Integrating Certain Products without Using Integration by Parts

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Abstract: In this paper, we will describe a novel method of integrating certain products without using the integration by parts formula. In a calculus class, the standard method of integrating products of functions such as polynomials, exponential functions, logarithms, and trigonometric functions is to use the integration by parts formula. Of course, every instructor teaches that this formula is equivalent to the well-known product rule for differentiation of functions. However, when a student uses integration by parts, the idea behind the product rule usually gets lost during the execution of the method. This probably happens because the student is trying too hard to concentrate on the actual mechanics of the integration by parts formula. In this paper, we will show how to use the product rule for differentiation to integrate a variety of products. We will also use the computer algebra system *Mathematica* to verify our results, and also to gain some new insights. What we have used is *Mathematica* version 7.0 on a *Windows 7* platform, but any other computer algebra system of reader's choice can be used for the purposes of the paper.

1. Introduction

In the opening section of the paper, we will review the traditional integration by parts formula in calculus (see [4], [7], and [8]). For suitable function *u* and *v* of variable *x*, this formula states the following:

$$\int u\,dv = uv - \int v\,du \tag{1.1}$$

The above formula (1.1) is traditionally used to integrate certain products of elementary functions in calculus. These functions include polynomials, exponential functions, logarithms, trigonometric functions, inverse trigonometric functions, hyperbolic functions, and inverse hyperbolic functions. Just to illustrate this method, let us consider the problem of integrating the function $x e^{2x}$.

Example 1.1. Calculate $\int x e^{2x} dx$ using integration by parts.

In order to match the given integral with the left-hand side of formula (1.1), one must make choices for *u* and *dv*. In general, when choosing *dv*, we try to pick an easier function to integrate. So in this example, either $dv = x\,dx$ or $dv = e^{2x}\,dx$ seems to be viable choices at the beginning. However, if we choose $dv = x\,dx$, then after integration we obtain $v = x^2/2$, thus increasing the original power of *x*, making the problem more difficult. Therefore, the correct choice for *dv* is $dv = e^{2x}\,dx$. This also means that $u = x$.

With these choices $u = x$ and $dv = e^{2x}\,dx$, one would obtain $du = dx$ and $v = e^{2x}/2$. Thus the formula (1.1) implies

$$\int xe^{2x}dx \;=\; \int u\,dv \;=\; uv - \int v\,du \;=\; \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx \;=\; \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

where C is an arbitrary constant.

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One difficulty of this method is that the student has to differentiate and integrate within the same problem, thus leading to some confusion. Every instructor teaches that the integration by parts formula is equivalent to the following product rule for differentiation in calculus, where *u* and *v* are differentiable functions of *x* (see [4], [7], and [8]).

$$\frac{d}{dx}(u\,v) \;=\; u\frac{dv}{dx} + \frac{du}{dx}v \tag{1.2}$$

However, the mechanics of the integration by parts formula really hides its underlying foundation, i.e. the product rule (1.2) of calculus. In this paper, we will reveal a method of integrating certain products of elementary functions, *directly* using this product rule. The next section shows how to use this new method to integrate products of polynomials and exponential functions.

2. Integrating Products of Polynomials and Exponential Functions

Let us reconsider the Example 1.1 in the previous section. However, this time we want to avoid using the integration by parts formula (1.1)!

Example 2.1. Calculate $\int x\,e^{2x}dx$ without using integration by parts.

We will first consider the problem of differentiating the function $x\,e^{2x}$ with respect to *x*, using the product rule (1.2):

$$\frac{d}{dx}(xe^{2x}) \;=\; x.2e^{2x} + e^{2x} \tag{2.1}$$

Now integrate both sides of equation (2.1) with respect to *x*.

$$x\,e^{2x} \;=\; 2\int x\,e^{2x}dx + \int e^{2x}dx \tag{2.2}$$

The equation (2.2) can now be solved for $\int x\,e^{2x}dx$ using simple algebra:

$$\int x\,e^{2x}dx \;=\; \frac{1}{2}x\,e^{2x} - \frac{1}{2}\int e^{2x}dx \;=\; \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

The above agrees with the answer obtained from the traditional integration by parts method, as indicated in Example 1.1 of the previous section.

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We will now consider how to integrate products of higher powers of x and exponential functions.

Example 2.2. Calculate $\int x^2 e^{2x}\, dx$ without using integration by parts.

Motivated by Example 2.1, let us differentiate the function $x^2 e^{2x}$ with respect to x: The product rule (1.2) yields

$$\frac{d}{dx}(x^2 e^{2x}) = 2x^2 e^{2x} + 2x e^{2x} \tag{2.3}$$

At this point, we can integrate both sides of equation (2.3) with respect to x, and use the result of the Example 2.1 for $\int x e^{2x}\, dx$. However, let us proceed independently as follows. Let us now consider the derivative of xe^{2x}, the last component of the right-hand side of equation (2.3):

$$\frac{d}{dx}(xe^{2x}) = 2xe^{2x} + e^{2x} \tag{2.4}$$

Subtract the equation (2.4) from (2.3) to obtain:

$$\frac{d}{dx}(x^2 e^{2x}) - \frac{d}{dx}(xe^{2x}) = 2x^2 e^{2x} - e^{2x} \tag{2.5}$$

Now integrate both sides of equation (2.5) to obtain:

$$x^2 e^{2x} - xe^{2x} = 2\int x^2 e^{2x} dx - \int e^{2x} dx \tag{2.6}$$

One can now solve equation (2.6) for $\int x^2 e^{2x} dx$ to obtain the desired result:

$$\int x^2 e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C \tag{2.7}$$

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It is possible to check the result given in equation (2.7) by standard integration by parts method or by using a computer algebra system (CAS) such as *Mathematica* (see [3], [6], and [9]). For example, the "**Integrate**" command of *Mathematica* can be used to calculate the integral in the above example:

Input: Integrate[x^2*Exp[2x],x]

The output can be obtained by pressing "**Shift-Enter**" at any place in the command line (see [3] and [9]). The output is as follows:

Output: 1/4 E^(2 x) (1-2 x+2 x^2)

One can observe that the above output is the same as the one given in equation (2.7) above.

As the final example of this section, let us consider higher powers of *x*, even higher than that of Example 2.2.

Example 2.3. Calculate $\int x^4 e^x \, dx$ without using integration by parts.

In this case, there are *four* equations that correspond to equations (2.3) and (2.4) of Example 2.2.

$$\frac{d}{dx}(x^4 e^x) = x^4 e^x + 4x^3 e^x \tag{2.8}$$

$$\frac{d}{dx}(x^3 e^x) = x^3 e^x + 3x^2 e^x \tag{2.9}$$

$$\frac{d}{dx}(x^2 e^x) = x^2 e^x + 2x e^x \tag{2.10}$$

$$\frac{d}{dx}(x e^x) = x e^x + e^x \tag{2.11}$$

Since we only want the integral of $x^4 e^x$, we have to eliminate all $x^3 e^x, x^2 e^x$, and xe^x terms from the right-hand sides of equations (2.8)-(2.11). We propose more than one way of doing this:

First approach is elementary, and is suitable for any level of calculus student: This involves adding four equations together, after multiplying each one by a suitable constant. In particular, first leave equation (2.8) unchanged. Then multiply equation (2.9) by -4 and call it equation $(2.9)'$. Next, multiply equation (2.10) by 12 and call it equation $(2.10)'$. Finally, multiply equation (2.11) by -24 and call it equation $(2.11)'$. Now add all the equations (2.8), $(2.9)'$, $(2.10)'$, and $(2.11)'$ together to obtain

$$\frac{d}{dx}(x^4 e^x) - 4\frac{d}{dx}(x^3 e^x) + 12\frac{d}{dx}(x^2 e^x) - 24\frac{d}{dx}(x^4 e^x) = x^4 e^x - 24 e^x \tag{2.12}$$

Finally, integrate both sides of equation (2.12) with respect to *x*, and solve for $\int x^4 e^x dx$ to obtain the required result:

$$\int x^4 e^x \, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x^4 e^x + 24 e^x + C \tag{2.13}$$

Second approach of elimination uses some matrix algebra, and might not be suitable for the beginning calculus student. In order to do this, first rewrite the equation (2.11) by moving the e^x term to the left-hand side:

$$\frac{d}{dx}(xe^x - e^x) \;=\; xe^x \tag{2.14}$$

One can now rewrite the equations (2.8), (2.9), (2.10), and (2.14) in the following matrix form (see [1], [2], and [5]):

$$\begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^4e^x \\ x^3e^x \\ x^2e^x \\ xe^x \end{pmatrix} = \begin{pmatrix} d(x^4e^x)/dx \\ d(x^3e^x)/dx \\ d(x^2e^x)/dx \\ d(xe^x - e^x)/dx \end{pmatrix} \tag{2.15}$$

In the above equation (2.15), let us denote the $4X4$ matrix by $\boldsymbol{A}$, the $4X1$ column vector on the left-hand side by $\boldsymbol{X}$, and the $4X1$ column vector on the right-hand side by $\boldsymbol{b}$. Our matrix $\boldsymbol{A}$ is a special triangular matrix, called a bidiagonal matrix, and it is invertible since each of its main diagonal entries is nonzero (see [1], [2], and [5]). Therefore, multiplying both sides of equation (2.15) by $\boldsymbol{A}^{-1}$, we obtain:

$$\boldsymbol{X} \;=\; \boldsymbol{A}^{-1}\boldsymbol{b} \tag{2.16}$$

We can calculate $\boldsymbol{A}^{-1}$ either by hand, or by using "**Inverse**" command of *Mathematica* (see [3] and [9]). At this point one might argue why not use the "**Integrate**" command of *Mathematica* to calculate the integral $\int x^4 e^x \, dx$ at once. However, bear in mind that the purpose of this investigation is to illustrate a novel approach, and the aim is far from getting the answer quickly. This matrix approach also illustrates alternate ways of using a modern CAS.

The following *Mathematica* commands calculates $\boldsymbol{A}^{-1}$ and displays the output in a row format:

Input: mat={{1,4,0,0},{0,1,3,0},{0,0,1,2},{0,0,0,1}};
Inverse[mat]

Output: {{1,-4,12,-24},{0,1,-3,6},{0,0,1,-2},{0,0,0,1}}

Therefore, the equation (2.16) now reads as the following:

$$\begin{pmatrix} x^4e^x \\ x^3e^x \\ x^2e^x \\ xe^x \end{pmatrix} = \begin{pmatrix} 1 & -4 & 12 & -24 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d(x^4e^x)/dx \\ d(x^3e^x)/dx \\ d(x^2e^x)/dx \\ d(xe^x - e^x)/dx \end{pmatrix} \tag{2.17}$$

By matching the (1,1) entries on both sides of the equation (2.17) yields

$$x^4e^x \;=\; \frac{d}{dx}(x^4e^x) - 4\frac{d}{dx}(x^3e^x) + 12\frac{d}{dx}(x^2e^x) - 24\frac{d}{dx}(xe^x - e^x) \tag{2.18}$$

Integrating both sides of equation (2.18) with respect to x leads to the required result:

$$\int x^4e^x dx \;=\; x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C \tag{2.19}$$

Note that the answers given by equations (2.13) and (2.19) are identical.

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It appears that the method of integration we are proposing by using a differentiation formula, namely the product rule (1.2), is more natural and intuitive than the traditional integration by parts method. However, in order to see the true value of the new method, let us integrate products of different types of elementary functions.

3. Integrating Products of Polynomials and Trigonometric Functions

In this section we will investigate how to use the new method to integrate products of polynomials and trigonometric functions such as $Sin(x)$ and $Cos(x)$.

Example 3.1. Calculate $\int x^2Sin(3x)\,dx$ without using integration by parts.

By following the method indicated in Example 2.1, the question is whether we should consider the derivative of $x^2Sin(3x)$ or that of $x^2Cos(3x)$. It must be clear that we should consider the derivative of $x^2Cos(3x)$, because the derivative of a cosine term gives a sine term, and a part of the integrand contains a sine term.

$$\frac{d}{dx}(x^2Cos(3x)) \;=\; -3x^2Sin(3x) + 2xCos(3x) \tag{3.1}$$

In a moment, we want to integrate both sides of equation with respect to x, to obtain the required integral of $x^2Sin(3x)$. However, in the process, because of the last term of the equation (3.1), we run in to $\int xCos(3x)\,dx$. Therefore, let us consider the derivative of the function $xSin(3x)$ first:

$$\frac{d}{dx}(x\,Sin(3x)) \;=\; 3x\,Cos(3x) + Sin(3x) \tag{3.2}$$

One can now eliminate the $xCos(3x)$ term between the right-hand sides of equations (3.1) and (3.2). For example, multiply equation (3.1) by 3, and add it to equation obtained by multiplying equation (3.2) by -2, yielding the following:

$$3\frac{d}{dx}(x^2Cos(3x)) - 2\frac{d}{dx}(xSin(3x)) \;=\; -9x^2Sin(3x) - 2Sin(3x) \tag{3.3}$$

Now integrate both sides with respect to *x*, to obtain:

$$3x^2 Cos(3x) - 2xSin(3x) \;=\; -9\int x^2 Sin(3x)\,dx \;-\; 2\int Sin(3x)\,dx \tag{3.4}$$

The above equation can now be solved for $\int x^2 Sin(3x)\,dx$ to obtain the desired result, where *C* is an arbitrary constant:

$$\int x^2 Sin(3x)\,dx \;=\; -\frac{1}{3}x^2 Cos(3x) \;+\; \frac{2}{9}x\,Sin(3x) \;+\; \frac{2}{27}Cos(3x) \;+\; C \tag{3.5}$$

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As before, the result given in equation (3.5) can be verified by using the "**Integrate**" command of *Mathematica*:

Input: Integrate[x^2*Sin[3x], x]

Output: -(1/27) (-2 + 9 x^2) Cos[3 x] + 2/9 x Sin[3 x]

After some simplification, it can be seen that the above *Mathematica* output agrees with the answer given by equation (3.5).

4. Integrating Products of Exponential Functions and Trigonometric Functions

In this section, we will consider the problem of integrating products of exponential functions and trigonometric functions such as $Sin(x)$ and $Cos(x)$ using the new method. For this particular situation, the new method seems to be more efficient and natural than the traditional integration by parts method.

Example 4.1. Calculate $\int e^{2x} Sin(3x)\,dx$ without using integration by parts.

Before demonstrating our non-standard method for this problem, it is important for the reader to recall the traditional method using integration by parts (see [4], [7] and [8]). The traditional method uses the formula (1.1) twice. At the outset, you can make either choice for *dv*, i.e. $dv = e^{2x}dx$ or $dv = Sin(3x)\,dx$. Then, if you choose $dv = e^{2x}\,dx$ in the first round of integration, the trick is to remember to choose the same choice $dv = e^{2x}dx$ in the second round of integration as well. Likewise, if you choose $dv = Sin(3x)\,dx$ in the first round of integration, you must make the similar choice $dv = Cos(3x)\,dx$ in the second round of integration (see [7] and [8]). These different choices might be overwhelming to the beginning student. In contrast, the new method, as illustrated below, seems to be a much more natural and direct method of integration:

First consider the derivative of the function $e^{2x} Sin(3x)$ with respect to *x*:

$$\frac{d}{dx}(e^{2x}Sin(3x)) \;=\; 3e^{2x}Cos(3x) + 2e^{2x}Sin(3x) \tag{4.1}$$

Now consider the derivative of the function $e^{2x}Cos(3x)$ with respect to *x*:

$$\frac{d}{dx}(e^{2x}Cos(3x)) \;=\; 2e^{2x}Cos(3x) - 3e^{2x}Sin(3x) \tag{4.2}$$

Since we want to find the integral of $e^{2x}Sin(3x)$, let us eliminate the $e^{2x}Cos(3x)$ term between the equations (4.1) and (4.2). In other words, we are treating the equations (4.1) and (4.2) like a 2X2 system with "variables" $e^{2x}Cos(3x)$ and $e^{2x}Sin(3x)$. Such systems can be solved by either the addition method, the elimination method, or by matrix methods (see [1], [2], and [5]). For example, multiply equation (4.1) by 2, and add it to the equation obtained by multiplying equation (4.2) by -3:

$$2\frac{d}{dx}(e^{2x}Sin(3x)) - 3\frac{d}{dx}(e^{2x}Cos(3x)) \;=\; 13e^{2x}Sin(3x) \tag{4.3}$$

Now integrate both sides of equation (4.3) with respect to *x* to obtain:

$$2e^{2x}Sin\,(3x) - 3e^{2x}Cos(3x) \;=\; 13\int e^{2x}Sin(3x)\,dx \tag{4.4}$$

Now solve the above equation (4.4) for $\int e^{2x}Sin(3x)\,dx$ to get the desired result:

$$\int e^{2x}Sin(3x)\,dx \;=\; \frac{1}{13}e^{2x}[2Sin(3x) - 3Cos(3x)] + C \tag{4.5}$$

It is very interesting to observe that as a by-product of our method, we can also calculate $\int e^{2x}Cos(3x)\,dx$. This is a significant advantage compared to the traditional method of integration by parts. In order to calculate $\int e^{2x}Cos(3x)\,dx$, eliminate the $e^{2x}Sin(3x)$ term between the equations (4.1) and (4.2) to obtain

$$3\frac{d}{dx}(e^{2x}Sin(3x)) + 2\frac{d}{dx}(e^{2x}Cos(3x)) \;=\; 13e^{2x}Cos(3x) \tag{4.6}$$

Integrating both sides with respect to *x* yields

$$\int e^{2x}Cos(3x)\,dx \;=\; \frac{1}{13}e^{2x}[3Sin(3x) + 2Cos(3x)] \;+\; C \tag{4.7}$$

Both integrals given by equation (4.5) and (4.7) can be verified by *Mathematica* to be correct.

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In the next section, we will consider how to utilize the new method to integrate odd powers of secant and cosecant functions.

5. Integrating Odd Powers of Secant and Cosecant Functions

Let us consider the problem of integrating $Sec^3(x)$:

Example 5.1. Calculate $\int Sec^3 x\, dx$ without using integration by parts.

First note that $Sec^3(x) = Sec(x).Sec^2(x)$, and $Sec^2(x)$ is the derivative of $Tan(x)$. Thus, according to the new method, we will consider the derivative of the function $Sec(x)Tan(x)$ with respect to x:

$$\frac{d}{dx}(Sec(x)Tan(x)) \;=\; Sec(x)Sec^2(x) \;+\; Sec(x)Tan(x).Tan(x) \tag{5.1}$$

Simplify the right-hand side of equation (5.1) to obtain

$$\frac{d}{dx}(Sec(x)Tan(x)) \;=\; Sec^3(x) + Sec(x)[Sec^2(x) - 1] \;=\; 2Sec^3(x) - Sec(x) \tag{5.2}$$

One can now integrate both sides of equation (5.2) to obtain

$$Sec(x)Tan(x) \;=\; 2\int Sec^3(x)\, dx - \int Sec(x)\, dx \tag{5.3}$$

Note that $\int Sec(x)\, dx \;=\; \ln|\, Sec(x) + Tan(x)\,| + C_1$ where C_1 is an arbitrary constant (see [7] and [8]). Using this, one can solve the above equation (5.3) for $\int Sec^3(x)\, dx$ to obtain the following, where C is an arbitrary constant:

$$\int Sec^3 x\, dx \;=\; \frac{1}{2}Sec(x)Tan(x) \;+\; \frac{1}{2}\ln|\, Sec(x) + Tan(x)\,| \;+\; C \tag{5.4}$$

For this example, the new method of integration seems to be a bit less complicated than the traditional method using integration by parts. The reader is advised to work out the same problem using integration by parts, in order to compare the new method with the old method (see [7] and [8]).

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The odd powers of cosecant functions can be handled in a very similar fashion. In order to better understand the mew method of integration using the product rule, the reader is invited to attempt the following exercise:

Exercise 5.1. Calculate the following integrals without using integration by parts:

(a) $\int Sec^5(2x)\, dx$ (b) $\int Co\sec^3(4x)\, dx$

6. Integrating Certain Three-fold Products of Elementary Functions

Let us consider the following problem of integrating the product of x, e^x and $Sin(x)$:

Example 6.1. Calculate $\int x e^x \, Sin(x) dx$ without using integration by parts.

First, consider the derivative of $xe^x Sin(x)$, using the product rule given in (1.2):

$$\frac{d}{dx}(xe^x Sin(x)) \;=\; xe^x Cos(x) \;+\; xe^x Sin(x) \;+\; e^x Sin(x) \tag{6.1}$$

Secondly, consider the derivative of $xe^x Cos(x)$:

$$\frac{d}{dx}(xe^x Cos(x)) \;=\; -xe^x Sin(x) \;+\; xe^x Cos(x) \;+\; e^x Cos(x) \tag{6.2}$$

Thirdly, consider the derivative of $e^x Cos(x)$:

$$\frac{d}{dx}(e^x Cos(x)) \;=\; e^x Cos(x) \;-\; e^x Sin(x) \tag{6.3}$$

Since we want the integral of $xe^x Sin(x)$, but not the integral of $xe^x Cos(x)$, we must eliminate the $xe^x Cos(x)$ term between the equations (6.1) and (6.2). Therefore, subtract the equation (6.1) from (6.2) to obtain

$$\frac{d}{dx}(xe^x Cos(x)) \;-\; \frac{d}{dx}(xe^x Sin(x)) \;=\; -2xe^x Sin(x) \;+\; e^x Cos(x) \;-\; e^x Sin(x) \tag{6.4}$$

However, the last two terms of the right-hand side of equation (6.4) are precisely given by equation (6.3). After making the substitution, one obtains

$$\frac{d}{dx}(xe^x Cos(x)) \;-\; \frac{d}{dx}(xe^x Sin(x)) \;=\; -2xe^x Sin(x) \;+\; \frac{d}{dx}(e^x Cos(x)) \tag{6.5}$$

Now integrate both sides of equation (6.5) to obtain

$$xe^x Cos(x) \;-\; xe^x Sin(x) \;=\; -2\int xe^x Sin(x) dx \;+\; e^x Cos(x) \tag{6.6}$$

One can now solve the equation (6.6) for $\int xe^x Sin(x) dx$ to obtain the required result:

$$\int xe^x Sin(x) dx \;=\; \frac{1}{2} e^x \left[x Sin(x) \;-\; xCos(x) + Cos(x)\right] \;+\; C \tag{6.7}$$

In this example, our new method of integration has a *clear advantage* over the traditional method of integration by parts. Let us verify the accuracy of the above integral (6.7) using the "**Integrate**" command of *Mathematica*:

Input: Integrate[x*Exp[x]*Sin[x], x]

Output: 1/2 E^x (Cos[x] - x Cos[x] + x Sin[x])

The above *Mathematica* output clearly agrees with the result given by equation (6.7).

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In order to sharpen the skills on the new method of integration, the reader is invited to complete the following exercise:

Exercise 6.1. Calculate without using integration by parts:

(a) $\int x e^{x} Cos(2x)\, dx$ (b) $\int x^2 e^{2x} Sin(3x)\, dx$

Conclusion

In this paper, we showed how to integrate a wide variety of functions without using the traditional method of integration by parts. Our method relies on the exclusive use of product rule for differentiation of functions. We do not claim the new method to always be more efficient than the traditional method. However, in many cases, the proposed method seems to be more natural and intuitive than the standard method. We showed how to use modern technology not only to verify our results, but also to find new insights to the existing methods, as illustrated by matrix calculations in Example 2.3. In order to better grasp the new method, the reader is requested to complete the exercises indicated in the paper. Following the spirit of the paper, the reader is also encouraged to carry out more calculations to discover other types of integrals that are suitable for the new method.

References

[1] DeFranza, J and Gagliardi, D. (2009). *Introduction to Linear Algebra with Applications*. New York, NY: McGraw-Hill Companies, Inc.
[2] Goldberg, J. (1991). *Matrix Theory with Applications*. New York, NY: McGraw-Hill Companies, Inc.
[3] Gray, T. and Glynn, J. (2000). *The Beginner's Guide to Mathematica, Version 4*. Cambridge, UK: Cambridge University Press.
[4] Larson, R. (2010). *Calculus of a Single Variable*. Mason, OH: Cengage Publishing
[5] Lay, D. (1994). *Linear Algebra and its Applications*. Redwood City, CA: Addison-Wesley Publishing Company
[6] Maeder, R. (1991). *Programming in Mathematica*. Redwood City, CA: Addison-Wesley Publishing Company
[7] Swokowski, E. (1991). *Calculus, 5th edition*. Boston, MA: PWS-Kent Publishing Company
[8] Thomas, G. (2005). *Calculus, 11th edition*. Upper Saddle River, NJ: Pearson Education.
[9] Wolfram, S. (2003). *Mathematica Book, 5th ed*. Cambridge, UK: Cambridge University Press