# A Forgotten Important Tool in the new Mathematics Curriculum: Recursion

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**Abstract**. The capabilities of graphing calculators are changing the way we teach as well as the content and scope of what is taught in basic mathematics. In the United States, following recommendations of the NCTM Standards, recursive procedures, traditionally excluded from pre-college mathematics, began to appear at different levels from elementary to introductory college courses. The ability to i) repeat an instruction or a set of instructions with a single keystroke, ii) to make the output of a calculation the input for the next, and iii) to be able to stop and analyze or modify an iterative process provides a tool of great interest from both the pedagogical and the practical point of view. At the elementary level modern elementary calculators, allow to easily define constant operations that can be used to discover patterns, to reinforce basic facts via the "guess and check" approach, to investigate the relations between basic operations or even their rate of growth, etc. It is however at the secondary level where recursive procedures, accessible via graphing calculators, have the potential for a major impact. In this presentation we show examples from different mathematical areas to illustrate (i) some recursive procedures, that can be easily implemented without programming, (ii) associated powerful mathematical models traditionally taught in upper levels to a selected group of students and now accessible at lower levels to most students, and finally (iii) how recursion, in some cases, provides an alternative problem solving approach less dependent on readymade formulas.

## I. Introduction

For many decades, the teaching of mathematics had few changes in content or methodology. During the last twenty years, however, technology has been having an impact in the mathematics classroom at different levels; this is due, mainly, to two key developments. On one hand, the advancements in computer algebra systems such as Mathematica, Maple, Derive, Matlab, that paralleled the dramatic increases in the speed and memory of personal computers, are having an effect on the workplace and upper level university studies. On the other hand, scientific calculators were replaced in 1986 by the first generation of graphing calculators which included numerical and graphical capabilities. The second generation of graphing calculators with true symbolic capabilities was born ten years later. These calculators, relatively inexpensive, sturdy and portable, have been finding their way into the classroom. In the United States, the initial endorsement for the use of computers and calculators in the mathematics classrooms appeared in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989, 1991), and was later followed by other mathematical societies (MAA, 1990; AMATYC, 1995; NCTM 2000). The integration of this technology in the teaching and learning of mathematics is promoting changes not only in the way we teach and asses, but also in the content and scope of what is taught in basic mathematics. Some of the premises that in the past guided the selection of topics and algorithms traditionally taught are no longer valid. Thus, iterative and recursive procedures that were excluded from school mathematics when programming was not required, began to appear in new books at different levels, from elementary to introductory college courses. From the beginning, the new high school curricula developed in the nineties (Core-Plus Mathematics Project, 1996, 1998; The University of Chicago School Mathematics Project, 1992; The North Carolina School of Science and Mathematics, 1992) as well as new textbooks at the pre-college (Brown, 1992) and college levels (Ferguson, 1994; The North Carolina School of Science and Mathematics, 1996) included examples of iterative and recursive procedures and models whose solutions require the use of recursion. If in addition we consider that the price of hand-held graphing technology (HHGT) has since decreased while their capabilities have improved considerably, one might expect that fifteen years later, iteration and recursion would be a well established tool in precollege mathematics. However, our analysis of a large sample of precalculus textbooks in the last decade including the best sellers at the time (Quesada and Renker, 2008), showed that very little is done with iteration and recursion among other HHGT capabilities; at the same time that the preparation in this topic of pre and inservice teachers does not seem to be at the needed level (Quesada and Dunlap, 2011). In the USA, an increasing number of college graduates, particularly in the sciences and engineering, are expected to be familiar with a growing number of new technologies and of software packages and computer algebra systems; often, the fast pace of these changes, makes difficult to properly measure their impact, and to learn the best way to integrate them.

Although graphing calculators such as the TI-84 include sequential functions, we will see that it is also possible to use iteration and recursion from the Homescreen thanks to two basic facts. The first is the ability to repeat an operation with a single keystroke, which is the basis for iteration and has long been present in four-function calculators. It should be noted that, as it will be seen in the examples, graphing calculators allow us to repeat a set of operations juxtaposed with colons. Elementary calculators such as the TI-Explorer Plus, and the TI-73, allow for the definition of up to four constant operations that when used iteratively, provide a mechanism to discover patterns, to review basic facts via the "guess and check" approach, to investigate the relations between basic operations or even their rate of growth etc. Clearly, iteration is an interesting tool from both the pedagogical and the practical point of view. Secondly, being able to make the output of a calculation the input of the next makes recursion accessible to anyone with a simple graphing calculator such as the TI-84. In addition, the ability to manipulate different data types (such as matrices and lists) and to perform complicated operations with a single keystroke facilitates the introduction at the secondary level of important models relevant to business and industry that in the past were taught in upper level college courses.

In the past (Quesada, 2007) we have used two criteria in order to decide if a mathematical model or tool should be considered to be taught at a particular level. First, one should evaluate how useful is the model or tool considered, in light of the variety and importance of applications that it can be used for. Secondly, one should establish how accessible is the degree of complexity of the concepts and calculations needed for the level where the model is going to be taught. To illustrate that iteration and recursion satisfy these criteria, in this note we present several examples from different areas of mathematics and for different students' levels, that exemplify: (i) iterative and recursive procedures, as well as, recurrence relations that can be easily implemented without programming, in the Homescreen; (ii) the existence of associated powerful mathematical models, traditionally taught in upper levels to a selected group of students, that become accessible at lower levels to most students; and finally, (iii) how recursion, in some cases, provides an alternative problem solving approach less dependent on ready-made formulas.

The syntax used in the commands and the screens provided in all the examples correspond to a Texas Instruments TI-Nspire, but they can be equally implemented on simpler calculators such as the TI-84. The screens included, sometimes in excess, will hopefully remove any doubts in reproducing the solutions provided. We make the convention of denoting by Figure i.r (i.l, i.t, i.b) the right (left, top, bottom) side of figure i, when the figure is split accordingly.

A recursive procedure can be introduced as a repetitive process in which in each step, the input value for any variable(s) is the output value(s) of the variable in the preceding step. Therefore, any recursive procedure starts by assigning initial values to the variables involved; this first step is called the initialization. After that, all is needed is the “command line,” which consists of a command or a sequence of commands concatenated by colons, specifying how each variable changes. In some cases, recursion can be seen as the “divide and conquer” approach. This is the approach in which the solution to a problem is stated in terms of the solution to the same problem with smaller size. For example, finding *n!* is the same as first finding *(n-1)!* and then multiplying this result by *n*. That is, the solution to the larger problem is expressed in terms of the solution to a smaller one.

We have selected different areas in mathematics and one or more examples within each area, to illustrate the use of recursion in number theory, convergence, modeling, and solving equations.

## II. Applications to Number Theory

Number Theory is a rich area for iterative procedures. In our first example we have included a couple of representative algorithms, the first of which we have used to introduce the process of recursion for pedagogical purpose, since students are familiar with the recursive definition.

Example 1. Write a recursive calculation of $a!, a \in \mathbb{N} \cup \{0\},$ in the Homescreen.
*Solution.* The factorial of a non negative integer *a* is defined recursively by

$$a! = \begin{cases} 1, & \text{if } a = 0,1 \\ a(a-1)!, & \text{if } a > 1 \end{cases}$$

As seen in figure 1.l, both the variable *a* representing the current value, and the variable *f* that stores the result, are initialized with 1. The second statement is a description of how the variables are modified using as input the output of the previous step, followed by a list with the two variable names. Using the list allows for the value of both variables to be displayed at each step, for otherwise only the last value calculated will be displayed. Each time the students hit the enter key, the commands contained in the “command line” are executed and the value of the number in turn with its factorial are displayed.

When solving equations graphically, knowing the relative rate of growth within and between families of continuous functions helps to recognize the existence of hidden solutions. Hence, it is important to make students realize how fast the factorial function grows, and having them comparing and contrasting their growth with that of the other continuous functions they are familiar with.

Our second example deals with the calculation of a square root using continuous fractions. This is a topic long gone from the curriculum, but a good example for recursion that some added interest now that the algorithm to calculate the square root has disappear from the curriculum.

Example 2. Find $\sqrt{39}$ using continuous fractions.

*Solution*. Let $x=\sqrt{39}$, then $x^2-36=3$, hence (x-6)(x+6)=3, and it follows that

$$x=6+\frac{3}{x+6}=6+\frac{3}{(6+\frac{3}{x+6})+6}=6+\frac{3}{12+\frac{3}{12+\frac{3}{x+6}}}=6+\frac{3}{12+\frac{3}{12+\frac{3}{x+6}}}\ \ddots$$

As figure 1.r illustrates, the implementation in homescreen is obtained by first initializing the variable *Ans* with 6, and then letting *Ans* take the place of *x*, i.e., writing

$$6\ \boxed{Enter}$$

$$6+\frac{3}{Ans+6}\ \boxed{Enter}\boxed{Enter}\cdots\boxed{Enter}$$

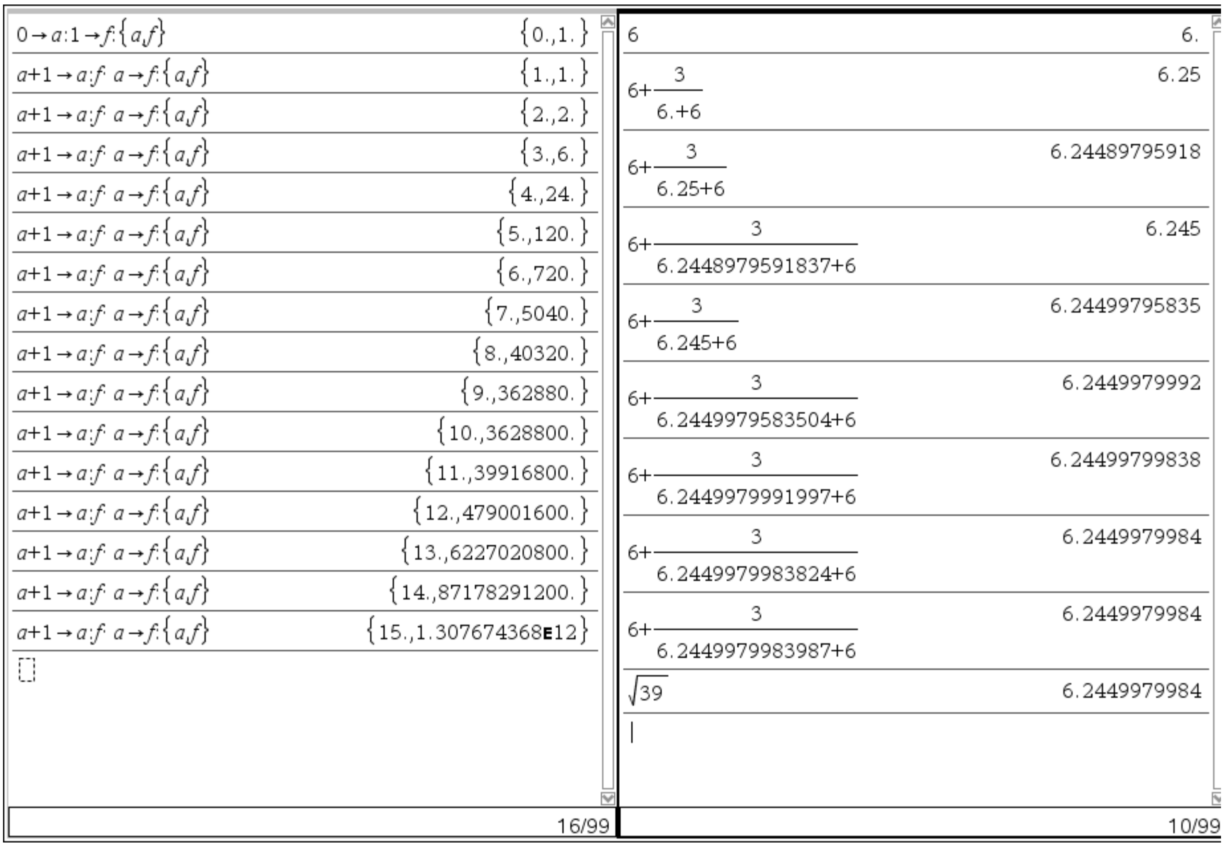



Figure 1

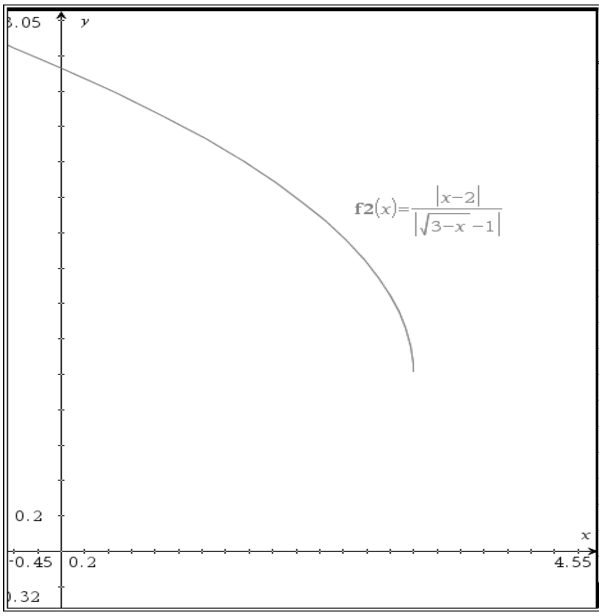
## III. On Convergence

Many of the important ideas from calculus arise from approximation problems, and the key concepts are better understood using convergent sequences that, as the next example illustrates, can be easily calculated via iteration. In the past we have consistently found students in the Advance Calculus course capable of calculating traditional limits, but lacking the basic conceptual understanding. We have conjectured that this may be the result of not exposing these students to a numerical approach for determining limits.

In this section we show examples to illustrate that recursion can be used to rapidly analyze the local and global behavior of a function.

Example 3. Explore the behavior of $f(x)=\dfrac{|x-2|}{\left|\sqrt{3-x}-1\right|}$ as $x\to 2$.

*Solution*. With precalculus students that have not had much numerical experience, we have found it advisable to start analyzing numerically the behavior of functions near a given value using sequences that converge to the value from the left and from the right via the *Table.* Initially, as seen in figure 2.r, we use fast sequences such as $p\pm 10^{-n}$. later on, we ask students to use slower sequences such as $p\pm 2^{-n}$. It seems that this arbitrary approach to a given real number from either side, so easy to implement now thanks to the data type *Table*, helps students to better grasp the ideas of approaching to a given value.




| x | f2(x):= abs(x−2)/(abs(√(3−x)−1)) |
|---|---|
| 1.9 | 2.04880884817 |
| 1.99 | 2.00498756211 |
| 1.999 | 2.00049987491 |
| 1.99999 | 2.00000500001 |
| 1.999999 | 2.0000004 |
| 1.9999999 | 2. |
| 2.1 | 1.94868329805 |
| 2.01 | 1.99498743711 |
| 2.001 | 1.99949987493 |
| 2.0001 | 1.99994999885 |
| 2.00001 | 1.99999500001 |
| 2.000001 | 1.99999948 |
| 2.0000001 | 2. |

Figure 2

Students seem to have no difficulty using the table to analyze the behavior of a function at a given value *a*, by observing the successive values of the function for each term of these sequences converging to *a* from either side, as seen in Figure 3.r. It is always important to call the students' attention to the fact that the graph of this function (Figure 2.l) seems to be continuous at 2, yet this value is not in its domain.

Once students have had enough "hands on" experience using the *Table*, is easy to introduce the use of a recursively defined sequences to analyze the local behavior of a function. For convenience, as seen in Figure 4, instead of approaching 2 first from one side and then from the other, we have chosen to display, via a list, the two sequences $f(2\pm 10^{-n})$ obtained when $x\to 2$ from both sides.

Our next example deals with the global behavior of a function.

Example 4. Investigate the behavior of $f(x)=\dfrac{2x-3}{\sqrt{3x^2+5}}$ as $x\to\pm\infty$.

*Solution*. To study the global behavior of *f(x)* as *x* approaches infinity, we let *x* take on the terms of any divergent sequence. Figure 3.c shows that, in this case, we have chosen

$\pm\{10^n\}$. The students can compare their answer with the decimal value of the closed form solutions $\pm 2/\sqrt{3}$.

| | |
|---|---|
| Define $f(x)=\dfrac{\lvert x-2\rvert}{\lvert\sqrt{3-x}-1\rvert}$ | *Done* |
| $1 \to n$ | 1. |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.04880884817,1.94868329805} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.00498756211,1.99498743711} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.00049987491,1.99949987493} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.00004999725,1.99994999885} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.00000500001,1.99999500001} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.0000004,1.99999948} |
| $\frac{n}{10} \to n:\{f(2-n),f(2+n)\}$ | {2.,2.} |
| | 9/99 |

Figure 3

In Figure 4.l, the students can see that the graph of the function approaches the horizontal asymptotes $y = \pm 2/\sqrt{3}$. The middle column in the table (Figure 4.r) shows that as $x \to \infty$ $(2/\sqrt{3}) - f(x) \to 0$, confirming numerically that the function approaches arbitrarily $2/\sqrt{3}$ as $x \to \infty$. Finally, after initializing *n*, the command line that is iterated consists simply in multiplying *n* by *10* and evaluating the function at the new value of *n* obtained (Figure 4.b).

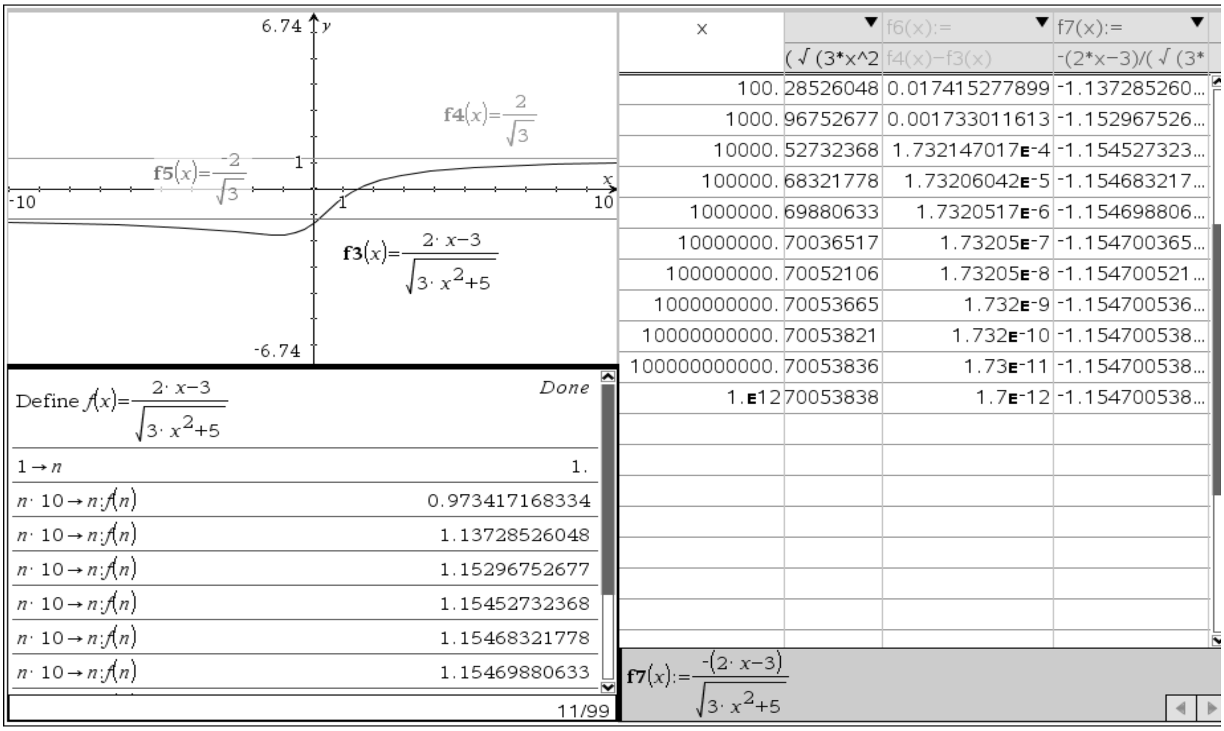


Figure 4

A word of caution is in order, namely, some students will easily exceed the precision of the calculator or computer when using this approach with the subsequent truncation errors. This, of course, provides an excellent opportunity to reinforce both the importance of the precision of the tool being used, as well as the inherent limitations of technology, however advanced it may be.

When the algebraic approach to the study of limits is formally introduced in calculus, students who have been previously exposed to the numerical approach, use it to get a sense of what the answer should be and/or to test their solution. It has been reported that the ability to double-check their answers seems to be of utmost importance to the students (Quesada, 1994).

In addition to analyzing the behavior of individual functions, at the precalculus level, students can start to compare the growth of different functions using this iterative approach (Demana & Waits, 1993). They may investigate for example the behavior of $x^6 / e^x$ as $x \to \infty$, and determine which of the two functions grow faster.

## IV. Iterative methods to Solve Equations

No selection of basic recursive procedures would be complete without a reference to solving equations via recursion. For the sake of brevity we have chosen only Newton's algorithm and the bisection's method because of their opposite converging speeds.

Example 5. Solve $x = \sin(x - \pi / 4)$ using: i) Newton's algorithm with an error $\varepsilon < 10^{-5}$; and ii) the bisection algorithm.

*Solution*. As shown in Figure 5.l we consider the function $f(x) = x - \sin(x - \pi / 4)$. First, a starting value close to the solution sought is determined by looking to its graph, in this case the value *1.2* is chosen to initialize $x_1$. Then its derivative is calculated, and the command line following the traditional algorithm is written as $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}, x_2 \to x_1$. The added instruction $x_2 - x_1 \to \varepsilon$, allows the user to stop when the precision sought is reached. Students appreciate how fast this method converges when they compared with any of the other basic methods (Picard, secant,…), but in any case it is important to remark that as seen in Figure 5.l the result does not change after only four steps, i.e., when for the 1$^{st}$ time all the decimals of the solution remain unchanged.

A group of honor students can be challenged to do a recursive program for the bisection method. As before a small interval, (-1.2, 0), containing the root sought is determined graphically. Then the coordinate *m* of the midpoint of the starting interval is determined, and the “if-then-else” command, available in the TI-Nspire, is used to decide in which endpoint of the interval in turn, the function has the same sign as *f(m)* (Figure 5.r). This end point is then replaced with *m*.

In the case of the TI-84, where inequalities are evaluated with *0=false* or *1=true*, one can take advantage of the Boolean constant *k* to decide which half of the current subinterval contains the zero using the command line:

*((l+r)/(2))→m:f(m)*f(l)>0→k:k*r+(1-k)*m→r:k*l+(1-k)*l→l:{l,r,r-l}*

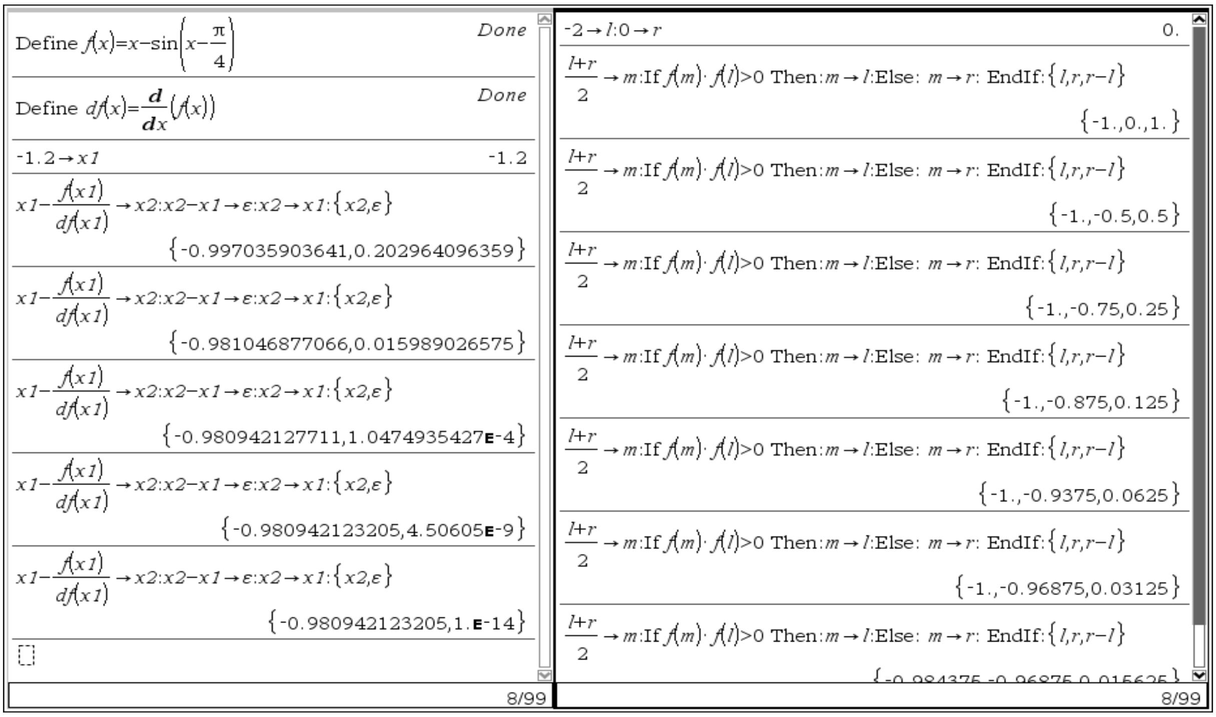


Figure 5

## V. On Modeling Population Growth

Example.

1. It has been estimated that the rate of growth of the population in a southern Canadian town during the last ten years has been of *6%* per year. If the town currently has $20,000$ members, what will be its size in four years assuming the same rate of growth? When will the population double?

2. After three years of steady growth, some changes on the environment reduce the growth to 4%, what will be the size of the population one year later?
3. Suppose that in addition to the initial population of 20,000 and rate of growth of 6% an average of 600 immigrants is settling in the community at the end of every year. What will the population be in three years?
4. Let the initial values of the population, the rate of growth and the amount of incoming immigrants be unchanged. If now the rate of growth begins to increase at a rate of *0.25%* every year, what will the population be in four years? If in addition to an increase in the population rate, an annual decrease of *3%* on the average of immigrants is expected, what will the population be in ten years?
5. Finally, if all the conditions in part four remain, estimate how long will the population take to reach between 110,000 and 120,000.

*Solution*. As figure 6.a illustrates, in a simple recursive approach, the variable *Ans* (that saves the result of the last calculation performed) can be used to store the amount at the end of each step; this becomes the basis for calculating the result in the next step. The process, as seen in figure 6.a, can be refined to include the variables $p$ for population and $t$ for time. After these variables are initialized, the body of the recursive process is inputted as the line of commands. It consists of the equations to update the values of the variables. The last calculation shows that it will take 12 years for the population to double, which confirms the rule of 72.

The answer to the second question, found in figure 6.b, illustrates how the discrete nature of the process facilitates a recall from the stack of commands (via copy and paste in the TI-Nspire) and an update of the rate of growth in the recursive equations before continuing the iteration.

| | |
|---|---|
| 2000 | 2000 |
| 2000·1.06 | 2120. |
| 2120.·1.06 | 2247.2 |
| 2247.2·1.06 | 2382.032 |
| $20000 \to p:0 \to t$ | 0 |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {1,21200.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {2,22472.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {3,23820.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {4,25250.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {5,26765.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {11,37966.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {12,40244.} |
| | 12/9 |

Figure 6.a

| | |
|---|---|
| $20000 \to p:0 \to t$ | 0 |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {1,21200.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {2,22472.} |
| $p\cdot 1.06 \to p:t+1 \to t:\{t,p\}$ | {3,23820.} |
| $p\cdot 1.04 \to p:t+1 \to t:\{t,p\}$ | {4,24773.} |
| $20000 \to p:0 \to t$ | 0 |
| $p\cdot 1.06+600 \to p:t+1 \to t:\{t,p\}$ | {1,21800.} |
| $p\cdot 1.06+600 \to p:t+1 \to t:\{t,p\}$ | {2,23708.} |
| $p\cdot 1.06+600 \to p:t+1 \to t:\{t,p\}$ | {3,25730.} |
| | 9/9 |

Figure 6.b

Similarly, to answer the third question, the initialization command is first recalled and executed, then the body of recursive equations is recalled and the amount of yearly immigrants, namely 600, is added to the calculation of $p$ (see 2$^{nd}$ part of figure 6.b).

| | |
|---|---|
| $20000 \rightarrow p:0 \rightarrow t:0.06 \rightarrow r$ | 6.E-2 |
| $p\cdot(1+r)+600 \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:\{t,p,r\}$ | $\{1,21800.000,0.063\}$ |
| $p\cdot(1+r)+600 \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:\{t,p,r\}$ | $\{2,23762.500,0.065\}$ |
| $p\cdot(1+r)+600 \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:\{t,p,r\}$ | $\{3,25907.063,0.068\}$ |
| $p\cdot(1+r)+600 \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:\{t,p,r\}$ | $\{4,28255.789,0.070\}$ |
| $20000 \rightarrow p:0 \rightarrow t:0.06 \rightarrow r:600 \rightarrow a$ | 600 |
| $p\cdot(1+r)+a \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:a\cdot 0.97 \rightarrow a:\{t,p,r,a\}$ | $\{1,21800.000,0.063,582.000\}$ |
| $p\cdot(1+r)+a \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:a\cdot 0.97 \rightarrow a:\{t,p,r,a\}$ | $\{2,23744.500,0.065,564.540\}$ |
| $p\cdot(1+r)+a \rightarrow p:t+1 \rightarrow t:r+0.0025 \rightarrow r:a\cdot 0.97 \rightarrow a:\{t,p,r,a\}$ | $\{3,25852.433,0.068,547.604\}$ |
| | 9/99 |

Figure 7

The first part of question four requires the introduction of a new variable *r* for the changing rate of growth. Figure 1.c illustrates the initialization of *r* and the necessary updates for the recursive equations. Next, the amount of immigrants becomes variable in the second part of question four, hence, a new variable *a* and the corresponding updates for *a* in the recursive equations are introduced (Figure 1.c).

Clearly, the equation depicting the solutions to the last part of the problem will be complicated for secondary. However, it is possible to get an estimate of when the population will reach some given amount. For this, we look at the scatter plot of a set of the (time, population) values obtained and determine the regression curve that best fit the data, in this case as expected is an exponential function (Figure 8.t). Finally, as seen in Figure 8.b, we have estimated the answer graphically by intersecting *y=115,000* with the graph of the exponential function obtained.

Several remarks are in order. First, the simplicity of this approach is striking when contrasted with the algebraic equations traditionally needed to find the solution, particularly of question four. Any doubts about this can be easily overcome by obtaining the equations needed for question four, and then comparing the level of algebraic sophistication used with that needed to obtain the recursive solution using this approach. Secondly, it has been said that to be able to write a program to solve a problem, one must have a good grasp of the solution. In order to solve a problem recursively, the student needs to identify the variables involved, their initial values, and the rules by which they change; just as they would need in order to write a program but with a minimal amount of syntax. Finally, the discrete nature of the iterative process facilitates the analysis, testing and modification of the solution.

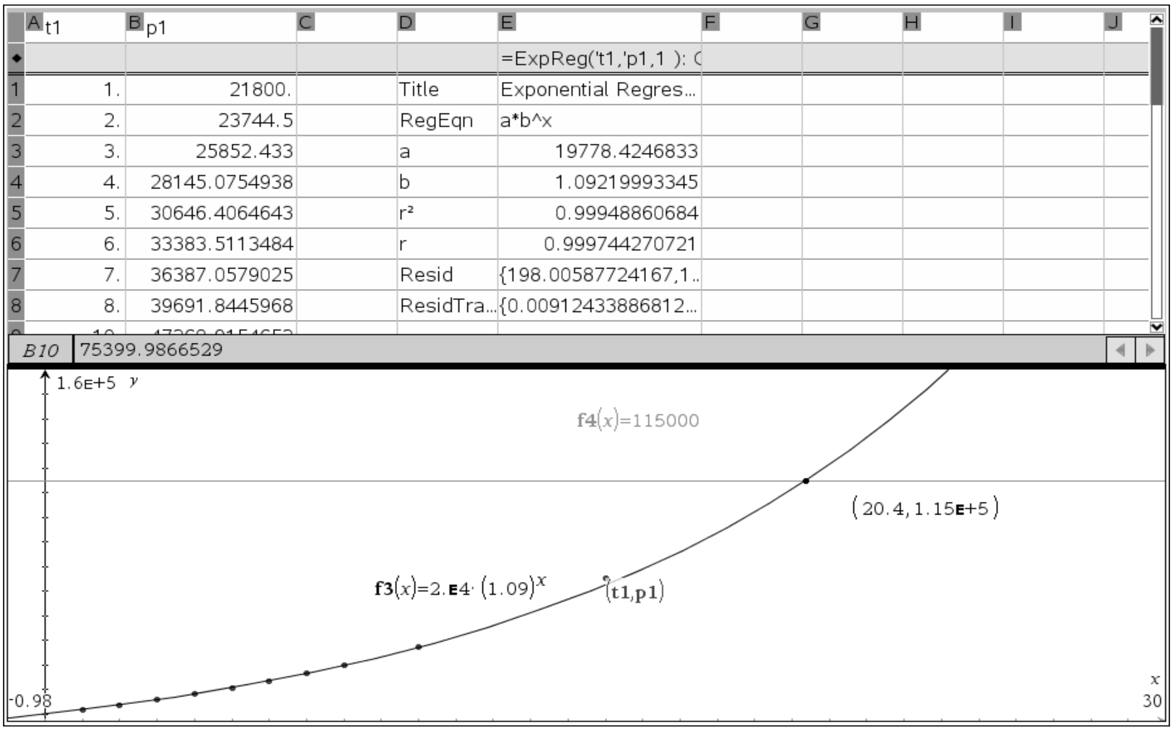


Figure 8

This approach can be used to analyze the growth of any population; but, it is not surprising that many secondary students are particularly interested when studying the growth of money under compound interest. As in the previous problem one may introduce variable yearly deposits as well as changing interest rates.

## VI. Conclusion

The examples shown in this paper were selected to give a sense of the possibilities for using iteration and recursion throughout the mathematics curriculum at different levels. They certainly do not exhaust the topic; recursive solutions to systems of equations, Markov chains, recursive relations, fractals, and many others, can be easily found through the curriculum.

Familiarity with iterative/recursive techniques will provide students with four main benefits. First of all versatility, since as we have seen, these tools can be used to solve a great variety of problems in different areas. Secondly, the accessibility to powerful models traditionally taught at upper levels to students of basic mathematics. Thirdly, as we saw in the last example, they can provide a simpler alternate problem solving approach. Last but not least, it is a good pedagogical tool since allows to easily analyze, test, and modify a solution.

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