE PYTHAGORAS THEOREM

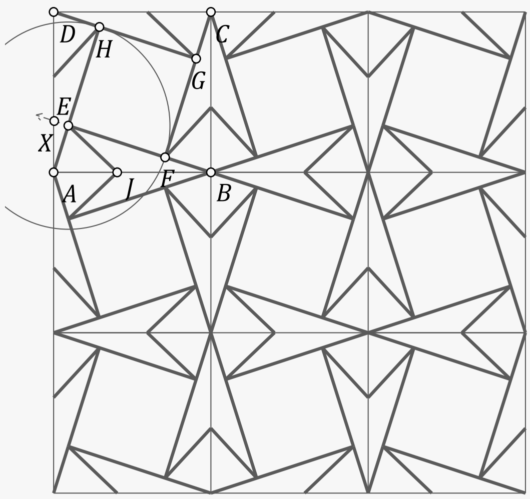
Start with a segment AB that is the intended side of the heptagon.
Construct a square □ABCD with side AB, and draw through it a slant line starting from one of the corners, here it is point B, and passing through the opposite side of the square.
From one of the neighboring points of B draw a line that is perpendicular to the slant line. Point of intersections of these two lines label as E.
The triangle ∆ABE inside the square is a rectangular triangle.
By constructing lines going through two the other vertices of the square and parallel to the sides of the triangle ∆ABE we obtain a construction that can be easily used to prove the Pythagoras theorem.



Further divisions of obtained figures and replication of the square can produce simple Islamic geometric ornaments. Here we show one of such subdivisions.
The triangle ΔABE was created in such a way that segments AE, and EH have the same length[4]*. This means the perpendicular sides of the triangle form ratio AE:EB = 1:2.*
The segment EJ was obtained by bisecting angle $\angle AEB$.
By dividing in exactly the same way the three remaining triangles we will develop a more complex pattern shown below left.
The proportions of the right triangle are quite essential. If we change them, then the pattern will have large, or small, empty squares and may look quite unbalanced (below right).

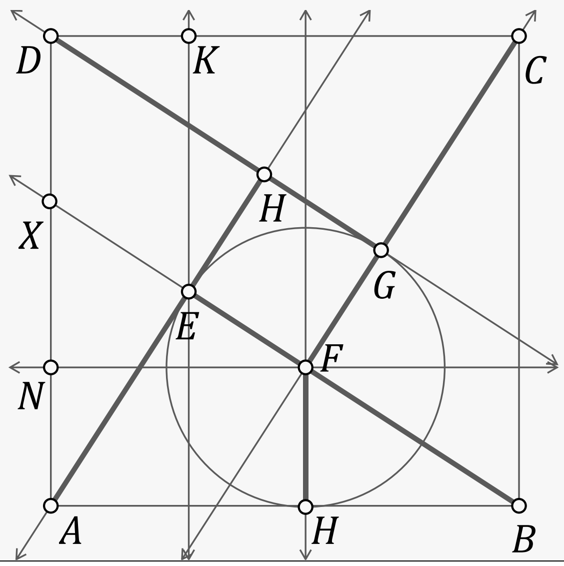






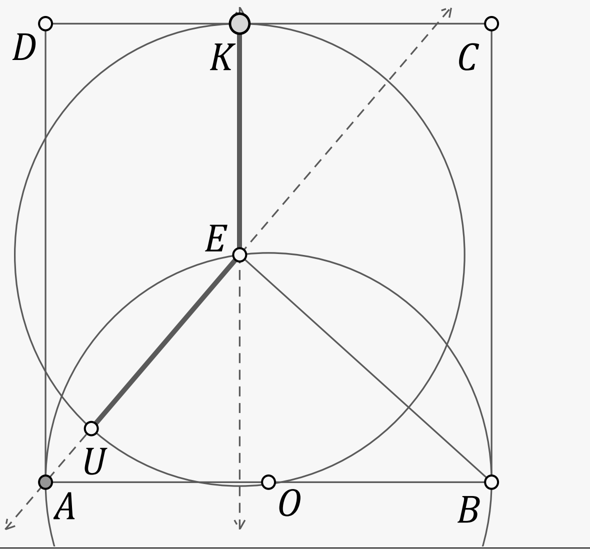
EXAMPLE 7A (CONT.)

We can easily imagine some other divisions of the right triangles in this example. One of them is shown in the picture to the right. In this particular construction EF=FH and $FH \perp AB$. This makes the sides AE and AH equal and creates the possibility for further subdivisions, this time the kite ◊AEFH. Unfortunately such proportions cannot be obtained by a straightedge and compasses only constructions. However, such proportions of the triangle ΔAEB can be easily obtained using a verging construction.
Before we describe such a construction, let us to note that if EF = FH then AE = EK (quick proof: if EF=FH then AE=AH=FN=EK). Therefore, we have to create a right triangle with AE = EK.



[4] The right triangle with AE:EB=1:2 proportions can be easily created knowing that shorter side of the triangle is equal $AE = AB\sqrt{5}/5$.

THE VERGING STEP
Construct a square □ABCD with side AB.
Find center of the base of the square, point O, and construct the circle ○(O,OA)
Select a point on the opposite side of the square, point K, and construct a line passing through K and perpendicular to AB. Label the point of intersection of this line with the circle ○(O,OA) as E.
Draw a line passing through points A and E. This will be the verging ruler. Point A will be the pole of the verging construction.
Construct segment EU equal to EK. This will be the diastema of our construction.
Finally, move point K until U=A. According to our earlier calculations we have obtained the required triangle.



The verging construction shown here was developed by one of the authors while writing this paper. It is slightly different and much simpler than solutions shown in literature (see[11]). One of the peculiarities of this construction is that it looks like a verging construction but it is not a verging construction in the sense of our definition. In a verging construction we require that the diastema must have a fixed length. Here, the segment *EU* changes its length as we move the point *K*.

Creation of curves using verging procedures

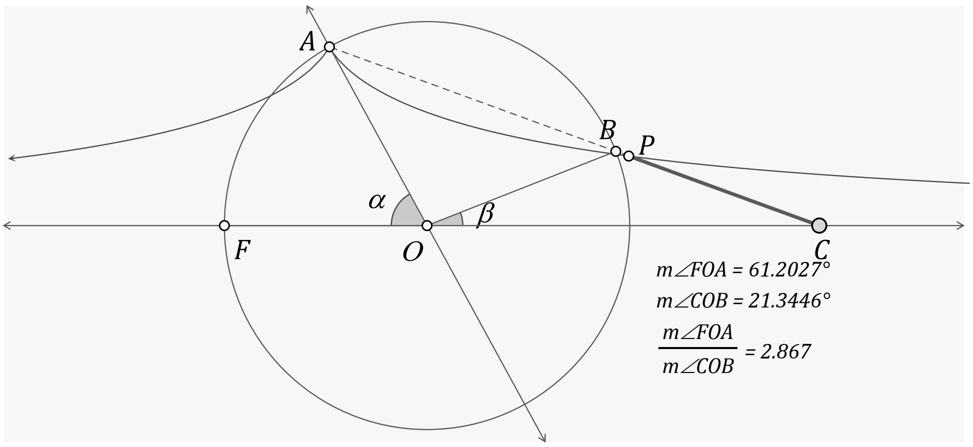


Figure 1 Conchoid of Nicomedes obtained while developing trisection of an angle

In literature related to ancient Greek geometry and medieval Islamic geometry we can find more examples of verging constructions. It is even more interesting to see what will happen if we trace the locus of some points in verging constructions. For example the locus of the touch point *P* from example 1 will draw a line that is a part of the curve known as *conchoid of Nicomedes* (see fig.1). In fact the point that we created while making the verging construction is the point of intersection of the conchoid Nicomedes with the circle. Ancient Greek mathematicians used verging procedures to create such curves as the conchoid of Nicomedes, or even conic sections.

Nicomedes, in order to create his conchoid, developed a special device that uses the verging concept and is able to produce an entire family of conchoids. The principle of this device is shown in the figure 2. It contains two sticks *ER* and *ES*, both the same length (measured here by the length of segment CD) and on the same line QE. Point Q is the pole and point E moves along the horizontal line. The family of conchoids was obtained by changing the location of the pole Q.

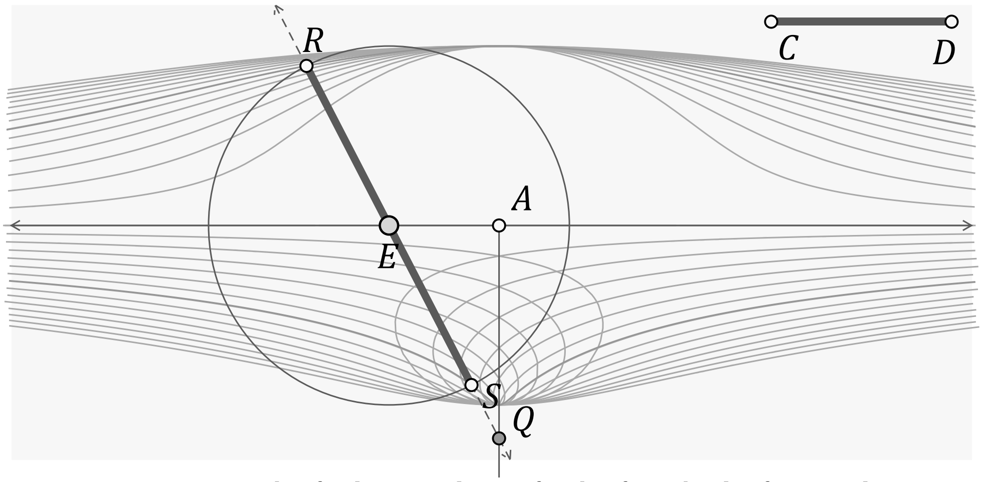


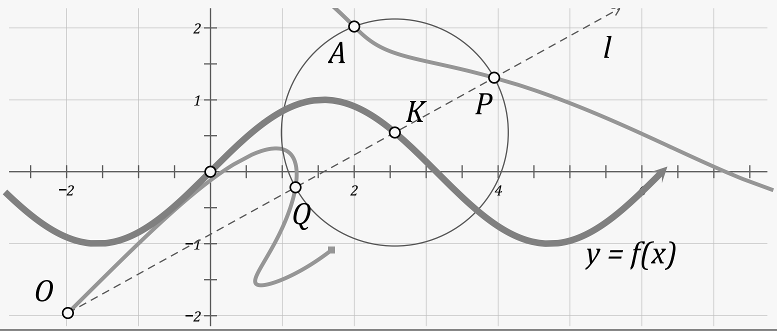
Figure 2 Principle of a device to draw a family of conchoids of Nicomedes

We can easily check that the curve generated by the touch point of the diastema in example 6 is also one of Nicomedes' conchoids (fig. 3) and the curve generated by the touch point of the diastema in example 7A is a modified form of a strophoid (fig.4). All these observations open a door to constructing curves using verging or quasi-verging procedures. Both ancient Greek and later medieval Muslim mathematicians were very creative in developing methods and devices to draw various types of curves.

In this paper, we will cite one more example that follows from example 7A. Let us start with a definition crated around 1670.

DEFINITION OF A STROPHOID

Suppose that we have a curve f, two fixed points A and O (the pole), and line l passing through O and intersecting with f in the point K. The locus of points P and Q such that PK = QK = KA is called a general strophoid.



Example 7A is very close to the sense of this definition (see fig. 4). The only difference is that we do not have there a point A, but we have a line DC. The rest is the same. The results can be quite interesting, depending on how the curve f looks and how points A, O and the curve f are located. The graphs in figures 5, 6, 7 and 8 depict a few strophoids. In each of these examples, the length of the diastema changes when we move point K.

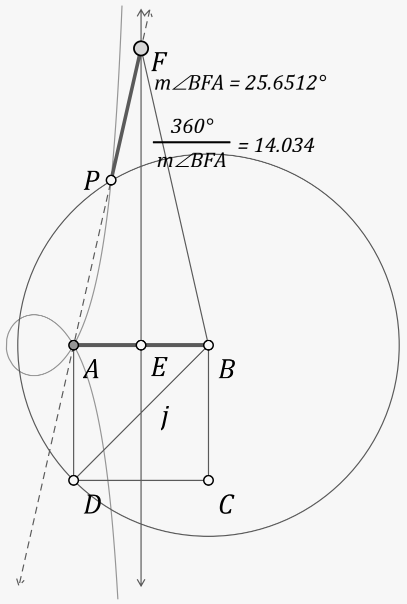


Fig. 3 Locus from example 6

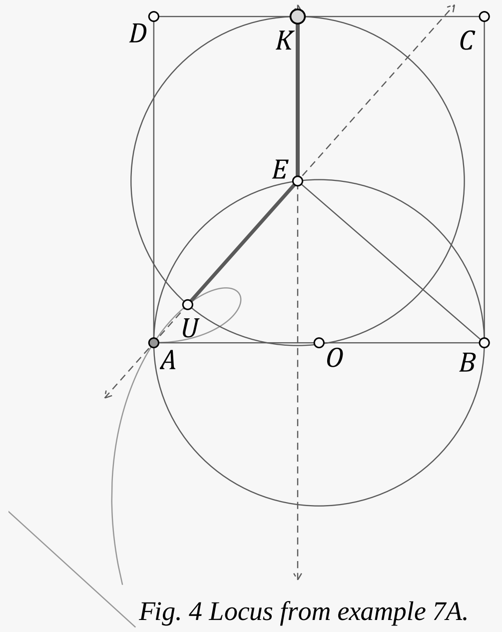


Fig. 4 Locus from example 7A.

.

EXAMPLE 8 STROPHOIDS

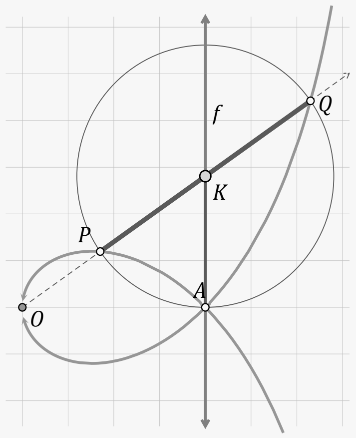


Fig. 5 Curve f is a line perpendicular to OA

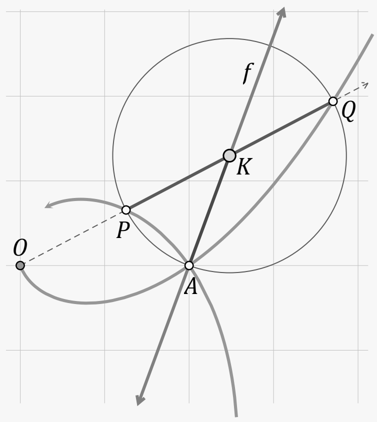


Fig. 6 Curve f is a slant line in respect to OA

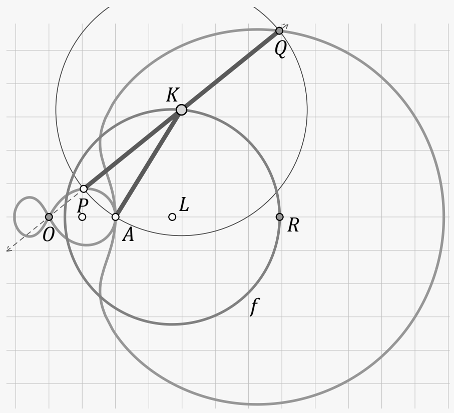


Fig. 7 Curve f is a circle

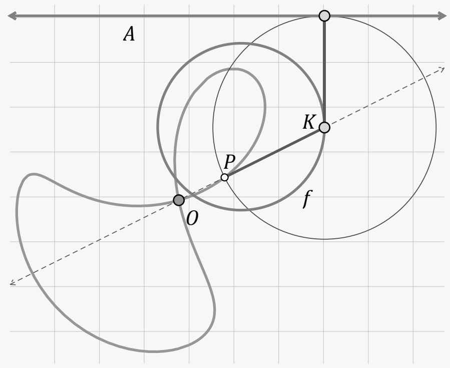


Fig. 8.Curve f is a circle, point A was replaced by line A

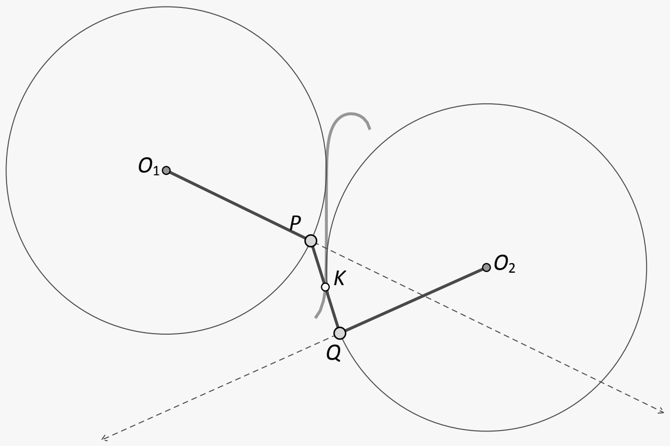
Linkages

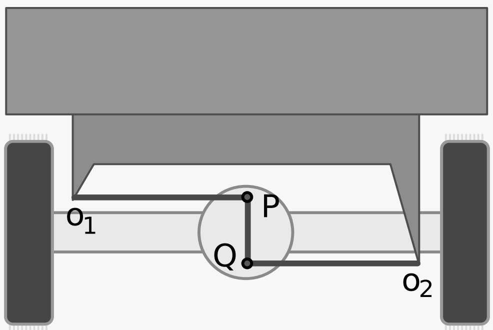
The concept of linkages was developed quite late but its origins remain in ancient Greek geometry and verging constructions. In verging constructions we had only one pole, one diastema and one verging ruler. In linkages we may have a few poles, a few diastemas, and a few verging rulers. The curves we create with linkages are more complex than those we created with a single pole and verging procedures in the previous chapter.

Probably the very first linkage was created by James Watt (1736 –1819) while working on his famous parallel motion project (year 1784). The objective was to produce a straight path out of two rotational movements. The simplest form of the Watt's linkage is shown in example 10.

EXAMPLE 10 WATT'S LINKAGE

Bars O_1P and O_2Q have the same length, K is the center of fixed-length bar PQ.



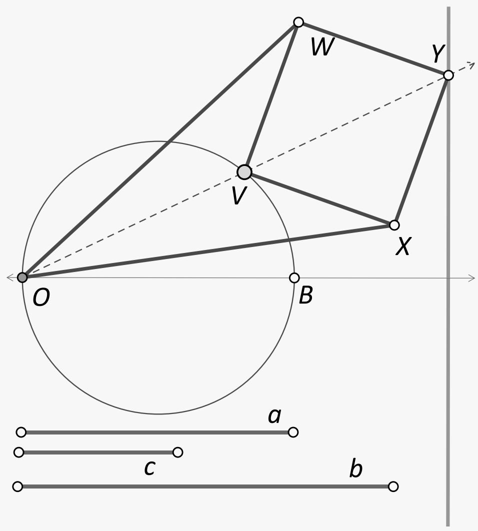


Application of Watt's linkage in car suspension

In the Watt's linkage we have two poles and two verging rulers, the two bars O_1P and O_2Q have exactly the same length, points P, Q are free to move on the edge of a circle with center O_1 and O_2 respectively. The bar PQ has a fixed length and its center K moves on what is supposed to be a straight line. As we can easily see, the path of point K is almost straight on a short distance and this was enough to apply this construction in many types of car suspensions.

EXAMPLE 11 CONSTRUCTION OF PEAUCELLIER CELL

The two points O and B are fixed ($OB = a$)
The two bars OX and OW have fixed length and $OX = OW = b$.
The four bars VW, VX, XY and YW have fixed length and $VW = WY = YX = XV = c$
Point V moves along a circle with diameter equal to OB.
As a result of these constraints point Y moves along a straight line perpendicular to the line OB.



The race to improve Watt's linkage was quite obstinate and there were many interesting developments along the way. However, the most successful linkage was developed in 1864, exactly eighty years later, by the French officer and engineer M. Peaucellier. His linkage, known now as a Peaucellier cell, became the source of a number of technical applications as well as developments in mathematics and kinematics (see example 11).

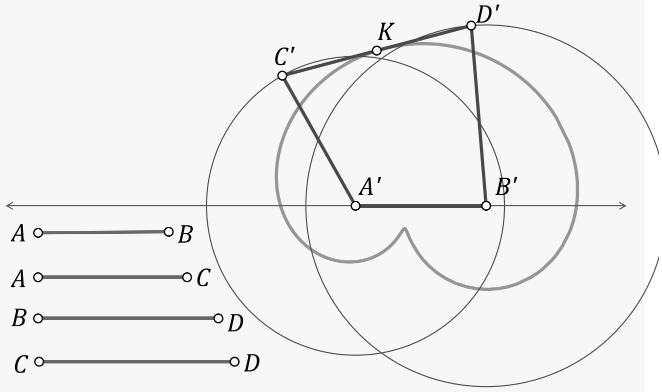
One of the developments in mathematics closely related to the Watt's linkage are so-called four-bar linkages and coupler curves. The idea of the four-bar linkage is the same as in Watt's linkage, but the bars may have different lengths. This is all, but this minor change produced a large family of curves that were classified by mathematicians, and their theory was developed.

Here is a brief idea of a four-bar linkage. We have:

1. A four-bar polygonal cycle
2. One of the bars is called a frame and it is a fixed part of the linkage
3. Another bar different than the frame is called a coupler
4. The whole device moves with respect to the frame
5. A coupler curve is a curve traced by a fixed point on a coupler.

EXAMPLE 12 CONSTRUCTION OF FOUR-BAR LINKAGE AND A COUPLER CURVE

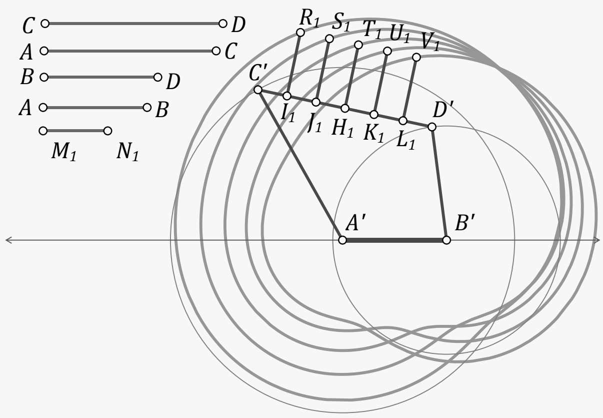
In this linkage
A'B '= AB,
A'C' = AC,
B'D' = BD,
C'D' = CD
K is the center of C'D'.



In mathematics any curve traced by a point on a linkage is referred to as a coupler curve. The linkage can be extremely complex with many bars and even with many fixed points. Multiple coupler curves are also discussed. The next example shows a linkage where 5 points are used to draw coupler curves.

EXAMPLE 13 FOUR-BAR LINKAGE WITH MULTIPLE COUPLER CURVES

In this linkage
A'B '= AB,
A'C' = AC,
B'D' = BD,
C'D' = CD
$M_1N_1 = R_1I_1$….



It is obvious that research of linkages and coupler curves is quite important to the development of modern technology. By analyzing coupler curves, we can tell if the movements of a particular part of a device works according to our plans and needs. Analyzing coupler curves is also a way to check if the movements of some points in a device are optimal for the person using it. For example a bicyclist, or a rower on a boat, have to move their legs and hands in such a way that they will not get quickly tired, and all movements will be optimal for their body.

Fig. 11Modern machinery uses extensively various types of linkages (*image from http://lefthandedcyclist.blogspot.com/*)

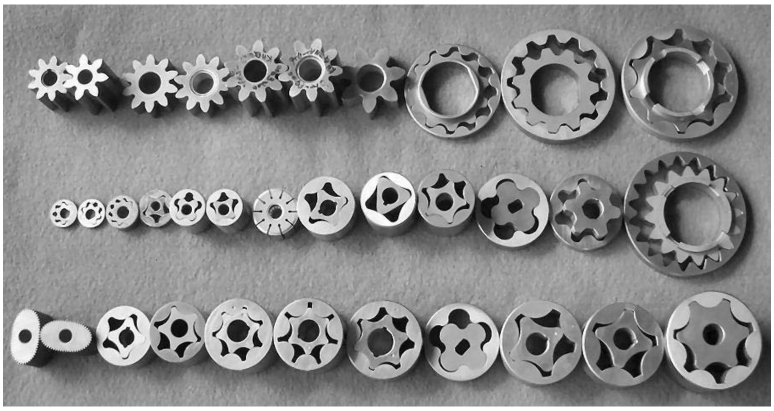


Fig. 12 Construction of many types of gears is based on the same elements of moving geometry invented and used by ancient Greek mathematicians

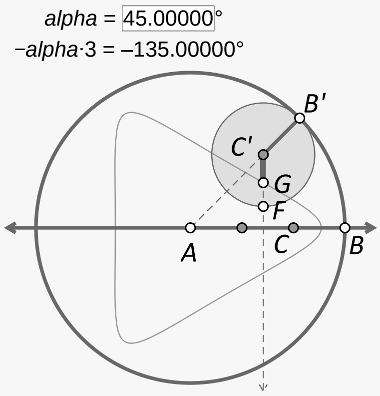
Gears

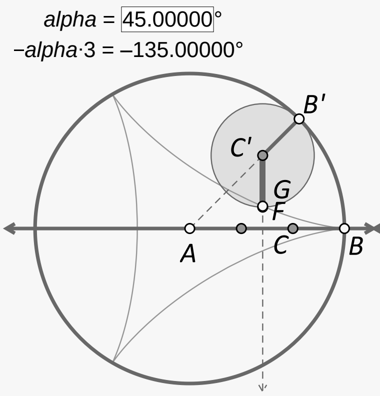
In a modern machinery we can find many parts that were constructed using ancient Greek moving geometry. One example of this are various gears (see fig. 12). If we examine closely their shape, we will find a number of well-known mathematical curves – epitrochoids, hypotrochoids, cycloids, epicycloids, ellipses, spirograph curves, etc. Let us briefly take a closer look at some of them. We will concentrate on a simple epitrochoid with only a few bumps. An epitrochoid is a curve created by a point attached to a circle moving inside of another circle. The point can be attached to the circle edge, or to its interior, or to a bar attached to the circle.

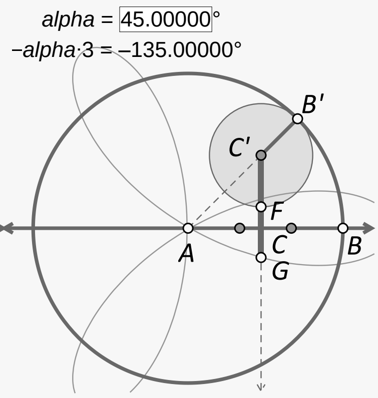
EXAMPLE 14 CONSTRUCTION OF AN EPITROCHOID

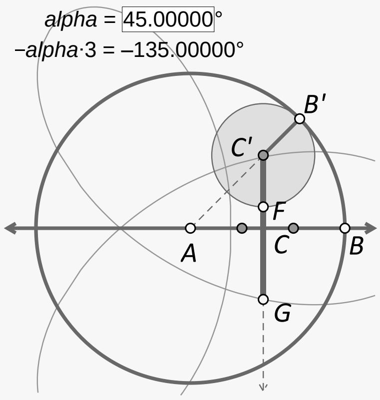
In this example we use two circles – a large one with the radius R and a small one with the radius r=R/3.
The small circle is rolling inside the large circle without sliding.
Therefore, the length of arc BB' is exactly the same as the length of arc B'F.
Point G is attached to the ray C'F. Depending on the distance of point G from the center of the small circle we can get a number of various curves.
Below are shown three more types of epitrochoids:

1. *Left G=F*
2. *Middle GC'=2r*
3. *Right GC' > 2r*





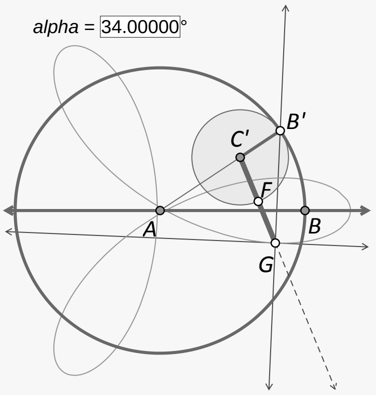


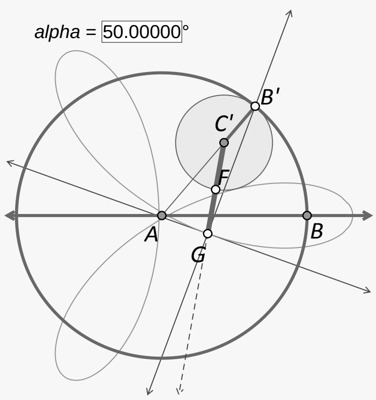


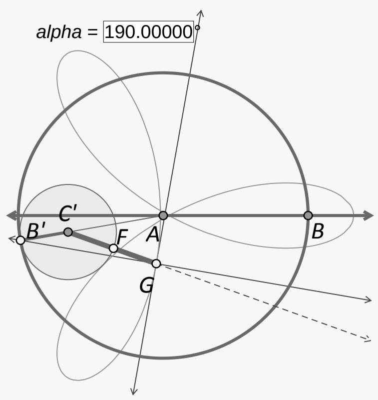
While constructing gears one of the problems is to find how quickly they will wear out while working. This requires to be able to find a tangent line (it shows the direction of the friction power) and the normal line (the direction of the pressure power on a gear) to a point on the curve. Both lines can be easily constructed.

EXAMPLE 15 TANGENT AND NORMAL LINES TO AN EPITROCHOID

Draw a straight line through the points G and B'. It can be proved that this line is normal to the curve in point G. Draw a perpendicular line through the point G to the normal line GB'. This will be the tangent line. The enclosed pictures show this construction for various points on the curve.



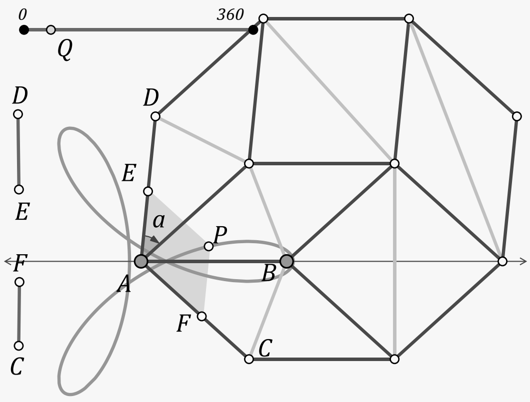


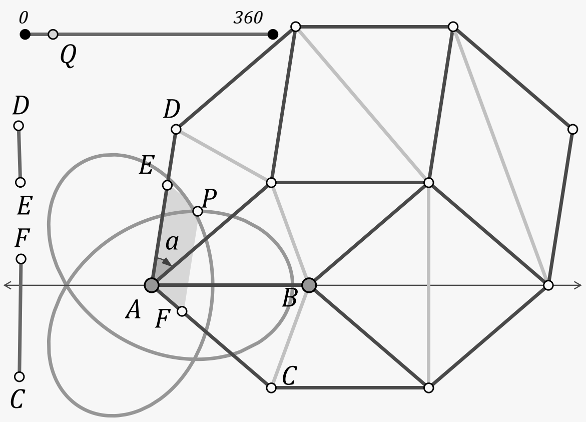


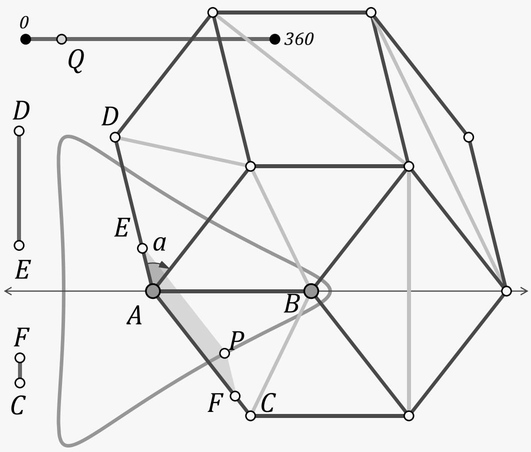
As we all know, parts of machinery are developed by various mechanical devices, where rather mechanical constraints are used than a mathematical formula of a created object. Therefore, we can think about a linkage that can be used to create various shapes of gears. One such example is shown below.

EXAMPLE 16 A LINKAGE TO CONSTRUCT EPITOROCHIDS

In this linkage points A and B are fixed points, all other points are movable.
All blue/dark segments have fixed length. All gray segments are rubber bands. The angle a changes from 0 to 360° (*point Q is used to simulate this change*).
The parallelogram AEPF has fixed lengths of its sides equal to the lengths of AD – ED and AC – FC respectively.
The vertex P of the parallelogram is used to draw the curve. Depending on the lengths of segments DE and FC, different shapes of an epitrochoid are obtained.







Summary

In this paper, we started from verging constructions in ancient Greek geometry. Then we explored the way the concept of moving geometry developed throughout the centuries. We started from very simple geometric constructions where one bar was moving along a line, or on a curve, and we went to complex constructions, so-called linkages, where many, sometimes hundreds, bars are moving. We explored also curves traced by selected points on our constructions (coupler curves). It is obvious that research both of linkages and coupler curves is quite important to the development of modern technology. By analyzing coupler curves we are able to tell if the movements of selected parts on a device work according to our plans and needs. In Watt's linkage it was important to obtain a straight-line movement on a short distance. However, to produce a very precise straight-line movement on a longer distance we needed a Peaucellier cell. In a modern bike it is important for a bicyclist to have movements optimized so his energy while driving his bike will be not wasted for unnecessary movements. Modern industrial robots need even higher precision of movements and optimization of the number of parts and their shapes. All this creates a great need to explore moving geometry.

Bibliography

[1] Archibald R. C., *Euclid's Book on Divisions of Figures*, Cambridge, University Press, 1915.

[2] Berggren J.L., *Episodes in the Mathematics of Medieval Islam*, Springer-Verlag, New York, 1986.

[3] Besant W.H., *Conic Sections Treated Geometrically*, Ninth edition revised and enlarged, London, George Bell and Sons, 1895 (Project Gutenberg)

[4] Bussey W.H., *Geometric Constructions without the Classical Restriction to Ruler and Compasses*, The American Mathematical Monthly, Vol. 43, No. 5 (May, 1936), pp. 265-280.

[5] Cajori F., *History of Mathematics*, The Macmillan Company, 1909

[6] Cundy M., Rollett A.P., *Mathematical Models*, Oxford University Press, 1954.

[7] Heath T. L., *The Works of Archimedes*, Cambridge University Press, 1897.

[8] Heath T. L., *A History of Greek Mathematics*, Vol. 1 From Thales to Euclid, Oxford, Clarendon Press, 1921

[9] Heath T. L., *Apollonius of Perga Treatise on Conic Sections*, Cambridge, University Press, 1896.

[10] Kempe A.B., *How to draw a straight line – a lecture on linkages*, MacMillan, London, 1872.

[11] Ozdural A., *Omar Khayyam, Mathematicians and "Conversazioni" with Artisans*, Journal of the Society of Architectural Historians, Vol. 54, No. 1 (Mar. 1995), pp. 54-71.

[12] Thomaidis Y., *Historical Digressions in Greek Geometry Lessons*, For the Learning of Mathematics 11, 2 (June 1991), pages 37-43, FLM Publishing Association, Canada.

[13] Yates R. C., *A Handbook on Curves and Their Properties*, Edwards Brothers Inc., Ann Arbor, 1952.