Students Will Excel If They Are Inspired

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics
and Statistics
Radford University
Radford, VA 24142
USA

Abstract

It is well-known that the requirement of teachers' content knowledge in secondary schools (middle or high schools) varies greatly from country to country (see [4] and [5]). It is also debated how much content knowledge a future math teacher should possess before he or she starts teaching. There are states in the U.S. that do not require future middle school teachers to finish the calculus sequence in college; the rationale simply being that middle school teachers do not need to teach calculus. On the other hand, becoming a future math teacher at a secondary school in South Korea virtually requires finishing a B.S. degree in mathematics. In this paper, instead of debating how much is enough, we introduce several examples to demonstrate how technological tools can expand students' knowledge of mathematics if teachers can properly inspire them. In order to reach this objective, teachers naturally need to have broader content knowledge.

1 Introduction

In Section 2, we start with simple demonstration of how figures can inspire students' understanding of the essence of Fundamental Theorem of calculus even to a group of seventh or eighth grade students in Beijing. In the event when a teacher has no knowledge of calculus, there will be no chance for middle school students to appreciate the theorem. In Section 3, we first see how seventh or eighth grade students react to the graphs of polynomial functions and how they are exposed to the idea that the graph of the limit of a sequence of continuous functions may not be continuous. This is a concept students normally learn from an advanced calculus class later on in college. Next, we see how a college entrance exams preparation problem looks like in China, and we will see how students can be inspired to know more if teachers are knowledgeable about the applications of such content. We further introduce a typical problem that was asked in a college entrance exams regarding 3D geometry in China and we point out that this is an area where a proper 3D software package can be used for exploration to enhance students' spatial understandings prior to the exam. The last example shows how a high school student from South Korea whom has never encountered the concept of recursive sequences and limits was still able to solve a problem I proposed to her.

2 Visualizing Fundamental Theorem of calculus

During the plenary speech given by Professor Lin, Qun at the second ATCM-China, he noticed that there were many seventh or eighth grade students from Beijing in the audience and he asked students how to measure the height of a building. Surely, students understood that it can be measured by using similar triangle concept such as the one shown in Figure 1.

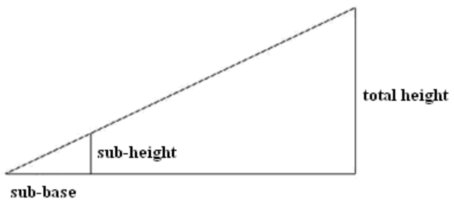


Figure 1 Similar triangles

He then asked students how one can measure the height of a mountain if its side view resembles the dark curve shown in Figure 2 and the curve is assumed to be $y = x^2$.

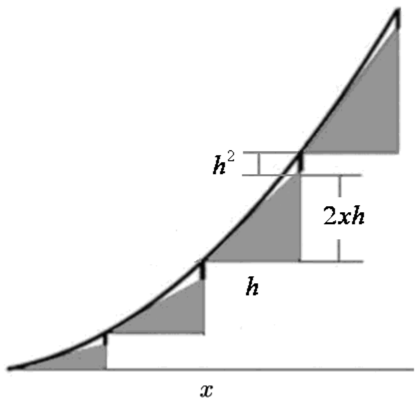


Figure 2 Total differences for $y = x^2$

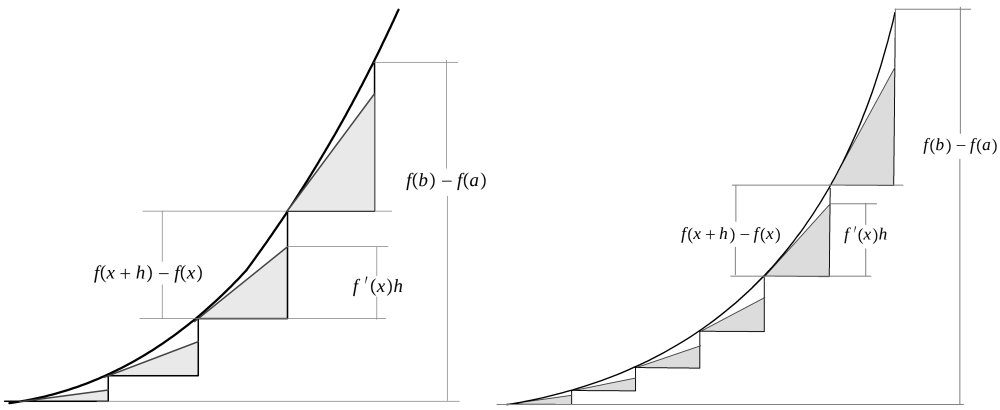
First, he reminded the students the differences at different nodes, x and $x + h$ for polynomial functions such as $y = x^2, y = x^3$ and $y = x^4$ respectively, can be worked out algebraically as follows:

$$(x+h)^2 - x^2 = (2xh) + \left(h^2\right), \tag{1}$$
$$(x+h)^3 - x^3 = \left(3x^2h\right) + \left(3xh^2 + h^3\right), \text{ and} \tag{2}$$
$$(x+h)^4 - x^4 = \left(4x^3h\right) + (6x^2h + 4xh^2 + h^4), \tag{3}$$

Not only did students see a pattern from calculations above, $2xh, 3x^2h$ and $4x^3h$, but also the Figure 2 is worth a thousand words. Professor Lin remarked that if we assume the graph in Figure 2 to be $y = x^2$, then the height of the triangle is $2xh$ and the height of the small gap is h^2 as indicated in Figure 2. We note here that students were briefly introduced that the meaning of a tangent line and the tangent line (seen in Figure 2 above) was drawn at the end point of each subinterval, and the slope of the tangent at x is called *the derivative of a function at x, and is denoted by $f'(x)$*. many students were wondering about the following:

1. Maybe the total height can be approximated by the sum of all the heights of shaded triangles, see Figure 3(a) below.

2. Furthermore, if we cut the original interval $[a, b]$ into enough small sub-intervals, the approximation will be better, see Figures 3(a) and 3(b). [Indeed, if we investigate this further by using a nice (smooth or differentiable) function $f(x) = x^2$ for $x \in [a, b]$, as demonstrated, $2xh$ is exactly $f'(x)h$ as seen in the Figures 3(a) and 3(b) below.]



Figures 3(a) and 3(b) Total difference for a function f when number of intervals increases

Students quickly responded by observing equations (1) to (3) that if $f'(x) = 2x$ for $f(x) = x^2$, $f'(x) = 3x^2$ for $f(x) = x^3$, $f'(x) = 4x^3$ for $f(x) = x^4$, and so forth, then $f'(x) = nx^{n-1}$ if $f(x) = x^n$. To sum up, students would like to say

$$f(b) - f(a) - \sum_{i=1}^{n} f'(x_i)h_i = \sum_{i=1}^{n} [(f(x_i + h_i) - f(x_i) - f'(x_i)h_i] \tag{4}$$

3. Computationally, if we take $f(x) = x^2, x \in [0, 1]$, and $x_i = (i - 1)\, h_i$, where $h_i = 1/n$ and $i = 1, 2, ...$, then it follows from the following simple calculations, middle school students can comprehend

$$f(b) - f(a) - \sum_{i=1}^{n} f'(x_i)h_i \tag{5}$$

$$= \sum_{i=1}^{n} ((f(x_i + h_i) - f(x_i) - f'(x_i)h_i) \tag{6}$$

$$= \sum_{i=1}^{n} h_i^2 = \frac{1}{n}. \tag{7}$$

Students have no problem conjecturing that $f(b)-f(a)$ can be approximately by $\sum_{i=1}^{n} f'(x_i)h_i$ when n gets large.

Therefore, we have the following Definition as discussed in [1] .

Definition 1 *If f is continuous over $[a,b]$ and differentiable over (a,b). We say that f' is Riemann integrable over $[a,b]$ if*

$$\left|\sum_{i=1}^{n}[(f(x_i+h_i)-f(x_i)-f'(x_i)h_i]\right| \tag{8}$$

$$= \left|\sum_{i=1}^{n}\left(\frac{h_i}{b-a}\right)\left(\frac{f(x_i+h_i)-f(x_i)}{h_i}-f'(x_i)\right)\right| \ll 1, \tag{9}$$

for any partition $a=x_1<x_2<...<x_n=b$, with $x_{i+1}-x_i=h_i$,where $i=1,2,...n-1$, and we write

$$\int_a^b f' = f(b)-f(a). \tag{10}$$

If we examine the equation (8) closely, it suggests if one of the expressions of $f(x_i+h_i)-f(x_i)-f'(x_i)h_i$ does not get small when the number of partitions increase, then the whole summation will not either. We note the Fundamental Theorem of calculus (FTC) is actually built in as a definition in this case. We make the following observations, more information can be found in [1].

1. To use FTC, we need to know antiderivative of a function.
2. There are functions that are not Riemann integrable such as $\frac{1}{\sqrt{x}}$ over the interval $[0,1]$. If we look closely by letting $h=-1/n$, $x=1/n$, then the expression $\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right|$ $=\left|\left(\frac{\sqrt{0}-\sqrt{1/n}}{-1/n}-\frac{1}{2\sqrt{n^{-1}}}\right)\right| = \frac{\sqrt{n}}{2}$can not be made as small as we wish. This example demonstrates if one of the terms,$\frac{f(x_i+h_i)-f(x_i)}{h_i}-f'(x_i)$, in equation (4) cannot be made small, then the function f' is not Riemann integrable.

3 Don't Underestimate Students' Abilities

3.1 Explore Concepts That Students Encounter Later

I also had an opportunity to give a hands-on workshop to the same group of seventh and eighth grade students at the 2nd ATCM – China. They had not encountered the concept of a function, but they did have access to graphics calculators at the time. So, I introduced to them the graphs of $y=x^n$, when $n\in\mathbb{N}$, which leads to concepts of even and odd functions. Next, I asked them to concentrate only on the domain of $[0,1]$ for the graphs of $y=x^n$, and asked them to observe the behavior of the graph when n gets larger; for example, they had no problem identifying the values of y gets closer and closer to 0 when n increases and $x\in[0,1)$; they also observed that there is almost a vertical spike at $x=1$. After experimenting with graphics calculators, they agree the graphs of $y=x^{20},x^{25}$ and x^{30} respectively look like the one in Figure 4 below, and they can identify which one is which for the graphs of $y=x^{20},x^{25}$ and

x^{30} respectively.

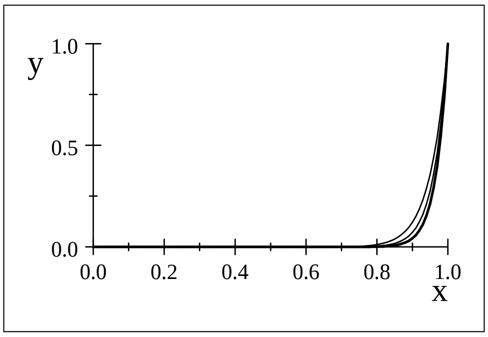


Figure 4 Graphs of $y = x^{20}, x^{25}$ and x^{30} (11)

Finally, I asked this brutal question:

$$\text{Can you conjecture what the graph of } \lim_{n\to\infty} x^n \text{ would be?} \tag{12}$$

I did not get into the definition of 'limit' of course, but they know n can be as large as they wish. They know the graph cannot have a vertical line at $x = 1$ (otherwise, it will not be a function), so what is happening here? I led them to the graph of a non-continuous function for $\lim_{n\to\infty} x^n$ when $x \in [0, 1]$. In other words, for $x \in [0, 1]$, we have

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & if \quad x \in [0, 1) \\ 1 & if \quad x = 1 \end{cases}, \tag{13}$$

whose graph we sketch in Figure 5.

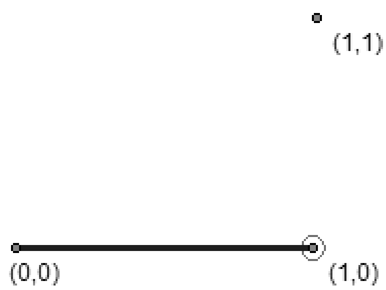


Figure 5 Graph of a discontinuous function

And students were surprised to see a sequence of functions $\{x^n\}_{n=1}^{\infty}$ that are continuous and yet its limit, $\lim_{n\to\infty} x^n$, is not. This is the concept of 'uniform convergence', which students normally learn in an advanced calculus class. This example shows that students would have less opportunities to learn more if teachers do not have adequate content knowledge.

3.2 A Preparation Problem for Chinese College Exam

One type of problem regarding the topic of 'symmetry' for senior high school students in China is to find the symmetry point of $P(x, y)$ on a curve $f(x, y) = 0$ with respect to the line of the form $Ax + By + C = 0$. We refer to the following graph and note $P'(x', y')$ is the desired point.

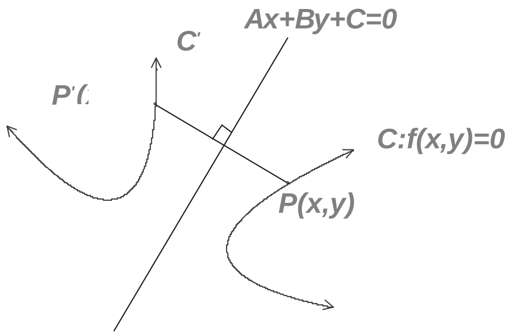


Figure 6 Finding a symmetric point with respect to a line

Most students in China have no problem finding x' and y' by using the property of symmetry as demonstrated in (14):

$$\begin{cases} A\dfrac{x+x'}{2} + B\dfrac{y+y'}{2} + C = 0 \\ \dfrac{y-y'}{x-x'} \cdot \left(-\dfrac{A}{B}\right) = -1 \end{cases} \Rightarrow \begin{cases} x' = x - 2A\dfrac{Ax+By+C}{A^2+B^2} \\ y' = y - 2B\dfrac{Ax+By+C}{A^2+B^2} \end{cases} \tag{14}$$

I was surprised because this concept was seldom discussed in a calculus textbook (at least in U.S.) that I know of, but this is precisely the idea mentioned in [3] when finding the general inverse for a parametric curve or surface with respect to a slanted line or a plane, which has many applications in Optics. We summarize the Theorem from [3] as follows:

Theorem 2 *Let the surface S be represented by* $\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$ *and $(x_0, y_0, z_0) = (0, 0, \frac{d}{c})$ be on the plane $ax + by + cz = d$. Then the reflection of S with respect to the plane $ax + by + cz = d$ is given by*

$$\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} \tag{15}$$

$$= \frac{1}{a^2+b^2+c^2} \begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix} \begin{bmatrix} x(s,t)-x_0 \\ y(s,t)-y_0 \\ z(s,t)-z_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

It is shown in [3] that the three dimensional formula does reduce to the corresponding one in 2D below.

$$\begin{aligned} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \frac{1}{a^2+b^2} \begin{bmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t)-0 \\ y(t)-(\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix}, \end{aligned} \tag{16}$$

which can be proved that the formula (16) coincides with equation (14).

Example 3 *We apply the formula (15) on a so-called Steiner's Roman Surface of the form* $\begin{bmatrix} \cos^2 s \sin 2t \\ \sin 2s \sin t \\ \sin 2s \cos t \end{bmatrix}$, *shown on the left of Figure 7. Then the reflection of S with respect to a plane (shown in Figure 7) can be found and shown on the right of Figure 7 .*

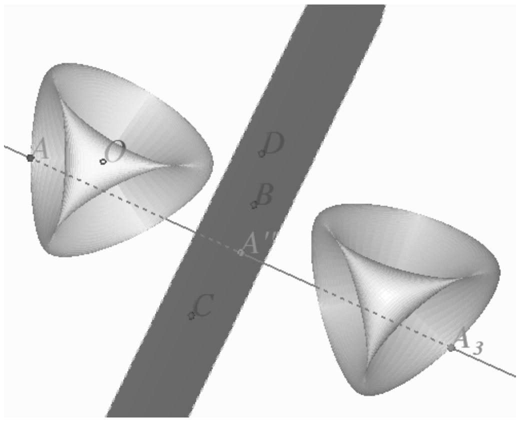


Figure 7 *A Steiner's Roman Surface, plotted by GInMA.* (17)

Remarks:

1. I was told that most teachers in China stopped at only deriving formula (14) or using formula (14) to find the symmetry of a simple curve with respect to $y = mx + b$ without exploring the applications of the formula (14).

2. Designated mathematics experiments periods are approved by Ministry of Education in China and should be implemented in high schools in China. However, many schools use those periods for preparing students for college entrance exams by doing more drills. Without a technological tool, it is clear that students would not be able to experiment finding the general inverse for a 2D parametric curve with respective to $y = mx + b$. Not to mention asking students to find the general inverse for a 3D parametric surface with respective to a plane by hand is impossible.

3. We note that the concept of finding the general inverse for a 2D parametric curve with respective to $y = mx+b$ is very much related to the concept of orthotomic curve as shown in [3]. Orthotomic curve for a curve C is the locus of the reflections of a light source P with respect to all tangents of C.

4. Caustic curve can be viewed as the envelope of light rays reflected or refracted by a curved surface or object (see definition from Wikipedia at http://en.wikipedia.org/wiki/Caustic) and see Figure 8 below. Caustic is also the locus of centers of curvatures of orthotomic curve. (see [3]).

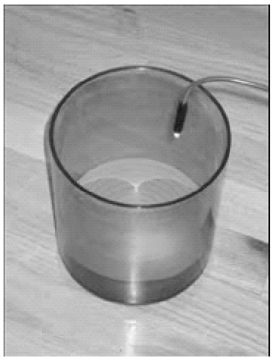


Figure 8 A Caustic Curve.

Here is an example we linked a mathematical formula to a concept learned in Physics. It is a clear that a teacher not only should have broader content knowledge in mathematics but also should expand their knowledge beyond mathematics. Furthermore, it is important to conduct mathematics experiments and design a curriculum where we can link mathematics experiments to applicable fields.

3.3 An Exam Problem About A Cube Being Cut By A plane

I was browsing through the types of college entrance exams problems from China one day and I encountered the following

Problem: Consider the following cube in the Figure 9

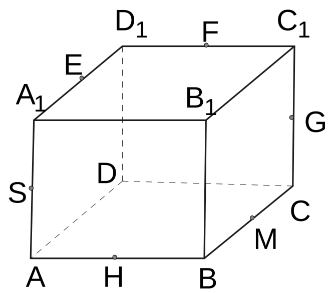


Figure 9 A cube

Suppose the points E, F, G, H, M and S are midpoints of their respective sides. Prove that E, F, G, H, M and S lie on the same plane, and further prove the plane is a regular hexagon.

My first reaction is it will be wonderful if students have a 3D geometry software and explore Figure 9 (in a mathematics experiment class), and have opportunities to rotate the graph (see

Figure 10). We leave this as an exercise to the readers to explore.

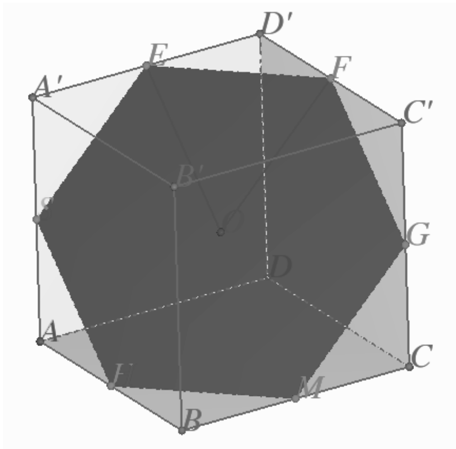
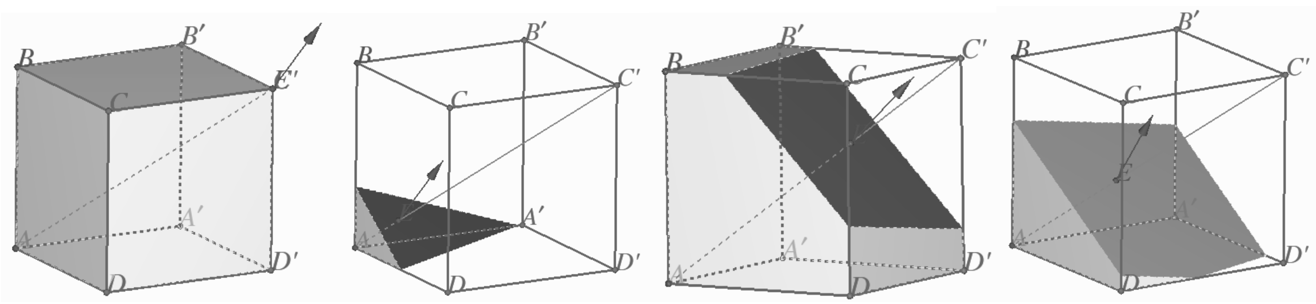


Figure 10 A hexagon and a cube

Instead, we propose a followed-up question: *Suppose we again start with a cube and intersect with a plane, sketch all possible intersecting figures.* The answer is that it could be a six sided polygons as we discussed earlier; in addition, it could be a point, a triangle, four or five sided polygons, see Figures 11(a),(b),(c) and (d) below:



Figures 11(a),(b),(c) and (d) Different possibilities when a plane cuts through a cube

Therefore, it is clear that there are great merits by allowing technological tools for experiment in a math class.

3.4 Surprised About A High School Student

I and a colleague encountered a high school student in Seoul, South Korea, during a lunch break at ICME 12. A couple of students sat next to us, we were first amazed how much they could understand from our English conversation, and next we heard one student was really good at math; without knowing how much knowledge the student had, I asked the following question:

$$\text{Given } a_{n+1} = \sqrt{a_n + 5}, \text{and } a_1 = 1. \text{ Find } \lim_{n\to\infty} a_n. \tag{18}$$

I did not expect the student to solve it because I was thinking she may not have learned the concepts of limit and recursive sequence, I later confirmed that the student had no prior knowledge on limit and recursive sequence. To my surprise, it took the student less than

5 minutes to get the right answer. In fact, she gave me a symbolic answer not numerical approximation. This was what she did, of course I polish up her answer as follows:

$$\begin{aligned} \text{If } \lim_n a_n &= a, \text{ then } a = \sqrt{a+1}, \text{ and proceed solving a quadratic equation} \\ a^2 &= a+1, \text{ which leads to } a = \frac{1 \pm \sqrt{5}}{2}. \\ \text{Since } a_n &> 0, \text{ we have } \lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}. \end{aligned} \tag{19}$$

The answer is perfect, but I did not get to show her the graphical interpretation of this problem, which could have inspired the student to learn more mathematics. We see in this case that there is an intersection between $y = x$ and $y = \sqrt{x+1}$, and the intersection is exactly the answer to this problem. See the graph below

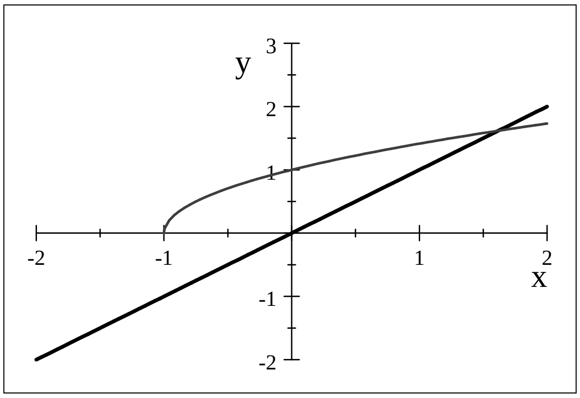


Figure 12 Intersection between $y = x$ and $y = \sqrt{x+1}$

This problem naturally is a starting place when introducing the Fixed Point Theorem in a mathematical analysis class. Naturally, we consider the followings:

1. What kind of function will have an intersection with $y = x$? This is easier to see: Suppose that f is a continuous function from an interval $[a, b]$ to $[a, b]$, then it follows from intermediate theorem that there exists a $c \in [a, b]$ such that $f(c) = c$.

2. What kind of function will have only one intersection with $y = x$? This leads us to the Fixed Point Theorem, which we state as follows

Theorem 4 *Suppose that f is a continuous function from an interval $[a, b]$ to $[a, b]$ and that $\delta < 1$ and that for every number $x \in (a, b)$ we have $|f'(x)| \leq \delta$. Then $f(x) = x$ has a unique solution for $x \in [a, b]$.*

It is worth noting that one can use graphical capability of a device to check if a function f satisfying $|f'(x)| < 1$, which can be difficult if calculated by hand. Furthermore, the idea of checking if a function function f satisfying $|f'(x)| < 1$ by graphing $y = f'(x)$ can be extended to higher dimensions. As expected, we have higher dimension Fixed Point Theorem as follows:

Theorem 5 *Suppose that f is a continuous function from a convex subset S of $\mathbf{R}^n$ into S and that for every point x in the interior of x we have $\|f'(x)\| \leq \delta < 1$. Then the function f can have at most one fixed point in S.*

We consider the following 2D example, which is discussed in [2]

Example 6 *Let* $f(x,y) = \left(1+\frac{1}{3}\cos xy, 1+\frac{1}{3}\cos(x+y)\right)$ *for* $0 \le x \le 2$ *and* $0 \le y \le 2$. *We write the coordinates of* $f(x,y)$ *as* $f_1(x,y)$ *and* $f_2(x,y)$ *for each point* (x,y) *in the square* $[0,2]\times[0,2]$. *We observe that*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{y}{3}\sin xy & -\frac{x}{3}\sin xy \\ -\frac{1}{3}\sin(x+y) & -\frac{1}{3}\sin(x+y) \end{bmatrix}.$$

We would check if

$$\left\| \begin{bmatrix} -\frac{y}{3}\sin xy & -\frac{x}{3}\sin xy \\ -\frac{1}{3}\sin(x+y) & -\frac{1}{3}\sin(x+y) \end{bmatrix} \right\| < 1$$

before applying Theorem . The norm for a 2 by 2 matrix will not be discussed here, but noting to obtain an upper bound for the norm of the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

as the point (x,y) *varies through the square* $[0,2]\times[0,2]$ *we need to maximize the function* g *defined by the equation*

$$g(x,y) = \frac{1}{6}\sqrt{\left(\begin{array}{c} 2\left(\sin^2 xy\right)y^2 + 2\left(\sin^2 xy\right)x^2 + 4\sin^2(x+y) + \\ 2\sqrt{\left(\begin{array}{c} \left(\sin^4 xy\right)y^4 + 2\left(\sin^4 xy\right)y^2x^2 + \left(\sin^4 xy\right)x^4 + 4\sin^4(x+y) + \\ 8\left(\sin^2 xy\right)y\left(\sin^2(x+y)\right)x \end{array} \right)} \end{array} \right)}. \tag{20}$$

With a graphical tool, we plot $z = g(x,y)$ and $z = 1$ together and we observe that the maximum value of the function g is less than 1 in Figure 13. We can therefore deduce from Theorem 5 that the function f can have at most one fixed point in $[0,2]\times[0,2]$. We leave it to readers to show that the fixed point of this function is approximately $(1.1789, .852)$ by solving the following system of equations:$1+\frac{1}{3}\cos xy = x$ and $1+\frac{1}{3}\cos(x+y) = y$.

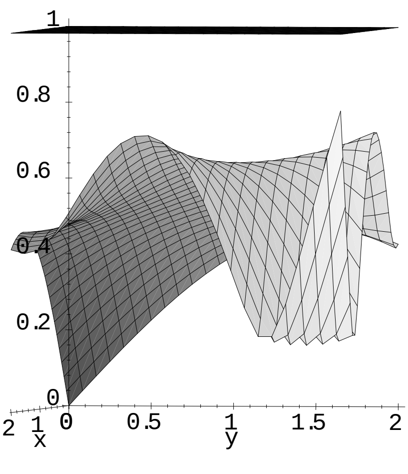


Figure 13 Graphs of $z = 1$ and $z = g(x,y)$

4 Conclusions

We often hear many debates or offer solutions to solve the problem of U.S. students performing poorly in an international competition in math and science. It does not matter whether we have a solution or which solution will work. However one thing is certain – unless we increase the content knowledge of pre-service teachers and allow more time for explorations, students will never benefit from teachings tailored to the test curriculum. We see that the mathematics curriculum in China is changing by allowing more technological tools into a math class. However, most schools are using those mathematics experiment periods to do more preparation problems for entrance exam purposes. In this case, unless appropriate authorities in China provide certain weights or incentives to schools who offer and conduct true experimenting mathematics classes, any technological tools will be stored in safe place with no use.

It is understandable that many educators in the U.S. try to make teachers accountable for students' success in a testing environment due to many reasons; however, one should note that the more U.S. curriculum is driven toward testing based, the sooner they are turning away opportunities allowing students to be more innovative or creative. The curriculum in China, South Korea and many other countries adopt high level of content knowledge in mathematics in their curricula. To become a secondary math teacher, he or she will need to pass courses such as Abstract Algebra, Real Analysis, Topology and etc. but we all know such curriculum demands so much theory and emphasizes too much memorization with too little applications is always being criticized as lacking creativity.

We have seen examples in this paper that visualization plays a pivotal role in stimulating students' interests in mathematics from an early age. Regardless if one would like to fix a problem in the U.S. or otherwise, there is no doubt that drills on doing repetitive exercises will not be the driving force for cultivating students' creativity. Technology is here not to increase a student's performance in a test; rather, it is here to increase one's knowledge in math and science. Educators in the East should not only be interested in knowing how to retain or improve their standings in international math and sciences competitions, but also think about how to seriously inspire more students to become interested in math and sciences from early ages. On the other hand, educators in the U.S. should no longer ignore the consequences of lacking mathematics content knowledge in pre-service schools trainings. In summary, to enhance mathematics education quality in the East and the West, we need to continue discussing the innovative use of technology in teaching, learning and research in mathematics. After all, broadening teachers' content knowledge through innovative use of evolving technological tools, will also broaden students' knowledge in STEM (Science, Technology, Engineering and Mathematics) areas, which definitely will provide them better job opportunities and good for societies as a whole.

Software Package
GInMA, 2012, Nosulya, S., Shelomovskii, D. and Shelomovskii, V.
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