The Other Role of Technology: Communication Between People

Paper Presented at ATCM 2012

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ABSTRACT. For the most part, the traditional role of technology in the teaching of mathematics has been focussed on its ability to solve, to evaluate, to produce images and even to guide students through stereotypical working steps that are supposed to be employed as one works through exercises that appear in the textbooks. But technology has another role that is sometimes neglected. It gives us new and efficient ways to talk with one another. It gives us opportunities to convey mathematics in a much more friendly form than we can find in any ordinary textbook. It allows instructor and student to exchange mathematical ideas even when they are not standing face to face in the same room. It can be a means of communication between people.

In some of my earlier presentations at ATCM and elsewhere, I have emphasized the value of screen capture videos in the creation of mathematical learning materials and the role of video will again be featured this year. I shall refer both to classroom video and to the video in products like my Virtual Calculus Tutor. However, the production of video is not the whole story and I shall not be confining my attention to this topic. My presentation will demonstrate some of the unusual textbooks that I have designed for reading on the computer screen and it will show how, in my two main video products, Virtual Math Tutor and Virtual Calculus Tutor, a balance between video material and document material can provide a very successful medium for helping students to understand mathematics.

I shall show how an on-screen book can be just as easy to browse through as a printed bound text and provide more help, more solutions to exercises, and more development of the material. It can do so without becoming cluttered and at a modest price. I should add that several of my works are provided for free as public domain items. Some of my products involve coordinated use of on-screen documents and video and I intend to demonstrate how these two media can work with each other to help a student to understand mathematical ideas.

The Different Roles of Technology

Introduction

Over the past 20 years, the use of technology in the teaching and learning of mathematics has come to be identified largely with the computer algebra system (CAS) technology that has provided us with some exciting opportunities to explore mathematics and to experiment with it. CAS technology also allows us to extend our activity to more useful and realistic problems than we could have brought into the classroom in the old pencil and paper days. There is no doubt that CAS technology has changed the face of mathematical study and it is to be expected that a focus on CAS technology should be a principal ingredient in a mathematics technology conference such as ATCM. In the same way, it is to be expected that the majority of commercial products that bring technology into the mathematics classroom are focussed on CAS technology.

There are, however, other ways in which access to technology can change the face of mathematical study and I believe that these other ways are being somewhat neglected by the community. One of these other ways is to use technology as a communications medium rather than for its CAS ability and it is this kind of application of technology that is the subject matter of my paper. I am in no way denying the immense value of CAS technology in the mathematical learning process but I do want to add that I believe that CAS technology has received more than its fair share of attention. While I believe that CAS technology, used in the proper context when it is relevant, is a wonderful tool in the teaching and learning of mathematics, I also see evidence of some slavish over-use of CAS technology that may be doing a disservice to mathematical learning. With this thought in mind, I shall begin my presentation by describing some ways in which I believe that CAS technology may have been misused.

Some Undesirable Effects on Mathematical Learning when CAS Technology is Misused

From the time when I began teaching university courses in mathematics 48 years ago, I have often expressed my concern that too many undergraduate mathematics courses confine themselves largely to the methods that must be used to obtain “answers” to problems. The courses are replete with problems that invite the students to *evaluate*, and to crank out “answers” rather than to demonstrate understanding and appreciation of mathematical principles. Such problems are largely unconcerned with the logical flow that is the bedrock of mathematics and the courses pay little or no attention to mathematical writing. All too often, the students learn to solve a problem without acquiring any appreciation of the relevance of the problem to the course being studied.

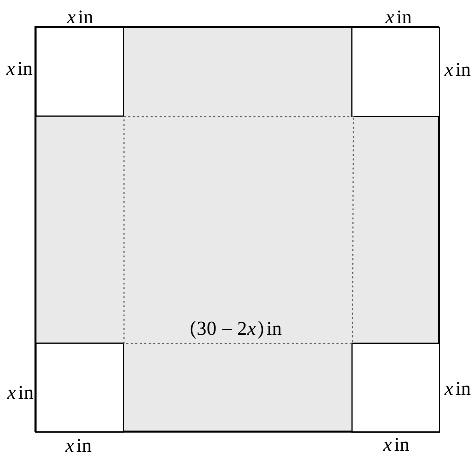
When CAS technology burst upon the scene about 20 years ago, I heard grand promises that all this was going to change. We were promised that, now that computers were available to do the manipulations, we could give greater emphasis to logical connections between ideas and to mathematical writing. However, I am sorry to have to report that, as far as I can see, many of those promises have proved to be empty.

Far from tilting the thrust of classroom activity away from the cranking out of answers and towards understanding of ideas, the effect of CAS technology has, all too often, been an even stronger emphasis on mechanical steps. To some extent, we have been turning our students into button pushers and spectators instead of students of mathematics. I think that the worst aspect of this misuse of computer technology involves the use of certain software products that are designed to steer students through stereotyped sequences of working steps that must be followed to obtain the desired “answer” to a problem. There are several such products on the market. They are raking in millions of dollars but they are doing nothing to help the cause of mathematical learning. One of the symptoms of our neglect of mathematical writing and the understanding of mathematical principles is the abundance of multiple choice and "put your answer in a box" testing that we see in so many mathematics courses. There is no doubt that, used in moderation, CAS technology is a wonderful tool but, unfortunately, I do not believe that it is always used in moderation. I believe that the degeneration of our undergraduate courses into mindless button pushing sessions is counter-productive to the learning process of mathematics and serves only to line the pockets of the companies that produce the software products that are supposed to teach students how to crank out "answers" to problems.

I offer the following examples as illustrations of the way in which I think that a great many mathematics courses are tilted towards mechanical steps and away from the learning of mathematics.

CAS Technology and the Box Problem

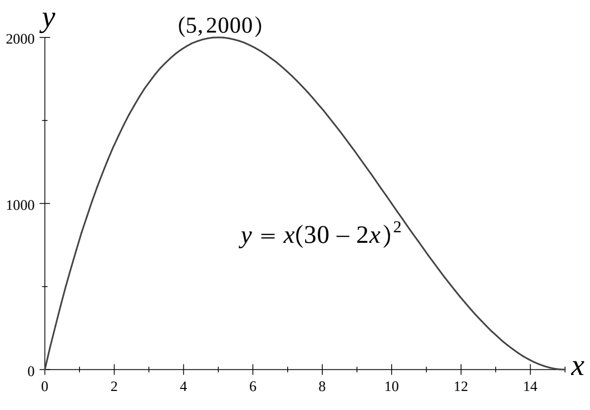
I had an interesting experience, several years ago, at the Perimeter Mathematics Technology Conference that takes place each year near Atlanta, Georgia where I live. I attended a presentation there that was given by the representative of a software company and discovered that the sole objective of that presentation was to market a software product. I was surprised to see that, in spite of the fact that the presentation was directed to the level of high school algebra, a principal item in that presentation was the well known "box problem" that we often teach in a beginning calculus course.



The box problem involves the removal of four equal squares from the corners of a square piece of cardboard and then folding the cardboard up along the dotted lines shown in the figure to make a box without a lid. The problem of finding the size of the cut out corners that will maximize the volume of the box is given in every beginning calculus course because it is a nice simple illustration of the application of elementary calculus to maxima and minima. In the event that the original square has a side of 30 inches and each corner cut-out has a side of x inches and the volume of the box is $f(x)$ cubic inches, then

$$f(x) = x(30 - 2x)^2$$

for $0 \leq x \leq 15$. The equation $f'(x) = 0$ holds when $x = 5$ and the maximum possible volume is $f(5) = 2000$.



My view, in teaching calculus, is that the box problem is not merely an *application* of the material being studied. It is also an *illustration* of the principles that we are meant to be conveying to our students. Since that conference presentation was directed to high school algebra, I had to ask myself why the problem was being considered at all. It seemed to me that its sole purpose was to give us a reason to buy the speaker's software. His approach to the box problem was to use a tracing feature in the software to approximate the top point on the graph $y = x(30 - 2x)^2$ and, of course, he gave a good estimate to the answer. But, when I asked why the problem was there, except to sell his software; when I asked what mathematical principle is being illustrated by the use of a tracing feature to estimate the maximum, the fervent believers in the room all turned on me with fangs bared. I slunk out of the room with my tail between my legs and have not been back to that conference. That event illustrates what I mean when I say that CAS technology is sometimes used slavishly for its own sake rather than because it can contribute to mathematical learning.

The Emphasis on Mechanical Working Steps in Elementary Calculus Courses
The majority of elementary calculus courses have degenerated, in recent years, into drills that do little more than regiment students into the correct mechanical working steps they must perform in order to arrive at the "correct" answers to the homework problems that will also be the problems that the student can expect to encounter in the tests and examinations. All too often, the student never appreciates why a particular problem is in the course or what mathematical principle is being illustrated by it. Nor do they understand the logical flow that tells us why the mechanical steps that they are using are actually valid. Many instructors will disagree with what I have said and will claim that they do, indeed, explain the connections in the classroom. To those instructors, I have one question: *Are your students expected to explain those logical connections in your examinations?* Anything that will not be included in the examinations will be seen by the students as outside the main scope of the course and will be ignored.

We should not be surprised, then, that the majority of students come out of their first calculus course well able to deal with problems that say *evaluate*, but with no knowledge of the ideas that are the bedrock of the calculus they are supposed to have studied. So, for example, students will be well able to work out expressions such as $\frac{d}{dx}\frac{x^2\sin\left(1+x^3\right)}{\sqrt{1+x^4}}$ but will be totally unable to explain why the product rule and quotient rule hold. They have become experts at evaluating trigonometric limits but have no idea why these trigonometric limits were studied, how they were derived from the important inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

that depends on radian measure of an angle or how they allow us to conclude that $\frac{d}{dx}\sin x = \cos x$. They come out of their first calculus course with little appreciation for such key ideas in calculus as the mean value theorem and no understanding of the way in which the existence of maxima and minima of continuous functions is a key ingredient in this theorem.

In my view, a careful balance needs to be preserved between mechanical working steps and appreciation of mathematical ideas and logical connections. There is a place for everything, and that includes applications of CAS technology. In my own calculus courses and in my *Virtual Calculus Tutor*, I make extensive use of CAS technology where I feel it is relevant. But I also strive to preserve a balance and to make it clear that our principal goal should be the understanding of mathematical ideas and that the acquisition of technical skills is important because it helps us to achieve this goal.

Technology for Communication

My Use of a Mathematical Word Processor as a Whiteboard

When I taught mathematics at the Ben Gurion University in Be'er Sheva, Israel more then forty years ago, I needed to provide my students with complete lecture notes because some of them were frequently called away from campus to perform military duty. I made the discovery that, by taking the trouble to produce complete lecture notes for my students, I could optimize the academic quality of my courses. Instead of having to divide their attention between creation of lecture notes and concentration on my lectures, the students could give all of their attention to the lecture and I could be sure that their study materials were complete and reliable. From that time, I have always provided my students with their lecture notes, even for lectures I have given in other countries.

In the old days, I needed to write my notes on mimeograph masters and then send the masters for printing several days before the lecture was given. That was the best we could do in those days but it was not an ideal approach because I believe that the best teaching takes place while the lecture notes are being created spontaneously. However, all that changed when the technology revolution burst upon us. I am happy to say that I have not stood before a board for more than 18 years. Instead, I use the wonderful mathematical word processing systems, Scientific Workplace or Scientific Notebook 5.5 to create my notes in the classroom as I teach. It takes me less than five minutes to make these notes available for download from my website. In my presentation, I shall illustrate they way in which I create my lecture notes but, for someone who is reading this paper, I can suggest that he/she should use a Windows computer to go to

> http://science.kennesaw.edu/~jlewin/VCT/excerpt-from-math-2203/
> excerpt-from-math-2203.html

to see a short video that shows a short excerpt from one of my actual calculus lectures at Kennesaw State University. This video is actually an excerpt from the real time recording I made of that lecture. Thus, not only does modern technology allow me to create my lecture notes while I teach, but it also allows me to record the entire process. As I have said in an earlier talk here at ATCM, I burn a daily CD for each of my students, giving them all of their lecture notes and the recording of the day.

Advantages of Electronic Versions of Textbooks

I have moved almost entirely towards electronic textbooks or sets of lecture notes in all of the courses that I teach. I believe that, not only are electronic books superior in many ways than traditional bound books, but they also cost much less. As a matter of fact, a good deal of my material is provided for free. In this section, I shall describe some of the benefits my students have enjoyed from my electronic textbooks and notes.

Electronic Books Provide Superior Browsing Ability

One of the concerns that have been expressed about electronic books is that they sometimes require the reader to scroll through the material in a linear fashion. When that happens, we lose all the advantages that spinally bound books provide over the scrolls that were used in ancient times. However, a well structured electronic book offers browsing ability that is even better than that of a spinally bound book.

The feature of an electronic book that provides it with browsing ability is the ability of such a book to contain hypertext links (hyperlinks). My electronic books contain a highly sophisticated hyperlink structure that includes a multi-level table of contents. They are supplied in the form of LaTeX files that are designed to be read on-screen by version 5.5 of the MacKichan Software[1] products including the free product, Scientific Viewer. I have chosen not to provide my documents in PDF form because the hyperlinking ability of PDF is not sufficient for the browsing ability that my books provide. I therefore do not enjoy the benefit of the hyperref feature in Scientific Workplace that could have created my table of contents automatically. However, my tables of contents, created painfully by hand, are better than the automatic tables of contents in PDF documents because my tables of contents can contain mathematical symbolism.

[1] www.mackichan.com

9.3.6 Some Applications of the nth Term Criterion for Divergence

A Ratio Criterion for Divergence

Testing the Series $\sum \frac{n!}{6^n}$

Divergence of the Series $\sum \frac{(2n)!}{(n!)^2}$

Divergence of The Series $\sum \frac{(-1)^n 4^n (n!)^2}{(2n)!}$

A Problem that We Cannot Solve Right Now: Test the Series $\sum \frac{(2n)!}{4^n (n!)^2}$

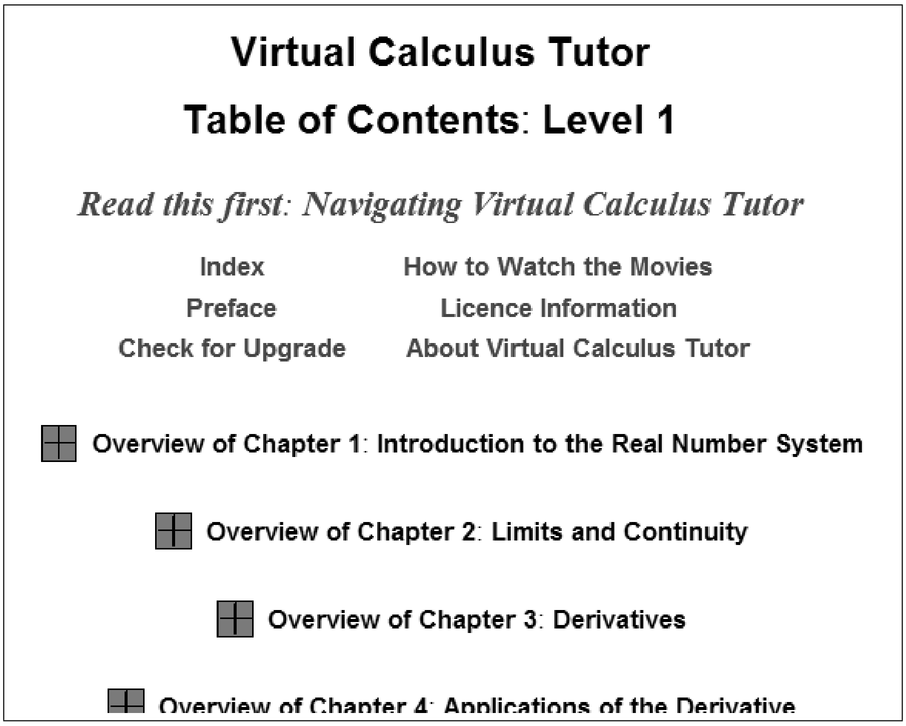
A Limit Form of the Ratio Criterion for Divergence

Divergence of the Series $\sum \frac{3^n}{n^{10}}$

Divergence of the Series $\sum \frac{(3^n)(n!)}{n^n}$

A slight disadvantage of my approach is that my material is readable only on Windows computers while PDF files would also be available to other platforms. There is a possibility that I may eventually be able to open my material to non-Windows computers by migrating it to the new Version 6 of the MacKichan Software family of products but, because of some technical difficulties, I do not anticipate migrating to Version 6 in the foreseeable future.

When one opens the table of contents in my Virtual Calculus Tutor, one sees the level one version.



Virtual Calculus Tutor

Table of Contents: Level 1

Read this first: Navigating Virtual Calculus Tutor

Index | How to Watch the Movies

Preface | Licence Information

Check for Upgrade | About Virtual Calculus Tutor

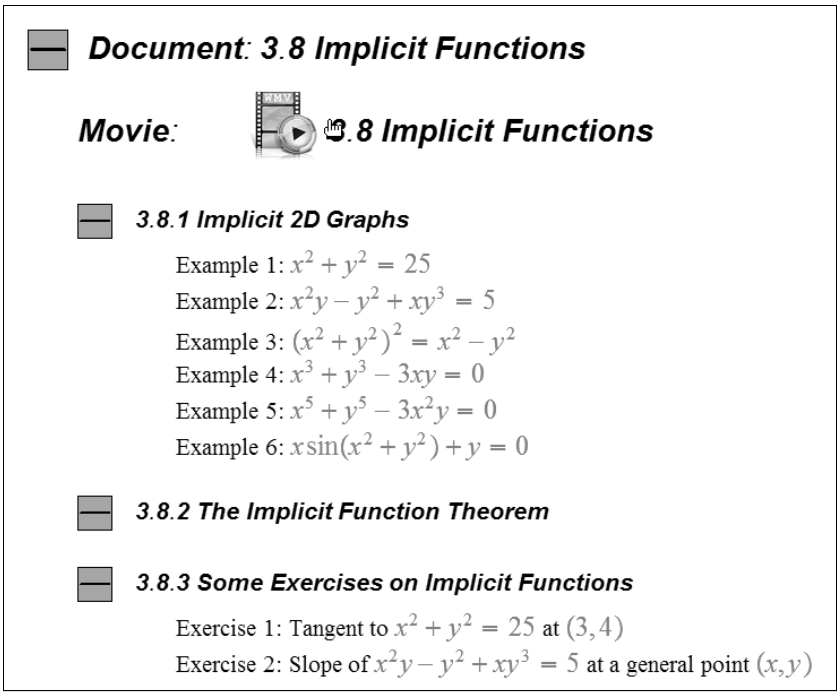
Overview of Chapter 1: Introduction to the Real Number System

Overview of Chapter 2: Limits and Continuity

Overview of Chapter 3: Derivatives

Overview of Chapter 4: Applications of the Derivative

This version shows the chapter headers and to left of each of these, is a box on which the reader can click to zoom in. Zooming in this way, the reader can browse the table of contents in four levels, zooming in to see more detail and zooming out to obtain a bird's eye view. So, for example, the level 4 table of contents shows an image like the following.


Document: 3.8 Implicit Functions

Movie: 3.8 Implicit Functions

3.8.1 Implicit 2D Graphs

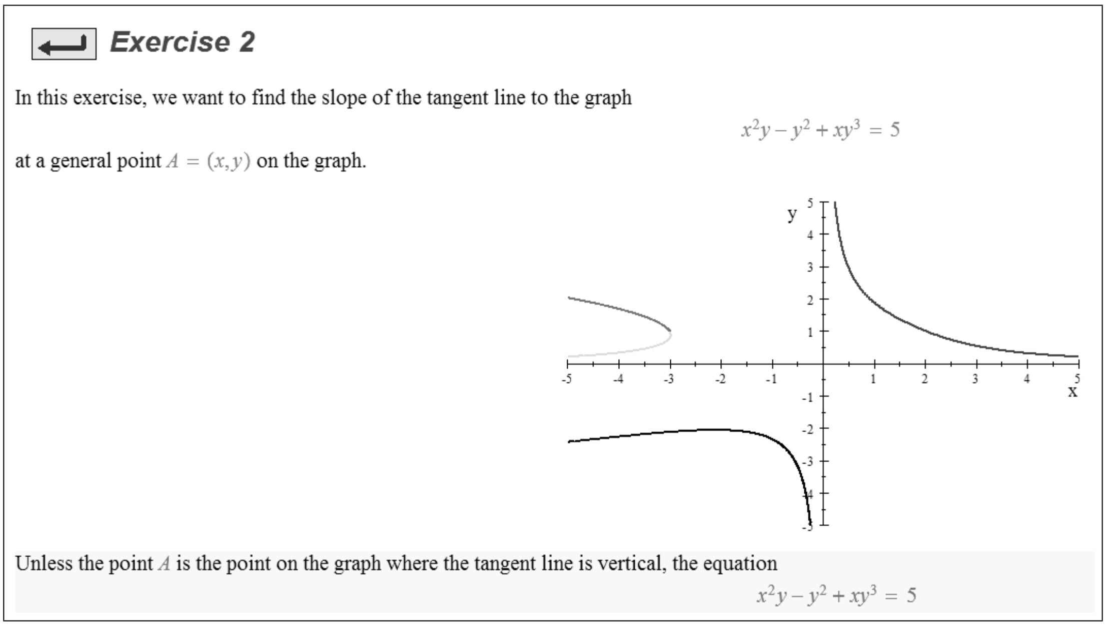
Example 1: $x^2 + y^2 = 25$
Example 2: $x^2y - y^2 + xy^3 = 5$
Example 3: $(x^2 + y^2)^2 = x^2 - y^2$
Example 4: $x^3 + y^3 - 3xy = 0$
Example 5: $x^5 + y^5 - 3x^2y = 0$
Example 6: $x\sin(x^2 + y^2) + y = 0$

3.8.2 The Implicit Function Theorem

3.8.3 Some Exercises on Implicit Functions

Exercise 1: Tangent to $x^2 + y^2 = 25$ at $(3,4)$
Exercise 2: Slope of $x^2y - y^2 + xy^3 = 5$ at a general point (x,y)

The level 4 table of contents shows the chapter headings, the section headings, the subsection headings, and the individual items in the text. To the left of any item that can be contracted, there is a contracting link that zooms out to a version of the table of contents that shows less detail. A click on an item in any level of the table of contents takes the reader directly to that item in the text. Thus, if the reader were to click on the link to Exercise 2 of Subsection 3.8.3 that is shown in the above image, then the cursor will jump to the actual item.


Exercise 2

In this exercise, we want to find the slope of the tangent line to the graph

$$x^2y - y^2 + xy^3 = 5$$

at a general point $A = (x,y)$ on the graph.

Unless the point A is the point on the graph where the tangent line is vertical, the equation

$$x^2y - y^2 + xy^3 = 5$$

As shown in the above figure, each item in the text is supplied with a backlink that takes the cursor back to the link to that item in the table of contents. All three of my electronic books, (Virtual Calculus Tutor, Virtual Math Tutor, and the on-screen version of my Interactive Introduction to Mathematical Analysis) have this feature that makes it easier to browse the book than to browse a printed bound book.

Electronic Books Can Show More Detail
Authors are constantly struggling with the conflicting needs of clarity and brevity. If we provide the kind of detail that we would like to include in a printed book, we produce a monster that weighs a ton, costs a fortune, and is actually harder to read because the length of each item makes it look intimidating. Furthermore, adding detail to a printed book raises the cost of production. However, in creating electronic books, we do not face this kind of dilemma. Extra detail can provided in such a way that the student clicks on a link to unfold it and, far from being intimidating, the detail actually looks friendly. And we can add as much detail as we like to an electronic book without increasing its cost.

So, for example, in my "Interactive Introduction to Mathematical Analysis", there are many points where, following the tradition in such books, I inform the reader that the proof of an assertion will be left to the reader *as an exercise*. In each case, the words “as an exercise” appear as a hyperlink on which the reader can click to see the complete details of the proof. The following image shows one of those hyperlinks.

6.4.2 Existence of a Largest Member

Every nonempty closed set that is bounded above must have a largest member.

Proof. Suppose that H is a nonempty closed set and that H is bounded above. In order to prove that H has a largest member, we shall show that $\sup H$ belongs to H. We write $\alpha = \sup H$, and, to obtain a contradiction, we assume that α does not belong to H. Then, of course, α belongs to the open set $\mathbf{R} \setminus H$, and, using this fact, we choose $\delta > 0$ such that

$$(\alpha - \delta, \alpha + \delta) \subseteq \mathbf{R} \setminus H.$$

Since no member of H can be greater than α and since no member of H can belong to the interval $(\alpha - \delta, \alpha + \delta)$, we see that the number $\alpha - \delta$ is an upper bound of H, contradicting the fact that α is the *least* upper bound of H. ■

In the same way we can show that every nonempty closed set that is bounded below must have a least member. We leave the proof of this fact as an exercise.

The next picture shows the details that are revealed when the reader clicks on the link.

Solution of an In-Text Exercise

Every nonempty closed set that is bounded below must have a least member.

Proof: Suppose that H is a nonempty closed set and that H is bounded below. In order to prove that H has a least member we shall show that $\inf H$ belongs to H. We write $\alpha = \inf H$ and, to obtain a contradiction, we assume that α does not belong to H. Then, of course, α belongs to the open set $\mathbf{R} \setminus H$ and, using this fact, we choose $\delta > 0$ such that

$$(\alpha - \delta, \alpha + \delta) \subseteq \mathbf{R} \setminus H.$$

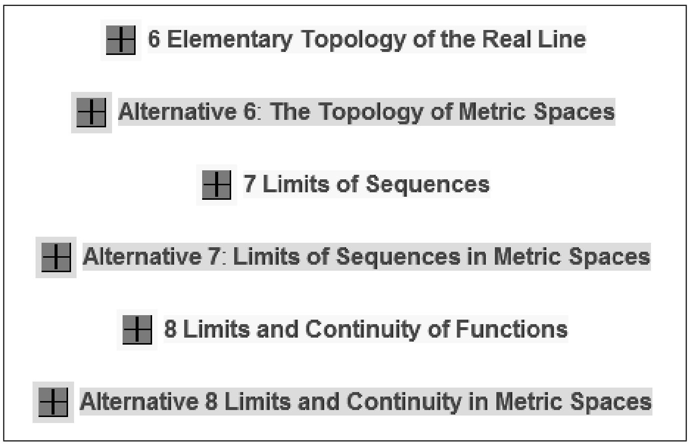
Since no member of H can be less than α and since no member of H can belong to the interval $(\alpha - \delta, \alpha + \delta)$ we see that the number $\alpha + \delta$ is a lower bound of H, contradicting the fact that α is the *greatest* lower bound of H. ■

Notice the back hyperlinks that take the reader back to the point in the text that he/she was reading before clicking on “*as an exercise*”.

In the same way, my products, Virtual Math Tutor and Virtual Calculus Tutor, give the reader the choice of seeing the lists of exercises either with or without complete solutions.

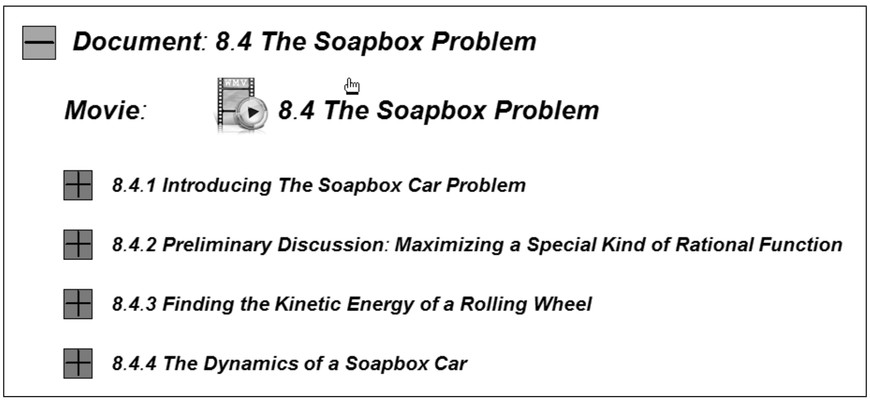
Electronic Books Can Be More Flexible
In some of my electronic books I am able to present the reader with a choice of how advanced the book should be. Many items are presented at a more elementary level with a hyperlink to an alternative deeper treatment of the topic. In some cases, I provide entire chapters at more than one

level. So, for example, in my Interactive Introduction to Mathematical Analysis, I have green and yellow entries in the table of contents as shown in the figure.

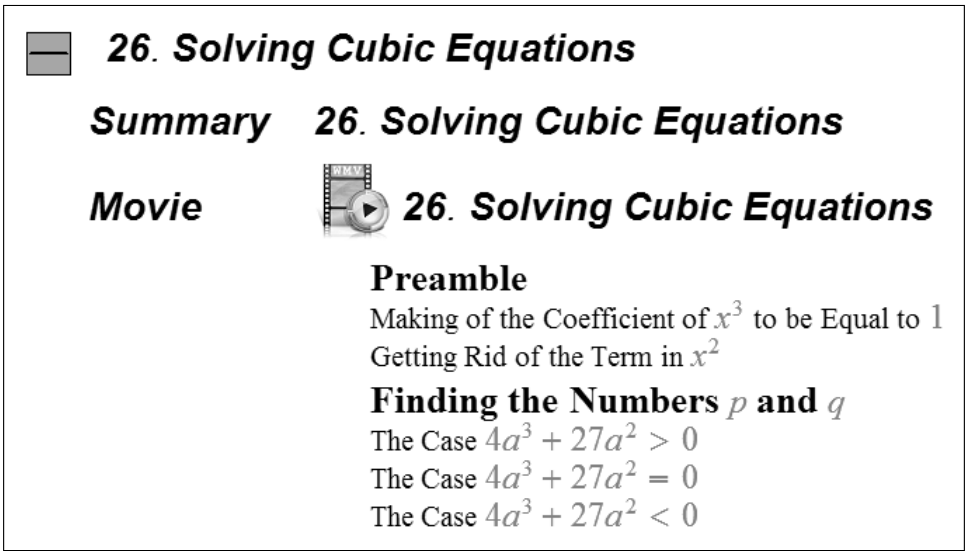


A typical first course in real analysis at an American university would teach the topology of the number line $\boldsymbol{R}$, limits of sequences in $\boldsymbol{R}$ and limits and continuity of functions of a real variable and the yellow Chapter 6 link leads to that version of the chapter. A more advanced course might replace the number line by the higher dimensional spaces $\boldsymbol{R}^k$ and by metric spaces and the green Chapter 6 link points to that version of the chapter.

In all of my electronic books I provide a variety of special enrichment topics many of which could never be included in a printed book. So, for example, my Virtual Calculus Tutor contains an optional chapter that presents an interesting problem in mechanics that is known as the Soapbox Problem



and even my more elementary Virtual Math Tutor contains a special enrichment chapter on the solution of cubic equations.



26. Solving Cubic Equations

Summary 26. Solving Cubic Equations

Movie 26. Solving Cubic Equations

Preamble

Making of the Coefficient of x^3 to be Equal to 1

Getting Rid of the Term in x^2

Finding the Numbers p and q

The Case $4a^3 + 27a^2 > 0$

The Case $4a^3 + 27a^2 = 0$

The Case $4a^3 + 27a^2 < 0$

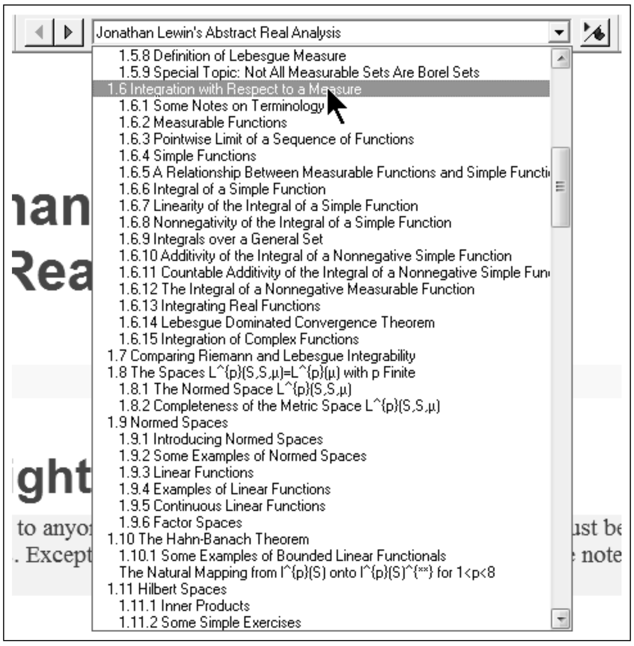
My real analysis text also provides many such enrichment links. For example, there is a link to the presentation of the full force of the change of variable theorem for Riemann integrals:

Suppose that u is a differentiable function on an interval $[a, b]$ and that its derivative u' is Riemann integrable on $[a, b]$. Then, given any function f that is Riemann integrable on the range of u, we have

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx.$$

Although the statement of this theorem is quite simple and intuitive, the proof is too difficult to include in the mainstream of an introduction to real analysis.

Electronic Books Provide an Easy Way to Create Courses.
Using a mathematical word processor, like Scientific Notebook, it is possible to produce electronic books very efficiently and rapidly. I have produced several booklets in this way for use in my advanced courses and I provide these booklets for free to my students. All of these booklets are designed for reading with any of the Version 5.5 MacKichan Software products, including the free product, *Scientific Viewer*. Although these booklets do not, as yet, contain the sophisticated multilevel tables of contents that are found in my commercial products, it is easy to browse them using the *Navigate* toolbar in the MacKichan products.



To view any of these booklets, download it and then extract it to a convenient folder in your hard drive. Then browse to that folder and open the *.tex document. My booklets are listed as follows:

(1) Jonathan Lewin, *An Introduction to Abstract Real Analysis*. This booklet contains over 220 pages and is used to groom students who intend to enter graduate school in mathematics. This booklet may be downloaded at

http://science.kennesaw.edu/~jlewin/analysis/
Jonathan-Lewin-Real-Analysis-Notes.zip

This document contains my courses in Lebesgue integration, functional analysis, integration on locally compact Hausdorff spaces, and the theory of Banach algebras.

(2) Jonathan Lewin, *A First Course in Set Theory* and *A Second Course in Set Theory*. This booklet contains about 100 pages. The first course introduces countability, set equivalence and the Bernstein theorem, the logical connection between the axiom of choice, the well ordering principle, and Zorn's lemma and some applications of these topics. The second course presents the Zermelo-Fraenkel system of axioms and shows how the axioms may be used to justify the set theoretic notions that we need in mathematics. Then it presents ordinals, cardinals, and the concept of cardinality. Finally, it shows how we can construct the system $\boldsymbol{R}$ of real numbers. This booklet may be downloaded at

http://science.kennesaw.edu/~jlewin/analysis/
Introduction-to-Set-Theory.zip

(3) Jonathan Lewin: *Introduction to Topology.* This booklet contains about 100 pages and presents the topological background that is needed to support the study of my *Abstract Real Analysis*. This booklet may be downloaded at

http://science.kennesaw.edu/~jlewin/analysis/
Jonathan-Lewin-Topology-Notes.zip

(4) Jonathan Lewin: *Introduction to Abstract Harmonic Analysis*. This booklet contains about 130 pages. It introduces topological groups, Haar measure, and the Banach Algebras $L^1(G)$ and $M(G)$ for a locally compact abelian group G. It includes some inversion theorems for

abstract Fourier transforms, the Pontryagin duality theorem, and the Weiner tauberian theorem which it applies to prove the "deep" Hardy-Littlewood tauberian theorems for infinite series. This booklet may be downloaded at

http://science.kennesaw.edu/~jlewin/analysis/
Jonathan-Lewin-harmonic-analysis-notes.zip

Electronic Books Can Include Video Content.
One of the principal features of my Virtual Math Tutor and Virtual Calculus Tutor is the set of videos that each contains. Virtual Calculus Tutor contains no less than 101 hours of video content that gives the reader the option of entering a virtual classroom in which I, the instructor, actually teach the material instead of just handing it to the student as reading material. Used correctly, the video versions of my books can be very helpful, especially to students who are struggling with the material. The video version of my Virtual Calculus Tutor also gives me the opportunity to use a CAS to provide animated graphs and other computer assisted tools to illustrate the material in a way that no printed book can achieve. Quite obviously, I can't provide these animations in this printed version of my paper but I shall demonstrate some of them in my presentation at ATCM. However, two short excerpts from my Virtual Calculus Tutor videos can be found at the following URLs:

http://www.math-movies.com/atcm/VCT-Demo-Excerpt.html

http://science.kennesaw.edu/~jlewin/ATCM/VCT-ATCM-2012-Excerpt
/VCT-ATCM-2012-Excerpt.html

A video that gives an overview of my Virtual Calculus Tutor can be found at the following URL:

http://www.math-movies.com/VCT-Demo-June-2012.html

Combining Mathematical Word Processing with E-Mail
The combination of efficient scientific word processing with e-mail opens the door to a level of communication between student and instructor that could never have existed before the computer revolution. In just a few minutes, a student can open a Scientific Notebook document that contains lecture notes and save a portion of the document to send to me as an attachment to e-mail. In this way, they can write up material to send to me for correction, or submit a question. When I receive the e-mail, I can reply almost immediately, putting my comments into a different color field so that the student can easily identify what he/she wrote and what I wrote in reply. The following figure shows an excerpt of one of those documents with my reply in a yellow field.

Problem.

since $(x-\delta, x+\delta) \cap S \setminus \{x\} = \emptyset$ does this mean that x actually belongs to S?

Given that S is a nonempty bounded set of real numbers and that S has no limit point, prove that S must have a largest member.

We are given that S is nonempty and bounded and that S has no limit point. I want to prove that $\sup S$ belongs to S.
We define $x = \sup S$.
To obtain a contradiction, we assume that x does not belong to S. Since S has no limit point we know that the condition for every $\delta > 0$, the interval $(x-\delta, x+\delta) \cap S \setminus \{x\} = \emptyset$, and so we have our desired contradiction because x actually belongs to S.

The fact that the number x is not a limit point of S does not mean that the condition $(x-\delta, x+\delta) \cap S \setminus \{x\} = \emptyset$ must hold *for every* $\delta > 0$. What you know is that *there exists* $\delta > 0$ such that $(x-\delta, x+\delta) \cap S \setminus \{x\} = \emptyset$.
Nor have you reached your desired contradiction at this stage. Note that you have not yet used the fact that $x = \sup S$. You get your contradiction by *choosing* $\delta > 0$ such that $(x-\delta, x+\delta) \cap S \setminus \{x\} = \emptyset$ and then using the fact that the number $x-\delta$ must

One of the great benefits of this kind of communication is that it is instant. The student does not have to wait until I next hold office hours. I receive these letters at all times of the day and week and, as I have said, I reply to them almost immediately.

Conclusion

The examples of my work that I have exhibited in this paper demonstrate that, although the dominant player in the application of computer technology to the teaching of mathematics has always been CAS technology, the communications features of technology can play a role that is just as important for us as CAS technology.

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