Develop Students' Visualization and Understanding of Functions Through Geometry and Pictures with Sketchpad 5

Scott Steketee
KCP Technologies
University of Pennsylvania Graduate School of Education
stek@kcptech.com

Steven Rasmussen
KCP Technologies
srasmussen@kcptech.com

Abstract: *Students understand the concept of function more deeply by using dynamic mathematics software to manipulate an independent point and observe the behavior of the dependent point. This approach gives students an important visual window on the behavior of functions, on domain and range, and especially on relative rate of change and on composition of functions.*

An important abstraction in students' understanding occurs when they realize that they can use a function to map an entire set of input values to a corresponding set of output values. By working with geometric rather than numeric functions, students can see this process as one of mapping a shape to a corresponding transformed shape, or mapping a picture to a corresponding transformed picture. By considering shapes or pictures not only as collections of points, but also as recognizable visual objects, students can more easily understand the important duality that functions can operate both atomically (transforming a single input value or point) and collectively (transforming an entire set of input variables).

Students can use Sketchpad 5 to define such transformations using isometries, similarity transformations, affine transformations, or arbitrary geometric constructions. Two particular classes of functions that interest them are transformations that remind them of those they see in popular media, and the transformations that artists use to paint a realistic 3D scene on a flat surface (anamorphic street art).

Introduction

Mathematics educators have engaged in serious discussion over many years about how we ought to teach algebra, and what elements we ought to emphasize [2, 3, 7]. There appears to be increasing agreement that the concept of function should serve a unifying role, providing a thread around which the algebra curriculum can be organized.

A problematic aspect of this developing consensus is the difficulties students have with functions and the related concepts of domain, range, covariation and rates of change, and composition. Most serious are the different understandings of *function* itself that students must develop and integrate. Functions can be variously regarded as recipes for action, as processes, as correspondences, and as objects to be acted upon directly, and a mature view of function encompasses all of these aspects, enabling a student to take different viewpoints in different contexts.

Besides the cognitive difficulties involved in simultaneously holding a variety of views of function, many students have similar difficulties with the concept of *variable* as well. This is not surprising, because variables like functions are used in a wide variety of contexts and for a wide variety of purposes. The term may refer to an unknown (unchanging) value, to a value that can take on a small number of discrete values, to a place-holder in a rule for calculating (such as $A = l \cdot w$), or to a quantity that actually varies. And it's no wonder that students have difficulty understanding covariation and rate of change in functions like $f(x) = x^2 - 1$ when they immediately think of the function as a Cartesian graph in the form of a parabola; this visual representation of the function contains no explicit representation of either variable.

This paper proposes that we begin to correct students' misconceptions of function by using dynamic mathematics software [5] to provide two crucial elements: opportunities to control the continuous variation of an independent variable, and visual representations of functions in which the behavior of a function is manifested through the motion of the variables on the computer screen.

There are important precursors to this approach. Dynagraphs [4] provide students the opportunity to drag independent variables with a mouse and observe the motion of independent and dependent variables on the screen, but with the limitation that Dynagraph variables are restricted to lines – an input axis for one variable and an output axis for the other. A line connecting the two variables serves as a visual cue that the two variables are connected, and helps students to observe the behavior of the function and to analyze the relative rates of change. "Parallel axes representations" [1] and "function diagrams" [8] are visually similar representations without the dynamic dragging of the variable. (The current paper elaborates ideas presented at ATCM 2010 [6].)

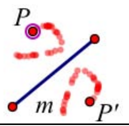
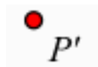
Geometric Functions

The main difference between Dynagraphs and the "Geometric Functions" described here is that the variables for a geometric function are not confined to an axis, but are free to move in the plane. In mathematical terms, geometric functions take $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, whereas dynagraphs take $\mathbb{R} \rightarrow \mathbb{R}$. At first thought, this seems sounds like folly: students are normally expected to have years of experience with $\mathbb{R} \rightarrow \mathbb{R}$ functions before they ever encounter $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ functions. However, students have abundant experience working in two dimensions, so $\mathbb{R}^2$ is hardly foreign to them, particularly if we don't insist on coordinatizing the plane. (In fact, we should consistently *avoid* coordinatizing the plane when introducing geometric functions.)

Using variables in two dimensions provides several very important advantages for students, advantages that are important to students' ability to develop and refine the concepts of variables and functions. First, the most natural way to manipulate variables on the computer is by dragging them, and dragging is a two-dimensional process, whether the input device is a mouse, a trackpad, or a touch screen. Second, the computer's output appears on a two-dimensional screen. Third, human visual perception, the sense of sight, is two-dimensional, as is our visual memory. We remember pictures far more readily than we do numbers or formulas.

In one sense, there is nothing new about this approach: students have studied geometric functions for years, and are already familiar with several families of such functions by the time they begin algebra. But we seldom associate geometric transformations with algebraic functions, even though the mathematical terms *transformation* and *function* are synonyms. We even adopt different terminology to describe the variables: independent and dependent variables in algebra, and pre-image and image in geometry. Most current algebra curricula fail to take advantage of students' prior experiences with transformations, and lose the opportunity to have students drag the independent variable, observe the variation of the dependent variable, form a visual image of the behavior of the function, and observe rate of change in the form of related motions on the screen.

The following table illustrates several corresponding features between functions in the numeric realm and functions in the geometric realm. Note that geometric variables are single moveable points, just as numeric variables can be thought of a single moveable numbers.

Variable	**Example Function**	**Independent Variable (Pre-image)**	**Dependent Variable (Image)**	**Function Notation**
Number	$f(x) = 2x - 3$	5	7	$f(5) = 7$
Point	Reflection Across m 			$r_m(P) = P'$

In this paper, we describe in detail specific geometric functions activities that we believe can provide students with these opportunities, paying particular attention to the transition from an action view of function to a correspondence view. Many of these activities are available on the web site of the Dynamic Number project (http://www.kcptech.com/dynamicnumber).[1]

Introducing Geometric Functions

With numeric functions, students often observe and record a function's behavior by creating a table of values. Looking at a table of numbers doesn't give a good sense of how a function behaves, so students often graph the values in the table. With a geometric function, students can trace both the input and output points, drag the input, and see an immediate image of the function's behavior. These traces, shown in Figure 1, are the equivalent of a table of values for a numeric function, but give a much better sense of the function's behavior.

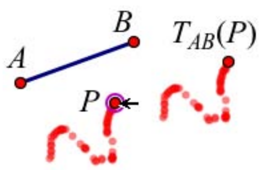


Figure 1: A Geometric "Table of Values"

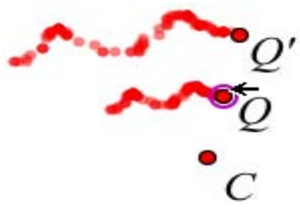


Figure 2: Relative Rate of Change

As students drag the independent variable in a function like the one shown in Figure 2, they can also observe the motion of the dependent variable relative to the independent variable. By dragging the independent variable, students exercise close control over both its speed and its direction, and can easily observe that in Figure 1 (a translation), the dependent variable's speed and direction exactly match that of the independent variable, but that in Figure 2, the variable's speed is greater than that of the independent variable, although the directions in which the two variables move are parallel. Because the traces consist of dots, they reveal not only the distance and direction but also the speed of dragging, so that students can use the greater spacing between dots to observe that the dependent variable is moving faster than the independent variable. The student's observations of

[1] This work was supported in part by the National Science Foundation Dynamic Number grant, grant #0918733. Opinions and views remain the authors', and not necessarily those of the NSF.

the behavior of the function are in part based on the observed motion of the variables, allowing the phrase "rate of change" to take on a meaning based on the changes observed on the screen.

Another particularly interesting aspect of function behavior is the existence and location of fixed points, that is, locations where the dependent variable coincides with the independent variable. Fixed points are very helpful in distinguishing one family of functions from another, and are particularly important when functions are applied repeatedly, as in iterated function systems.

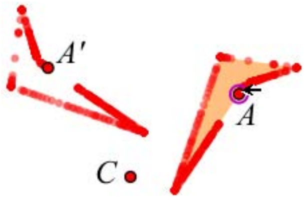


Figure 3: Point A Restricted to a Domain

In Figure 3, the student has restricted the domain of the independent variable to a polygon. This domain restriction helps the student visualize the behavior of the function, since the features of the traced dependent variable are no longer a result of free form dragging. As before, the traces make it clear that both variables moved more quickly during one portion of the movement (along one side of the polygon) that they did during the remainder of the movement.

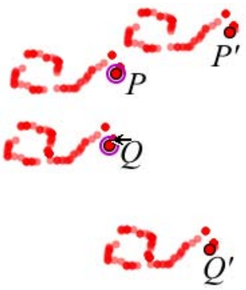


Figure 4: Functions from the Same Family

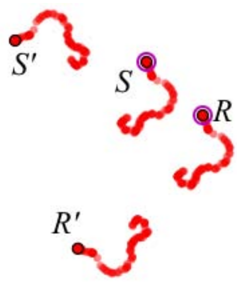


Figure 5: Functions from Different Families

Figures 4 and 5 each show two different functions. The behavior of the two functions in Figure 4, as shown by the traces left on the screen, allows students to conclude that these two functions come from the same family. This is particularly clear because the student selected both independent variables and dragged them in tandem, so that the shapes traced out were the same. Similarly, students can use the behavior of the functions in Figure 5 to conclude that these functions come from different families.

As students construct and investigate geometric functions, they should use the vocabulary of functions whenever possible. *Independent* and *dependent* variables, *domain*, *range*, *restricted domain, relative rate,* and *fixed point* should become familiar terms in the geometric context, to help students make sense of the same vocabulary in the numeric/symbolic context. Similarly, students should become familiar with function notation by using it whenever the function can be explicitly characterized (*e.g.*, $T_{AB}(P)$ when the function is a translation by the vector from A to B).

The examples in this section are based entirely upon similarity transformations. We choose these particular transformations because they are easy to understand and they are already familiar to students, but there is no fundamental reason to exclude other kinds of geometric functions, nor is there a fundamental reason to avoid using compass-and-straightedge constructions that duplicate geometric transformations. For instance, if point *A* is the independent variable and point *B* is fixed, constructing the midpoint of the segment is equivalent to dilating point *A* by 0.5 about point *B*. Nonetheless, we will continue using similarity transformations for the time being, taking advantage of their simplicity and familiarity, until later in this paper.

As these examples suggest, there are several powerful advantages in using the geometric realm to introduce students to functions:

- Students create a variety of functions easily, starting from a blank screen.
- Students describe functions by their behavior and the relative rates of change of their variables.
- Students restrict the independent variable to a domain to aid their observations.
- Students classify functions into families based on their behavior.

Composition with Geometric Functions

In this section we consider the ways in which geometric functions can be useful to students who are learning about function composition. Composition seems to be particularly difficult for students to understand. It's likely that this difficulty arises partly from students' poor skills in observing and analyzing the behavior of functions, and partly from a weakness of the visual representation on which students are most reliant (the Cartesian graph). The Cartesian representation is particularly unhelpful in understanding composition.

One strength of a Cartesian graph is the way in which it portrays the two variables on orthogonal axes, with the independent variable on the horizontal axis and the dependent variable on the vertical axis. But this strength becomes weakness when the dependent variable of one function is used as the independent variable of the other. This "intermediate" variable, which is central to the definition and behavior of the composed function, finds itself with no place to go on the Cartesian graph, and so the visual representation on which students are most reliant (some might say over-reliant) becomes useless for one of the most important concepts related to function. This difficulty manifests itself in students' general lack of understanding of composition and in their all-too-common inability to understand, and explain the difference between, $f(g(x))$ and $g(\mathrm{f}(x))$.

By concentrating students' attention on the behavior of functions, geometric functions can help to overcome these difficulties. Students begin by creating two different functions and observing the behavior of each. They drag one independent variable, trace the path of its dependent variable, generate a record of that behavior, and then drag the second independent variable along the path traced out by the first dependent variable. The visual representation and the kinetic experience of dragging the second independent variable along the traces, provides students with a much more vivid and meaningful experience than they can get from the numeric analog.

Students working in the numeric realm can certainly choose a value for the independent variable, evaluate the first function, use the result as input to the second function, and pair this final numeric result with the number chosen initially, and repeat the entire process for some number of numeric inputs. This process is laborious, doesn't require attention to behavior and rate of change, and cannot be used to produce a visual representation of the role of the intermediate variable.

With geometric functions and The Geometer's Sketchpad, students can take the additional step of actually merging the independent variable of the second function to the dependent variable of the first. This is analogous, in the numeric realm, to substituting the formula for one function into the formula for the other. In the geometric realm the operation of merging is concrete, infused with meaning, and results in an immediately visible behavior: dragging the independent variable now causes both the intermediate variable and the dependent variable to move at rates and in directions that reveal their nature. The algebraic substitution in the numeric realm produces no visual representation, no immediate behavior to observe. For many students, the substitution is mere symbol manipulation, devoid of meaning; for all students, it's much more abstract than the geometric operation of merging the independent variable of one function to the dependent variable of the other.

The more concrete nature of geometric composition has another important consequence: students can look at the visual representation and generate a verbal description of the composed function that corresponds exactly to the function notation we'd like them to learn as a shortcut. Thus the composed function in Figure 6 can be read as *P'' is the rotation of the translation of P*, and a direct translation converts this to the appropriate symbolic form, either as $P'' = R(T(P))$ or $P'' = R \circ T(P)$. Students' ability to generate the symbolic form by reading it directly from the visual representation helps to eliminate the confusion students often have in using and understanding function notation to describe composition.

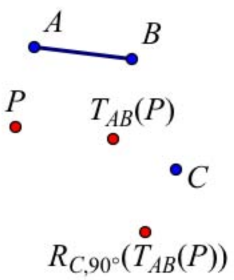


Figure 6: A Rotated Translated Image

Thus the geometric realm provides students important advantages in understanding composition of functions f and g to create $g \circ f(x)$:

- Students observe the behavior of both functions directly, by dragging independent variables while tracing the dependent variables.
- Students drag the independent variable of the second function along the trace of the dependent variable of the first, giving visual, kinetic meaning to the abstract idea of composition.
- Students merge the independent variable of the second function to the dependent variable of the first, performing a specific action that creates the composition.
- Students drag the independent variable, observe the behavior of both component functions, and see how they contribute to the behavior of the composed function.
- Students use their understanding of the composition they created, and the visual representation that resulted, to describe the composition in words that correspond directly to the proper function notation.

Geometric Functions as Mappings

In this section we consider ways in which geometric functions may help students to move from a process view of functions to what has been called a correspondence view,in which students see the correspondence between a set of input values and a set of output values as a thing in itself. In the activities described above, students transform many values of the independent variable by dragging it over time; at any one instant the function operates on only a single value of the independent variable. Both variables may leave traces, but those traces are ephemeral: students cannot manipulate them, and they can disappear either over time or as a result of various user actions.

In what follows, we'd like students to move from an "atomic" view to a "collective" view, and begin to conceive of a function operating simultaneously on an entire set of values of the independent variable, creating a correspondence between the points belonging to the domain and the points belonging to the range. This is one step toward thinking of the function as an object that can be modified and acted upon.

We are not obliged to restrict geometric functions to similarity transformations. We can use other constructions, provided that we identify a moveable point as the independent variable and another point as the dependent variable. Students can restrict the independent variable to a domain and use the **Locus** command to create in a single act the entire range corresponding to the domain. In Figure 7, a shearing transformation was applied to point *A*, which was then restricted to a polygonal domain. The figure shows the result of selecting both variables and choosing the **Locus** command.

As another example, Figure 8 shows a construction in which point *P* on a horizontal line drives the construction of point *P′* in such a way as to guarantee that distances *PP′* and *FP′* are always equal, satisfying the locus definition of a parabola. The **Locus** command uses the set of points on the restricted domain (the horizontal line) to construct the corresponding set of dependent points that form the range (the parabola). In the context of geometric functions, the parabola might be called the image of the line or the range that corresponds to the line as a domain. This sense, that the function maps the straight horizontal line to the curved parabola, is a new way of thinking even for students who have constructed this very locus outside the context of a function as a mapping.

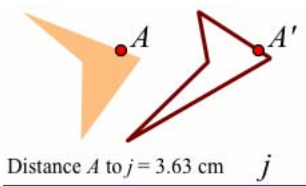


Figure 7: Horizontal Translation by the Distance from *A* to *j*

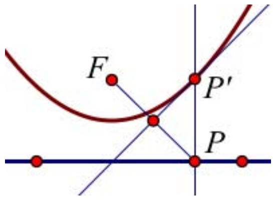


Figure 8: The Image of the Line is a Parabola

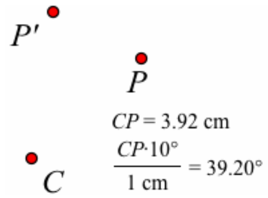


Figure 9: Rotation Depends on Distance

This idea, that any construction relating an independent point variable to a dependent point variable defines a function, opens a world of opportunities for students to explore the way in which a function maps an entire domain to a range. Sketchpad implements such functions as custom transformations, defined by the two point variables. These transformations can be applied to a variety of domains, including straight objects, circles, polygons, graphs, loci, and even pictures.

Figure 9 shows a construction in which Sketchpad has been used to rotate point *P* about point *C* by an angle that is proportional to the distance *CP*. By selecting points *P* and *P′* and choosing **Transform | Define Custom Transform,** the student defines a function that encapsulates the way in which P' depends on P. In figure 10, the student has applied the function to a circle, and is dragging point P around the circle to verify that the range (the image of the circle) does indeed correspond to the domain (the circle). In Figure 11, the student has applied the same function to a picture. In Figure 12 the student has introduced a parameter into the calculation that defines the function, thus making it easy to modify the function itself.

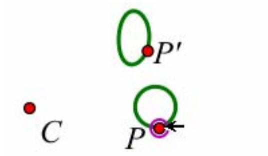


Figure 10: The Function Applied to a Circle

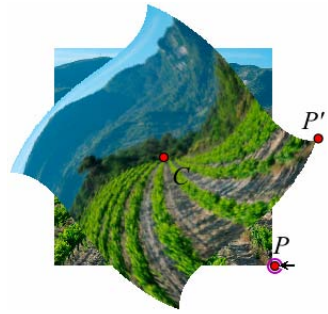


Figure 11: The Function Applied to a Picture

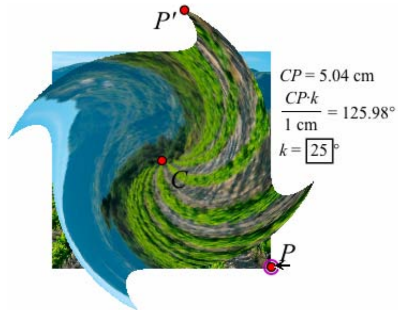


Figure 12: A Parameter Modifies the Function

By modifying the function, and by observing and explaining the results of doing so, the student is beginning to think of the function as an object, as something to be operated upon in order to modify the mapping that takes the entire domain to the entire range. This is a new way of thinking about functions, and represents an important shift in students' understanding.

We do try to develop this understanding, of a function as an object to be operated upon, in the traditional algebra curriculum. For instance, we may have students graph the function $f(x) = mx + b$, ask them to modify m or b, and then ask them to describe the effect on the graph. But we seldom ask them to explain how the graph encapsulates the way in which the function maps the domain (the x axis) to the range (the y axis). The result is often that students approach the investigation in a mechanical way, so that the language of their explanation identifies the independent variable as the gradient m and the dependent variable as the line. This is hardly the function we want them to be thinking about, and it should be no surprise that confusion often results.

By developing the concept of function as object in the geometric realm, as in this example of the picture transformation, the function itself remains in the foreground: the defining variables P and P' are still present, and the mapping of the original picture (the domain) to the transformed image (the range) is emphasized. By varying the parameter, it is obvious to students that they are operating on the function, and they can describe how changing the parameter changes the function.

A picture makes a very nice set of points to which to apply a function, for two reasons. One is that even if the range overlaps the domain, the student can still visualize the effect of the function because it's easy to remember the shape and features of the original picture. Photographs, art, and other visual imagery are popular in part because they're easy to remember; our sense of sight is attuned to such images. Secondly, students live in a very visual world, a world in which they see changing images constantly on their cell phones, video games, and televisions. By creating and operating upon functions to generate their own interesting dynamic images, we can catch their imaginations, and give them an idea of the way in which the computer graphics that surrounds them is generated through mathematical functions.

Anamorphic street art is another nice example of transforming a picture. Figure 13 shows Julian Beever posing with one of his paintings, portraying what appears to be a garden installed in a hole in the sidewalk. Beever paints on the sidewalk, distorting his painting so that it appears to be an actual three-dimensional scene when viewed from one specific location — the position of the camera used to take the photograph.

Figure 13: The Artist Posing[2]

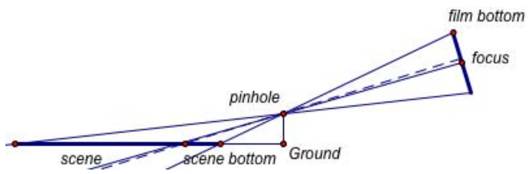


Figure 14: The Pinhole Camera Construction

Figure 15: The Sidewalk From Above

It's an interesting Sketchpad challenge to model a pinhole camera as in Figure 14 and use the Sketchpad construction to find the relationship between a point on the sidewalk and a point on the camera's film (or equivalently, a point on the retina of the viewer's eye). Once the basic model is built, it's possible to construct the two variables (a point on the scene and a point on the film) in either order. By constructing the film point from the scene point, the resulting function can be used as a camera to transform the distorted pre-image on the sidewalk into the realistic image seen through the pinhole. By constructing the scene point from the film point, the resulting function works like a projector to transform the realistic pre-image on the film into the distorted image on the sidewalk. This function was used to produce Figure 15, reconstructing an overhead view of the distorted image that Beever must have painted on the sidewalk.

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[2] Image by lrargerich at http://www.fotopedia.com/items/flickr-2424682377. Creative Commons license Attribution 3.0 Unported (CC BY 3.0)

Artistically inclined students may enjoy using the camera model to start with a picture of their own and run the model in projector mode to figure out what they need to paint on the sidewalk to create a corresponding three-dimensional illusion. This could even turn into a nice project for a class to do on the sidewalk outside their school.

Conclusion

By beginning with geometric functions to introduce basic concepts of independent and dependent variables, domain and range, covariation and rate of change, and function families, students can begin with their study of function with concrete experiences of function behavior and useful visual representations that form important connections between the geometric and the numeric/symbolic realms. By returning to geometric functions as they begin the study of composition, identity functions, and inverses, students can gain the tools they need to make sense of these difficult concepts. By interspersing geometric functions with functions from the numeric/symbolic realm in their later study, students can better develop their understanding of how functions transform entire domains, and of how they can be viewed as objects that can themselves be operated upon.

Throughout, these dual approaches can help students to integrate their views of algebra and geometry, enriching both realms by making connections between them.

References

[1] Arcavi, A., & Nachmias, R. (1989). Re-exploring familiar concepts with a new representation. *Proceedings of the 13th International Conference on the Psychology of Mathematics Education,* 3, 174-181.

[2] Chazan, D. (2000). *Beyond Formulas in Mathematics and Teaching*. New York: Teachers College Press.

[3] Chazan, D., & Yerushalmy, M. (2003). On Appreciating the Cognitive Complexity of School Algebra. *Research Companion to the Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

[4] Goldenberg, E. P., Lewis, P., & O'Keefe, J. (1992). Dynamic Representation and the Development of a Process Understanding of Function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy, MAA Notes No. 25* (pp. 235–60). Washington, D.C.: Mathematical Association of America.

[5] Jackiw, N., (2009). *The Geometer's Sketchpad V5.* Emeryville, CA: Key Curriculum Press.

[6] Steketee, S., (2010). Variables and Functions: Using Geometry to Explore Important Concepts in Algebra. *Electronic Proceedings, Fifteenth Asian Technology Conference in Mathematics*. Retrieved from http://atcm.mathandtech.org/EP2010/invited/3052010_18464.pdf

[7] Usiskin, Zalman (1998). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Schulte (Eds.), *The Ideas of Algebra*, *1988 NCTM Yearboo* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

[8] Wah, A., & Picciotto, H. (1994). *Algebra: Themes, Tools, Concepts* [online textbook]. Retrieved from http://www.mathedpage.org/attc/attc.html