How integration of DGS and CAS helps to solve problems in geometry

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Abstract

The use of dynamic geometry systems (DGS) and computer algebra systems (CAS) changed teaching geometry at all school levels considerably. To solve a problem students first visualize it by DGS then by changing parameters the problem is interactively modified and geometry properties like invariant points, lines, circles etc. are recognized. Using these knowledge a conjecture is stated and classically proved or disproved.
But sometimes we do not have a *key idea* to find a proof (or the locus). Then the use of CAS can help. By the theory of automated geometry theorem proving we are able to prove many such theorems.
Thus the integration of DGS and CAS is useful and helps to solve problems. This approach is demonstrated in a few examples of elementary geometry in a plane and space.

1 Introduction

The use of mathematical software has a big influence on methods of teaching mathematics at schools. For example searching for loci of points of given properties belongs to the most difficult problems at school mathematics but new technological tools facilitate it considerably.
In basic geometry courses at the university we use both classical methods and new technologies which contemporary mathematical software offers. To solve a problem we usually start with DGS to demonstrate a geometric situation. Dynamic features of this software allow us to state and consequently verify conjectures. This means that the conjecture is (numerically) verified in infinitely many situations and with a high probability we can say whether the conjecture holds or not. But numerical verification is not infallible. Therefore rigorous mathematical proof is needed. To prove a statement we use a classical proof if it is possible. But sometimes we do not have a *key idea* how to prove it. Then computer algebra systems come on the scene. With the use of CAS we are able to prove most of problems which occur in school mathematics.
It turns out that integration of both systems — CAS and DGS — into one mathematical software, is needed. Such trends are seen in most mathematical software. Nowadays we witness incorporation of some features of CAS (e.g. plotting implicit curves and surfaces, proving theorems) into GeoGebra, on the other hand we see installing of dynamic elements of DGS (e.g. sliders) into CAS.

We will show the utilization of CAS and DGS in proving theorems and derivation of new formulas. We will also demonstrate the use of new technologies in searching for the loci of points of given properties. All these techniques were used in the geometry courses which I held at the university in the last few years [6].

2 The use of DGS helps to solve a problem

The use of dynamic geometry systems helps to solve geometry problems considerably. In DGS we visualize geometric objects and interactively change their parameters. By this we recognize invariant elements like points, lines, circles etc. and another important properties of geometric objects. This knowledge helps us in proving or disproving conjectures. We should realize that DGS work on numerical basis and therefore a rigorous proof is needed. We will demonstrate it in the following example [4]:

Example 1: *Let $ABCD$ be a hinged closed quadrilateral with fixed base AB for which $a = |AD| = |BC|$ and $b = |AB| = |CD|$. Determine the locus of the intersection G of the sides BC and AD when the vertices C and D move into all possible positions if $a < b$ or $a > b$., Fig. 1.*

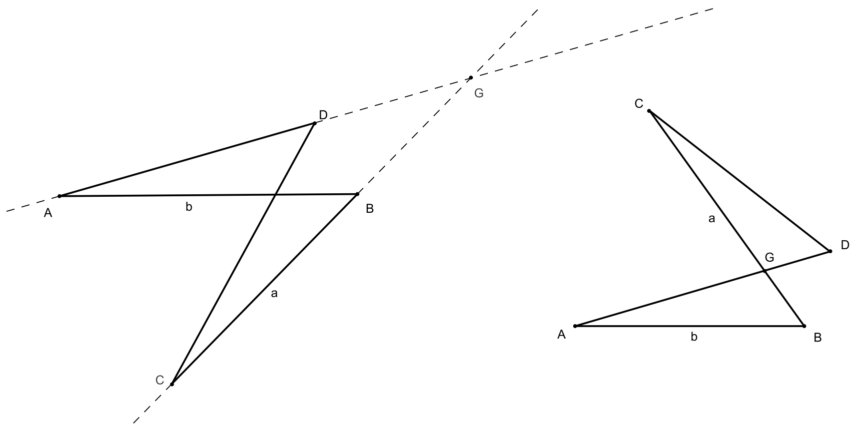

Figure 1: Determine the locus of G
if $a < b$ (left) or $a > b$ (right)

First construct a quadrilateral $ABCD$ of a given properties in GeoGebra. We can move with the vertex D (mover) to achieve infinitely many positions of $ABCD$ and observe the point G (tracer). If $a > b$ it seems that G is moving along a curve which is similar to a circle. Clicking on the icon Locus we find out that the locus of G is similar to an ellipse with foci at A and B. We can state a conjecture:

If $a > b$ the point G moves along an ellipse with foci at A and B.

Now we verify the conjecture in DGS. We will explore the sum of lengths of segments AG and BG, Fig. 2. The sum of $|AG| + |BG|$ is still constant which is in accordance with the definition of an ellipse. Hence with the high probability G moves along an an ellipse.
Classical proof. We take advantage of the knowledge which we obtained using DGS and

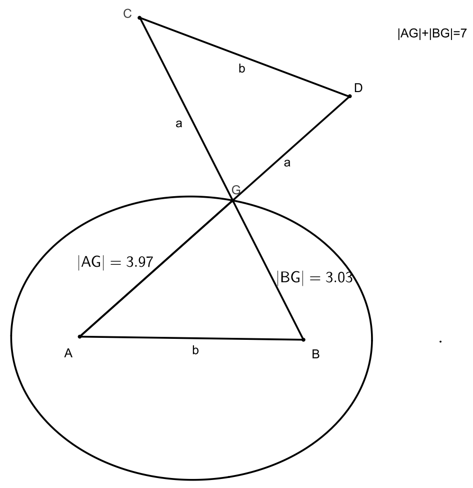

Figure 2: Verification, case $a > b$: G moves along an ellipse with foci A, B.

prove the statement classically. Suppose that $a > b$. Let us explore the sum of lengths $|AG| + |GB|$. We observe that triangles ABG and DCG are congruent as $CABD$ is an isosceles trapezoid with equal legs of the lengths b and diagonals a. Then

$$|AG| + |GB| = |AG| + |GD| = |AD| = a.$$

Hence the locus G lies on the ellipse with foci at A and B and the length of the major axis a.

Now suppose that $a < b$. Observation in GeoGebra tells us that the locus is similar to

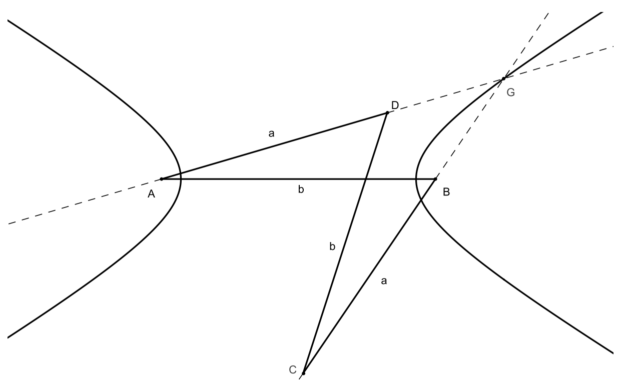

Figure 3: Verification, case $a < b$: G moves along a hyperbola with foci A, B.

hyperbola with foci at A and B, Fig. 3. We can state a conjecture:

If $a < b$ the point G moves along a hyperbola with foci at A and B.

Verification in DGS confirms out conjecture — measuring the value of the difference $|AG| - |GB|$ we find out that it is constant.

Classical proof. Let us explore the value of the expression $|AG| - |GB|$. Observe that the triangle ACG is isosceles with $|AG| = |CG|$. This implies $|GB| = |GD|$. Hence

$$|AG| - |GB| = |AG| - |GD| = |AD| = a.$$

We can conclude that the locus G lies on hyperbola with foci at A and B and the length of the major axis a.

Automated approach. We will find the locus equation. In a rectangular coordinate system we denote $A = [-b/2, 0]$, $B = [b/2, 0]$, $C = [r, s]$, $D = [u, v]$, $G = [x, y]$, Fig. 4. Then the

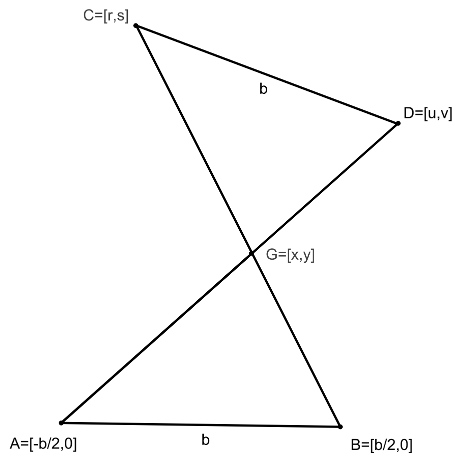

Figure 4: Searching for locus equation

description of the geometric situation by algebraic equations is as follows:

$|BC| = a \Rightarrow h_1 := (r - b/2)^2 + s^2 - a^2 = 0,$

$|CD| = b \Rightarrow h_2 := (u - r)^2 + (v - s)^2 - b^2 = 0,$

$|DA| = a \Rightarrow h_3 := (u + b/2)^2 + v^2 - a^2 = 0,$

$G \in AD \Rightarrow h_4 := uy - vb/2 + yb/2 - xv = 0,$

$G \in BC \Rightarrow h_5 := ry + sb/2 - yb/2 - xs = 0.$

To find the locus equation we need to "get rid" of the variables u, v, r, s in the system of equations $h_1 = 0$, $h_2 = 0$, $\ldots$, $h_5 = 0$. Elimination of u, v, r, s, gives in CoCoA[1]

```
Use R::=Q[u,v,r,s,a,b,x,y];
I:=Ideal((r-b/2)^2+s^2-a^2,(u-r)^2+(v-s)^2-b^2,(u+b/2)^2+v^2-a^2,
uy-vb/2+yb/2-xv,ry+sb/2-yb/2-xs);
Elim(u..s,I);
```

the polynomial which leads to the equation

$a^2b^2y^2(a^4 - a^2b^2 - 4a^2x^2 + 4b^2x^2 - 4a^2y^2) = 0.$

[1]program CoCoA is freely distributed at `http://cocoa.dima.unige.it`

We suppose that $a > 0$, $b > 0$. The case $y = 0$ can also be ruled out since in this case the quadrilateral $ABCD$ degenerates into a segment. Hence we get the only equation

$$4(b^2 - a^2)x^2 - 4a^2y^2 = a^2(b^2 - a^2). \tag{1}$$

We see that (1) is the equation of an ellipse or hyperbola in accordance with the sign of $b^2 - a^2$. If $a > b$ it leads to the equation

$$\frac{x^2}{(\frac{a}{2})^2} + \frac{y^2}{(\frac{\sqrt{a^2-b^2}}{2})^2} = 1$$

of an ellipse, and if $a < b$ we get the equation

$$\frac{x^2}{(\frac{a}{2})^2} - \frac{y^2}{(\frac{\sqrt{b^2-a^2}}{2})^2} = 1$$

of a hyperbola.

Remark: We have seen three kinds of "proofs" — verification in DGS (which is not a proof), classical proof and automated (computer) proof. In the example above we recommend to prove the statement at schools classically with the help of verification in DGS. Automated proof was added only for illustration to complete the methods of proving. Note that automated proof gave us information, that we have to distinguish *two* different cases.

3 Deriving new statements

Derivation of new statements belongs to mighty tools of computer algebra systems. Many theorems were found using derivation. By CAS we can derive new geometric statements which follow from the given assumptions [7]. The method is based on elimination of suitable variables in the system of algebraic equations and inequations which describe a given problem. To obtain the desired elimination ideal we usually use Gröbner bases method [2]. Another well-known method which is often used, is the method Wu–Ritt using characteristic sets [9]. After finding a new formula by CAS we can make sure by DGS that the formula is correct. We will demonstrate this approach in an example from elementary geometry in a plane.
Every student knows that if we draw a plane quadrilateral with side lengths a, b, c, d and diagonals e, f with a ruler and compass, we need to know five elements, for instance a, b, c, d, e. The sixth length f depends on the other lengths. The question is how does f depend on a, b, c, d, e ? We will solve it in the next example:

Example 2: *Given a plane quadrilateral $ABCD$ with side lengths a, b, c, d and diagonals e, f. Determine a relation holding among a, b, c, d, e, f.*

We adopt a rectangular coordinate system such that $A = [0, 0]$, $B = [a, 0]$, $C = [m, n]$, $D = [p, q]$, Fig. 5. Then

$b = |BC| \Rightarrow h_1 : (m - a)^2 + n^2 - b^2 = 0,$

$c = |CD| \Rightarrow h_2 : (p - m)^2 + (q - n)^2 - c^2 = 0,$

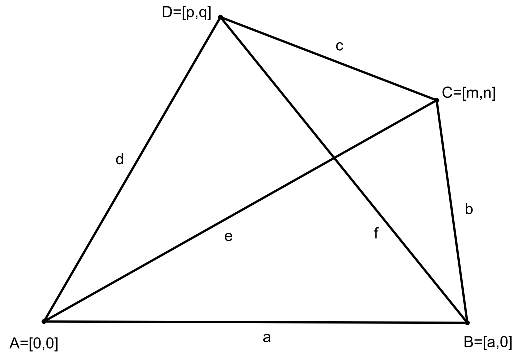

Figure 5: Searching for relation holding among the lengths a, b, c, d, e, f of $ABCD$

$d = |DA| \Rightarrow h_3 : p^2 + q^2 - d^2 = 0,$

$e = |AC| \Rightarrow h_4 : m^2 + n^2 - e^2 = 0,$

$f = |BD| \Rightarrow h_5 : (p - a)^2 + q^2 - f^2 = 0.$

Thus we have a system of five equations $h_1 = 0$, $h_2 = 0$, $\ldots$ $h_5 = 0$ which describe a quadrilateral $ABCD$. In the next step we eliminate coordinates m, n, p, q in this system to obtain an equation in variables a, b, c, d, e, f. In CoCoA we enter

```
Use R::=Q[a,b,c,d,e,f,m,n,p,q);
I:=Ideal((m-a)^2+n^2-b^2,(p-m)^2+(q-n)^2-c^2,p^2+q^2-d^2,m^2+n^2-e^2,
(p-a)^2+q^2-f^2);
Elim(m..q,I);
```

and get the only polynomial which leads to the equation

$$a^4c^2 - a^2b^2c^2 - a^2b^2d^2 + a^2b^2e^2 + a^2c^4 - a^2c^2d^2 - a^2c^2e^2 - a^2c^2f^2 + a^2d^2f^2 - a^2e^2f^2 + b^4d^2 - b^2c^2d^2 + b^2c^2f^2 + b^2d^4 - b^2d^2e^2 - b^2d^2f^2 - b^2e^2f^2 + c^2d^2e^2 - c^2e^2f^2 - d^2e^2f^2 + e^4f^2 + e^2f^4 = 0.$$

This is the relation holding among all six mutual distances a, b, c, d, e, f of four vertices A, B, C, D of a plane quadrilateral $ABCD$. This relation is called the Euler's four points relation [3].[2]

Now we will verify the Euler's four point relation in DGS. In GeoGebra we draw a quadrilateral $ABCD$ and denote its sides and diagonals by a, b, c, d, e, f. We express the value of the polynomial which is on the left in the Euler's four point relation, Fig. 6. Changing the positions of the vertices A, B, C, D the value of the polynomial on the left of the Euler's four points relation is still equal to zero. But be careful! If we set rounding in GeoGebra on higher decimal places we see, Fig. 7, that the relation does not hold! Of course the result is not true, the Euler's four points relation is still valid. The reason is that GeoGebra is working on numerical basis and it may happen that the result is not exact. That is the reason, why a rigorous mathematical proof is needed.

[2]Euler's four point relation is a special case of the Cayley–Menger determinant for the volume of a simplex [1].

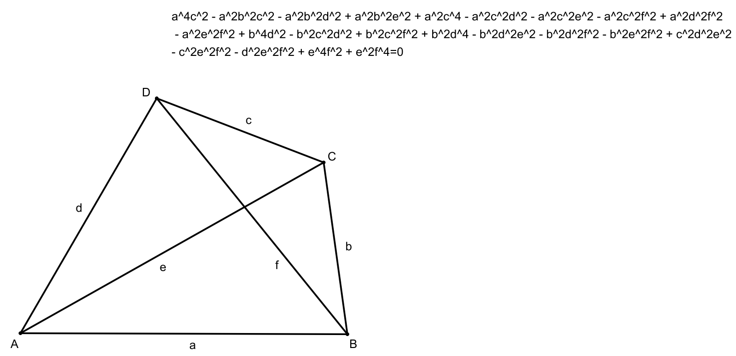

Figure 6: Verification of the Euler's four point relation — exact result

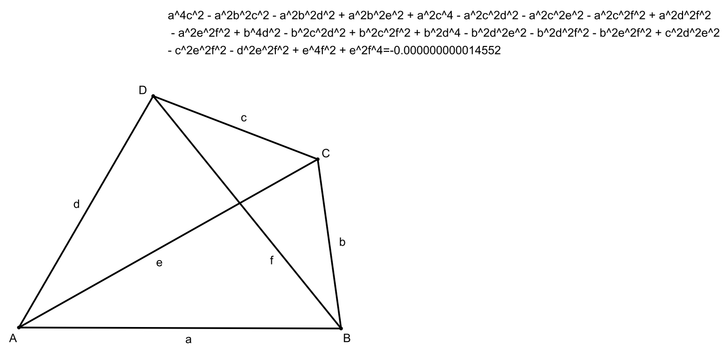

Figure 7: Verification of the Euler's four point relation — inexact result

4 Searching for loci

Searching for loci of points of given properties belongs to difficult topics in mathematics at schools. The use of DGS and CAS can facilitate it considerably. To determine a locus we can keep the following steps:

- Demonstrate the problem in DGS. Dragging a point (usually called a mover) we can guess what locus is formed by a point which is dependent on a mover. This point is usually called a tracer.
- Use the command Locus to demonstrate in DGS the locus and to state a conjecture.

Since DGS works on numerical basis it is necessary to make rigorous mathematical proof. It usually means to find the locus equation or to find characteristic properties of the locus. Then:

- Try to prove classically that the locus given by DGS is the real locus. By this we take advantage of DGS to determine possible geometric invariants and subsequently use them in the proof.

If we do not succeed then we try to find the locus by CAS — we call it automated approach. To do this two following steps are necessary:

- Express a geometric situation in terms of algebraic equations and inequations.
- Eliminate appropriate variables from a system of algebraic equations and inequations to obtain the locus equation.

To demonstrate the previous steps in searching for loci we will give two examples — one is on a Cassinian ovals, the second on an asteroid.

4.1 Cassinian oval

Determine the locus of the point L in plane whose product of distances to two given points A and B is a given constant a.

First let us demonstrate the locus in DGS. To construct the locus in GeoGebra we will apply the power of a point with respect to a circle, Fig. 8. Given points A and B and

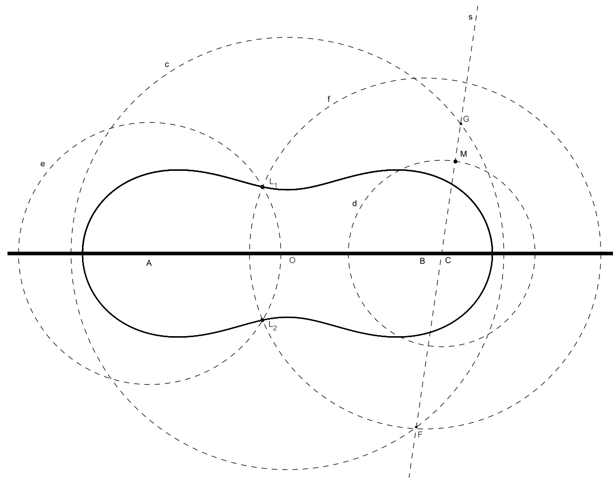

Figure 8: Construction of an Cassinian oval — the case $a > b$

let C be a point on AB. Let s be a line through C which cuts a circle c centered at O, the midpoint od AB, at points F and G. From the theorem on the power of a point with respect to a circle $|CG||CF|$ = constant for all positions of the line s which turns about the point C if we move the point M along a circle d. We construct circles e, f centered at A and B with radii $|CF|$ and $|CG|$. Then points of intersection L_1 and L_2 belong to the locus since $|AL_1||BL_2| = |CF||CG|$.
It is well-known that the locus is a *Cassinian oval* [8]. Changing the radius of the circle c or the position of the point C we get various forms of Cassinian ovals.

Now we will find the locus equation. Adopt a rectangular coordinate system with $A = [-b, 0]$, $B = [b, 0]$ and locus point $L = [x, y]$, Fig. 9. Suppose that $|LA||LB| = a^2$. This can be translated into an analytic form

$$\sqrt{(x+b)^2+y^2} \cdot \sqrt{(x-b)^2+y^2} = a^2.$$

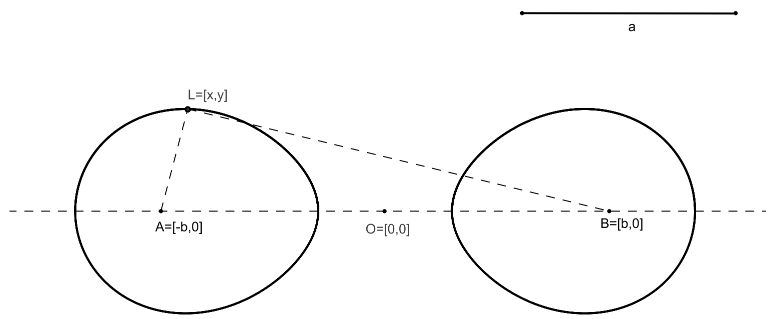

Figure 9: For the Cassinian oval $|LA||LB| = a^2$ holds — the case $a < b$

From this we get the equation of a Cassinian oval

$$(x^2 + y^2)^2 - 2b^2(x^2 - y^2) = a^4 - b^4. \tag{2}$$

We see that the Cassinian oval is an algebraic curve of 4th order in x, y.
For $a = b$ we get from (2) *lemniscate of Bernoulli*, Fig. 10, with the equation

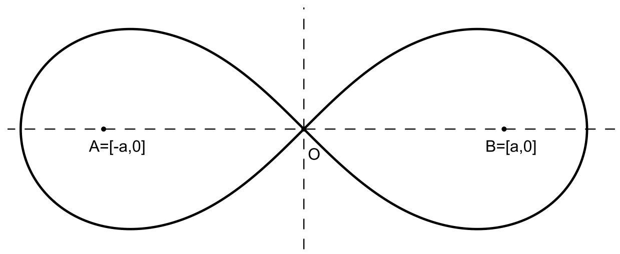

Figure 10: Lemniscate of Bernoulli: $(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0$

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0. \tag{3}$$

Let us extend a Cassinian oval into three dimensional space:

Determine the locus of the point L in 3D space whose product of distances to two points A and B is a constant a.

We omit construction in a space since we do not have appropriate dynamic geometry software to do this.

Let $A = [-b, 0, 0]$ and $B = [b, 0, 0]$ be two points, and a a given positive real constant. We are to find the locus of $P = [x, y, z]$ in 3D space so that $|PA||PB| = a^2$. The translation into analytical geometry gives

$$\sqrt{(x + b)^2 + y^2 + z^2} \cdot \sqrt{(x - b)^2 + y^2 + z^2} = a^2.$$

From this we get

$$(x^2 + y^2 + z^2)^2 - 2b^2(x^2 - y^2 - z^2) = a^4 - b^4. \tag{4}$$

This is the equation of a Cassinian oval in space.

Similarly as in the planar case we distinguish three types of Cassinian ovals in 3D. For $a = 9$ and $b = 8$ we get a Cassinian oval, see Fig. 11.

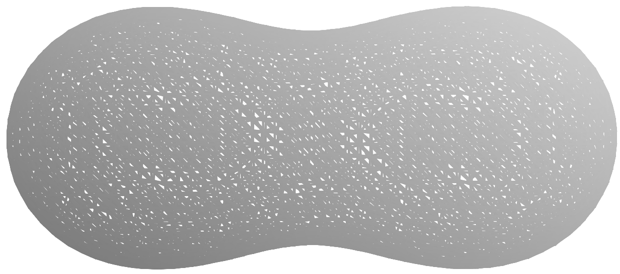

Figure 11: Cassinian oval in 3D — $a = 9$ and $b = 8$

If $a = b$ then we get a special case of a Cassinian oval — 3D version of the well-known Bernoulli lemniscate with the equation

$$(x^2 + y^2 + z^2)^2 - 2a^2(x^2 - y^2 - z^2) = 0, \tag{5}$$

see Fig. 12.

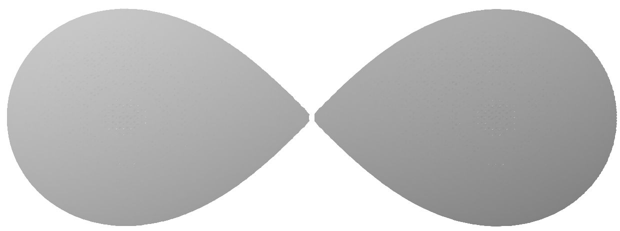

Figure 12: Bernoulli lemniscate in 3D — $a = b$

For the choice $a = 8$ and $b = 9$ the Cassinian oval in 3D consists of two separate parts, Fig. 13.

Figure 13: Cassinian oval in 3D — $a = 8$ and $b = 9$

4.2 Asteroid

Next part of this section is devoted to asteroid. In the basic course of geometry various definitions of an ellipse are taught. The following bar construction is well-known:

A segment PQ of a constant length d moves with its endpoints P and Q along two perpendicular lines. Then the locus of an arbitrary point M of the segment PQ is an ellipse.

To find the locus equation we will proceed as follows:
Let two perpendicular lines be axes x and y. Denote $P = [p, 0]$, $Q = [0, q]$, $M = [x, y]$ and let $|MQ| = a$, where $0 < a < d$, Fig. 14. Describing a geometric situation we get:

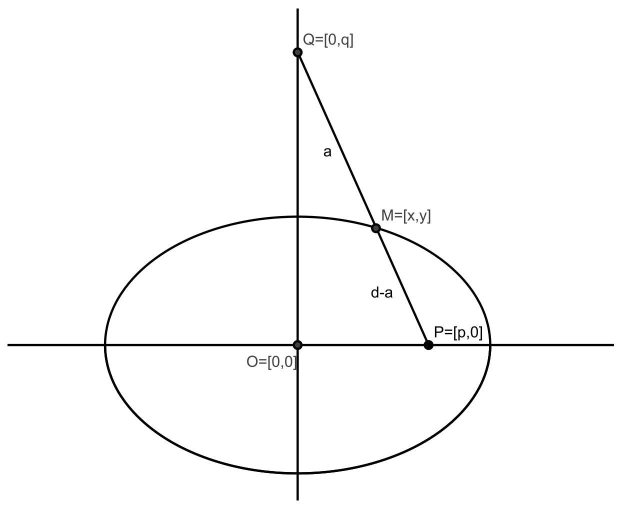

Figure 14: A bar construction of an ellipse

$\frac{MQ}{MP} = \frac{a}{a-c} \Rightarrow h_1 := a(x-p) - x(a-d) = 0 \wedge h_2 := ay - (a-d)(y-q) = 0,$
where MP and MQ are signed distances. Further

$|PQ| = d \Rightarrow h_3 := p^2 + q^2 - d^2 = 0.$

The elimination of variables p, q in the system of polynomial equations $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ gives

```
Use R::=Q[x,y,a,d,p,q];
I:=Ideal(p^2+q^2-d^2,a(x-p)-x(a-c),ay-(a-d)(y-q));
Elim(p..q,I);
```

the locus equation

$$\frac{x^2}{a^2} + \frac{y^2}{(d-a)^2} = 1. \tag{6}$$

We see that the locus is an ellipse with the axes a and $d - a$.
Changing the position of the point M on the segment PQ we draw the locus for each position of M. To depict it in GeoGebra, we use by the right button of the mouse the icon Trace On clicking on the figure of the locus. Moving with M we get the following family of ellipses, Fig. 15.

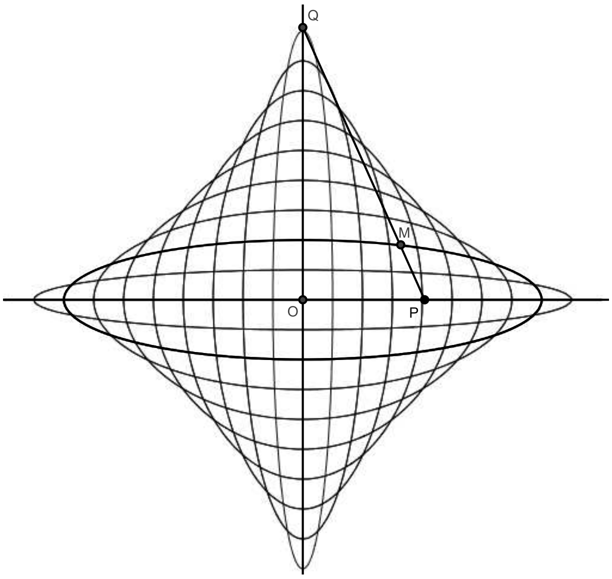

Figure 15: A bar construction of an ellipse for various points M on PQ.

A question now arises: What is the envelope of this family of ellipses?

Let us briefly describe what is the envelope and how to obtain it [5].
The envelope of a one parameter family of curves is a curve which is tangent to every curve of the family.
The equation of the family may be given in an implicit form as $F(x, y, t) = 0$, where t is a parameter. To find the equation of the envelope, it is necessary to eliminate the parameter t both from the equation of the family $F(x, y, t) = 0$, and its partial derivative with respect to the parameter $\partial F(x, y, t)/\partial t = 0$. This is guaranteed for those points for which $(\partial F(x, y, t)/\partial x)^2 + (\partial F(x, y, t)/\partial y)^2 \neq 0$. If both $\partial F(x, y, t)/\partial x$ and $\partial F(x, y, t)/\partial y$ are zero, then the envelope can have a singular point here.

To compute the envelope of the family of ellipses we can use the equation (6) with a as a parameter and d as a constant. After removing fractions in (6) we get

$$F := (d-a)^2x^2 + a^2x^2 - a^2(d-a)^2 = 0. \tag{7}$$

Computing derivative we obtain

$$\frac{\partial F}{\partial a} := -2x^2(d-a) + 2ay^2 - 2a(d-a)^2 + 2a^2(d-a) = 0. \tag{8}$$

Elimination of a from the system $F = 0$ and $\partial F/\partial a = 0$ gives

```
Use R::=Q[x,y,a,d];
I:=Ideal((d-a)^2x^2+a^2y^2-a^2(d-a)^2,-2x^2(d-a)+2ay^2-2a(d-a)^2+2a^2(d-a));
Elim(a,I);
```

$$d^2x^2y^2(x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 3x^4d^2 + 21x^2y^2d^2 - 3y^4d^2 + 3x^2d^4 + 3y^2d^4 - d^6) = 0.$$

We suppose that $|PQ| \neq 0$, hence $d \neq 0$. Factors $x = 0$ and $y = 0$ are extraneous and may be ruled out. Remaining equation can be written into the form

$$(x^2 + y^2 - d^2)^3 + 27d^2x^2y^2 = 0 \tag{9}$$

which is the equation of an *asteroid,*

The following problem is closely tied with the last one:

A ladder is leaning against a wall. Determine the envelope of all positions of the ladder.

We suppose that a ladder has a constant length d. First we draw the situation in GeoGebra, Fig. 16, where the end points of the ladder at one position are denoted by P and Q. We

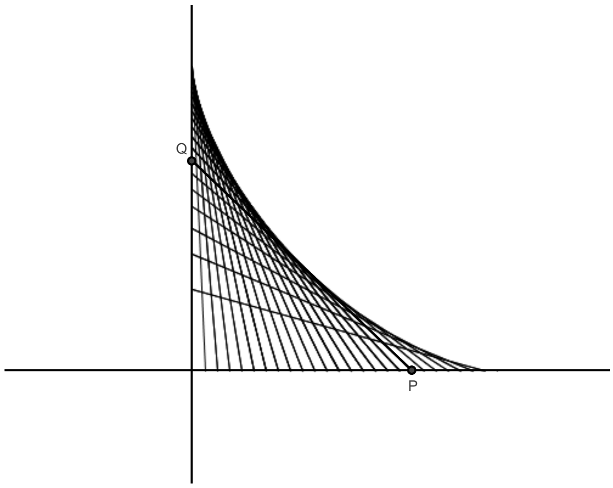

Figure 16: What is the envelope of all positions of a ladder leaning against a wall?

see that PQ is a tangent to a searched curve. We are searching for such a curve whose tangents intersect two perpendicular lines at two points with constant distance d. Let as compute the curve equation.

Choose a rectangular system of coordinates such that $P = [p, 0]$, $Q = [q, 0]$, Fig. 17. Then it holds:

$|PQ| = d \Rightarrow h_1 := p^2 + q^2 - d^2 = 0.$

The tangent $t = PQ$ touches the searched curve at the point $X = [x, y]$. Then

$X \in t \Rightarrow h_2 := qx + py - pq = 0.$

To compute the envelope of a parametric system of lines with p as a parameter we have to express q from $h_1 = 0$ and substitute it into h_2. Then

$\partial(x\sqrt{d^2 - p^2} + yp - p\sqrt{d^2 - p^2})/\partial p = 0 \Rightarrow h_3 := (xp + d^2 - 2p^2)^2 - y^2(d^2 - p^2) = 0.$

The elimination of p, q in the system of polynomials h_1, h_2, h_3 gives

```
Use R::=Q[x,y,p,q,d];
I:=Ideal(p^2+q^2-d^2,qx+py-pq,(xp+d^2-2p^2)^2-y^2(d^2-p^2));
Elim(p..q,I);
```

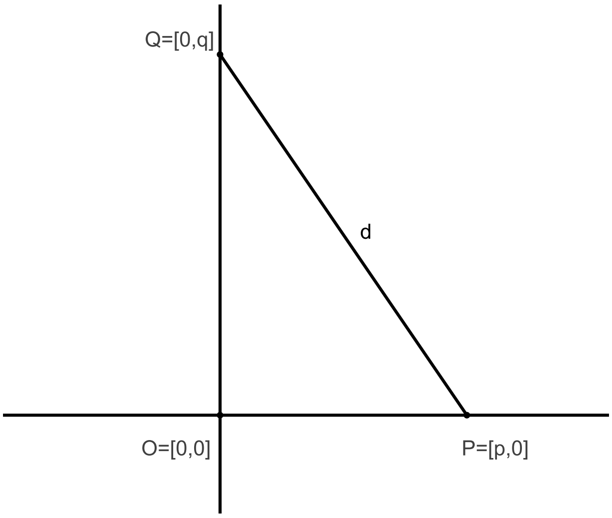

Figure 17: Searching for locus envelope

$d^2(x^6+3x^4y^2+3x^2y^4+y^6-3x^4d^2+21x^2y^2d^2-3y^4d^2+3x^2d^4+3y^2d^4-d^6)(x^6+7x^4y^2-17x^2y^4+9y^6-3x^4d^2+13x^2y^2d^2-15y^4d^2+3x^2d^4+7y^2d^4-d^6)=0.$

After removing the extraneous factors (the last factor does not obey our conditions, further we suppose that $d \neq 0$), we get the equation

$x^6+3x^4y^2+3x^2y^4+y^6-3x^4d^2+21x^2y^2d^2-3y^4d^2+3x^2d^4+3y^2d^4-d^6=0$

which can be written as

$$(x^2+y^2-d^2)^3+27d^2x^2y^2=0$$

which is (9). We obtained the same curve as in previous problem — the asteroid.

Remark: The derivative of the function $f = x\sqrt{a^2-p^2}+yp-p\sqrt{a^2-p^2}$ by p is $\partial f/\partial p = \frac{-2xp}{2\sqrt{a^2-p^2}}+y-\sqrt{a^2-p^2}+\frac{2p^2}{2\sqrt{a^2-p^2}}$.

After removing radicals we get the polynomial

$$(xp+d^2-2p^2)^2-y^2(d^2-p^2).$$

There are several definitions of an asteroid. We will conclude this section with the following one [4]:

Consider a circle k with two othogonal diameters x, y. Denote the feet of perpendiculars from an arbitrary point P of k to the diameters x, y by A, B. Then the locus of G which is on AB such that PG is perpendicular to AB is an asteroid.

We could show in a similar way as in the previous examples that the locus equation is the equation of an asteroid, Fig. 18.

Conclusion

We tried to show that integration of DGS and CAS is useful and helps to solve problems. Nowadays, besides well known and powerful mathematical software like Maple or Mathe-

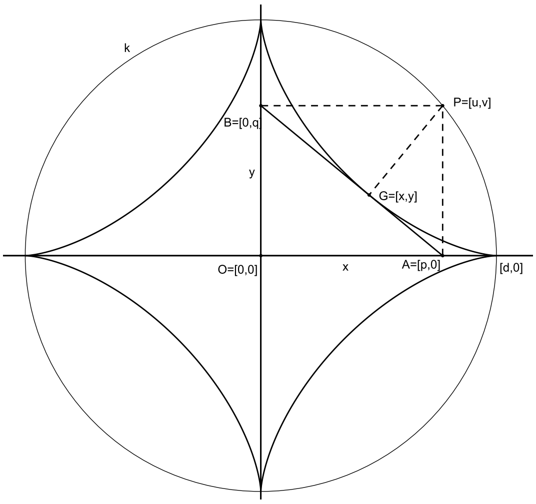

Figure 18: The locus of G when P moves on a circle k is an asteroid

matica (CAS) and Cabri or Sketchpad (DGS), we can also use free open source software like CoCoA, Maxima, Sage, Singular (CAS) and Geogebra (DGS) which is immediately accessible to everybody. Such situation calls for corresponding changes in the mathematics curriculum at schools because mathematical software opens new windows into still unreachable parts of modern mathematics.

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