Developing Concepts in Linear Algebra and Analytic Geometry by the Integration of DGS and CAS

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Abstract: *Facilitation of the Computer Algebra Systems (CAS) in pedagogic purposes is an incentive for any mathematics researcher, yet integration of the Dynamic Geometry Software (DGS) with CAS in teaching mathematics is even a greater challenge. This paper, throughout created applets in the DGS GeoGebra and the CAS wxMaxima, sustained by additional materials, refers accurately to implementation of the integrated DGS and CAS in obtaining new teaching approaches in courses of Linear Algebra and Analytic Geometry. Created teaching/ learning resources aim to facilitate the transition period from the upper- secondary to lower- tertiary level of this course. For this reason two- and three- dimensional, dynamic worksheets have been designed and furthermore, they have been implemented in the mathematics classroom. The goal of creating these interactive dynamic worksheets is to develop important concepts in Linear Algebra and Analytic Geometry, about which students have no previous knowledge. Thus, the paper presents an in-depth research in discovering new concepts for teaching and learning this essential mathematics discipline.*

1. Introduction

This paper is a continuation of my previous work including the papers [1] and [2], as well as the presentation [3], each one of them being a part of my doctoral studies in the area of mathematics education. The paper [1] recognizes both the students and the teachers' needs for computer facilitation during the mathematics classes together with the development of study/ teaching materials in electronic version. The second one, paper [2] basically considers a comparison of the gymnasium and university curricula in the course of Linear algebra and analytic geometry, followed by a discussion and concluding remarks suggesting possible ways in overcoming the difficulties the students encounter during the transition period. The third one [3], supports these remarks by providing some examples in the DGS GeoGebra and the CAS wxMaxima. Therefore, [3] is also a continuation of the paper [1]. While [1] describes computer support in mathematics education in general, [3] is exactly specified in the field of Linear Algebra and Analytic Geometry. In addition of the explained preceding effort, this paper presents much deeper investigation in several concepts in this mathematics discipline. It is of interest to note that a strong connection between geometric and algebraic methods is implemented to introduce some basic concepts and relate them to more advanced ones.
To clearly present the results of this research, alike as in [3], I am providing visualized presentations [7] in the DGS GeoGebra, on the web page:
www.math.hu-berlin.de/~filler/donevska/applets .

2. Elementary Linear Algebra and Analytic Geometry

One usual approach in introduction to Elementary Linear Algebra and Analytic Geometry, especially to the theory of matrices and determinants is by introducing students with a completely new theory. The teacher gives definitions for basic terms about two by two determinants, writes the formulas and properties, explains their implementation in solving systems of equations, then extends the approach for three by three determinants and generalizes the idea for n by n determinants. Thus, the preface of the course is absolutely unknown for the students, who study the course for the fist time. It is more than evident that in this kind of introduction to the course, the teacher does the entire work during the class and (s)he has the main role in the classroom. Thus, according to this teaching method, (s)he is the focus of the mathematics didactics research. The proposed approach in this paper has the student and his/her activities in its focus. The paper presents the initiative that it would be much better if students can use their own previous knowledge in building a new one, while teachers only facilitate the learning process. As an output of this learning by doing approach the students realize that something they have previously learned can be implemented in constructing other theories in some mathematical disciplines that they didn't know about before. This means that instead of teaching students how to use determinants and apply them in calculating area and volume, for example, we may teach them in the *complete opposite direction* (as explained below, in 2.1). Using student's geometry knowledge in calculation of area and volume (even though it may be only some basic knowledge) and guiding students step by step, until they assemble the concept for matrices and determinants mostly on their own. Thus, the main idea is for students to use geometry knowledge for measuring lengths and calculating area and volume; and *by exploring* (using dynamic geometric software) learn how to write the same data in algebraic way with matrices and determinants. The DGS GeoGebra provide students' interactive attitude during the class. It can be expected that in this way of discovering by themselves, students obtain more permanent knowledge.

2.1 Introduction to Theory of matrices and determinants

2x2 determinants

Required previous students' knowledge in 2D Geometry is:

- measuring lengths,
- calculating areas of plane figures, such as: square, rectangular, parallelogram, triangle.

Students know that the area of the *square with side 1unit* (the square *ABCD* on Figure 1.) equals 1 squared unit. This value of 1 unit is exactly equal to the identity 2x2 determinant. So, a determinant can be defined as a number, which in this case is equal to the area of the unit square, i.e. 1.

$$A = 1 = 1 - 0 = 1 \cdot 1 - 0 \cdot 0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \tag{1}$$

This is also a very convenient moment for defining the identity matrix I_{2x2}, as a squared scheme consisted of ones (elements at the main diagonal) and zeroes (the two remaining elements), i.e.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

Further on, students can notice how changing these elements, affects the value of the determinant. The relation between the identity matrix and its determinant is given with the following formula:

$$\det I_{2\times 2} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \tag{3}$$

Now, let us draw the square *ABCD* in the coordinate plane, such that its vertex *A* is placed at the origin and the vertices *B* and *C* lie on the *x*- and *y*- axis, consequently. This geometric presentation is provided by GeoGebra, Figure 2 and Applet 1. It demonstrates that the coordinates of the vertices of the square *ABCD* are: *A*(0, 0), *B*(1, 0), *C*(0, 1) and *D*(1, 1). The coordinates of the points *B* and

C are of special interest, because their coordinates are actually the elements of the two columns in the matrix I_{2x2}. Thus, the instructor may define the two-dimensional unit vectors as follows:

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

Applying this teaching method, the student knowledge in geometry gets a different, algebraic meaning.

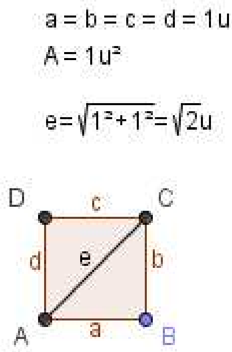

Figure 1. Area of the unit square via geometric means and students' previous knowledge

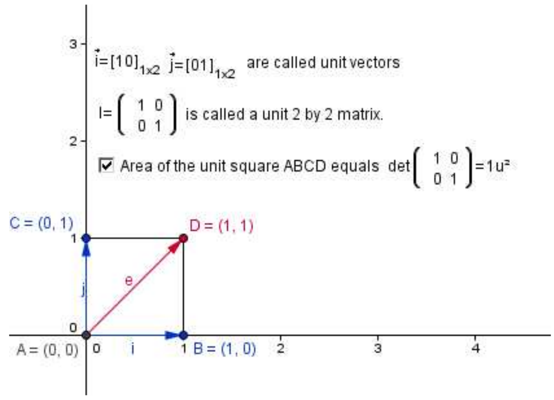

Figure 2. (Applet 1.) Notation of vertices of a unit square via algebraic means and introduction to unit vectors and the identity 2x2 matrix

Thanks to the dynamic properties of GeoGebra this idea of introduction to matrices and determinants may be extended to any *square with side a*. In order to obtain the square with side *a*, the student simply drags the terminal points *B* and *C* of the unit vectors $\vec{i}$ and $\vec{j}$ along the axes. Now, the area of the square is:

$$A_{ABCD} = a^2 = a^2 - 0 = a \cdot a - 0 \cdot 0 = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} \tag{5}$$

Thus, it can be written as a squared scheme consisted of two equal real numbers at the main diagonal and zeros (the two remaining elements). This leads to the term for the 2x2 matrix:

$$M_{2\times 2} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{6}$$

and its determinant:

$$\det M_{2\times 2} = \det \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2 \tag{7}$$

Visualized presentation of the square with side *a*, including the coordinates of its vertices, in the coordinate plane, is provided on Figure 3 and Applet 1. The same Figure 3 and Applet 1 offer a possibility to extend the idea furthermore, for any *rectangular* with sides *a* and *b*. The student changes the coordinates of the points *B* and *C* along the axes, such that sides of the rectangular differ in length, i.e. *B(a,0)* and *C(0,b)*. The area of this rectangular is:

$$A_{ABCD} = ab = a \cdot b - 0 = a \cdot a - 0 \cdot 0 = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \tag{8}$$

The corresponding matrix can be written as:

$$M_{2\times 2} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{9}$$

and its determinant can be calculated by the formula:

$$\det M_{2\times 2} = \det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = a\cdot b \tag{10}$$

While using the Applet 1 and dragging the points *B* and *C*, the student can simultaneously see how the elements of the matrices change. At the same time as the area of the square (or the rectangular) changes, the value of the corresponding determinate, changes as well.

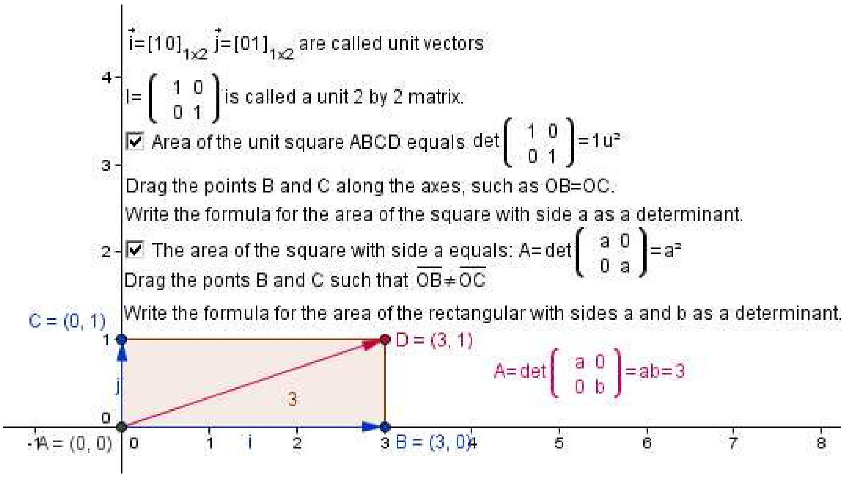

Figure 3. Calculating area of a square with side *a* and a rectangular with sides *a* and *b* by determinants (Applet 1.)

One very useful property of GeoGebra is that it calculates the area of any polygon. Applying this property, particularly for calculating area of a parallelogram (even though it may easily be done as shown on the Applet 2), the instructor may save some time during the class. This is a significant advantage, especially when calculating area of polygonal regions is not the main focus of the lecture, but teaching linear algebra. This GeoGebra property is used on the Figure 4. and Applet 2. The aim here is, by using the area of the *parallelogram*, to obtain the term for any two- dimensional matrix. In order to achieve this, the question is: What happens if the points *B* and *C* are not dragged along the axes, but anywhere on the coordinate plane? Thus, one of the coordinates of these two points will no longer be 0, but any other real number. Let us denote these coordinates as: $B(a,b)$ and $C(c,d)$. The vertex *A* stays at the origin $A(0,0)$. The coordinates of the vertex *D* can be found as addition of two vectors, by the property of a parallelogram. So, its coordinates are $D(a+c,b+d)$. This geometric presentation and the area of the parallelogram *ABCD* (Figure 4. and Applet 2) have their own algebraic interpretation, which is:

$$A_{ABCD} = (a+c)\cdot(b+d) - 2bc - 2\frac{ab}{2} - 2\frac{cd}{2} = ab - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{11}$$

This is exactly the definition for the second order determinant. The two- dimensional matrix is defined as:

$$M_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{12}$$

where $a, b, c, d \in R$ and its determinant is:

$$\det M_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{13}$$

Finally, the Figure 4 and Applet 2 can be used for calculating the area of any *triangle ABC* (the vertex *A* does not have to be placed at the origin, but it can be any point on the coordinate plane) as half of the area of the parallelogram and writing it in algebraic form as follows:

$$A_{\Delta ABC} = \frac{1}{2}(ad - bc) = \frac{1}{2}\begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{14}$$

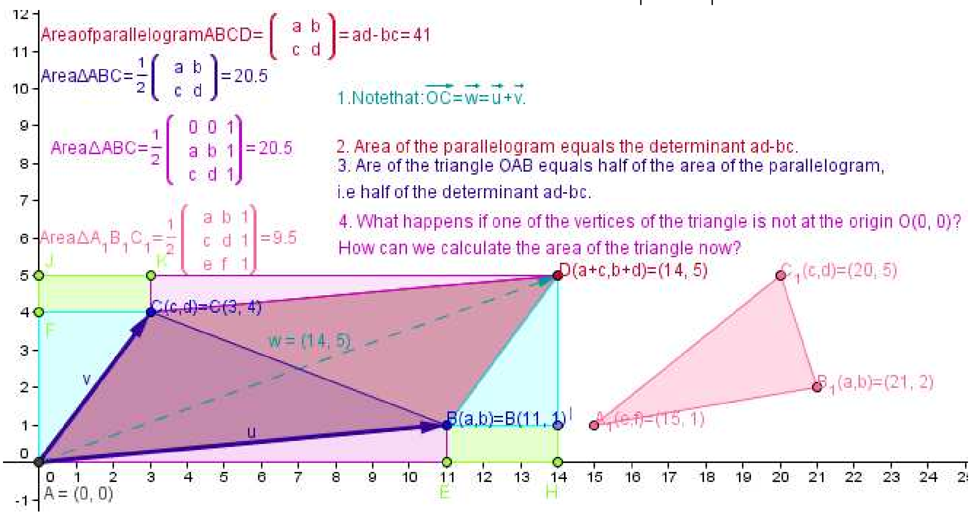

Figure 4. Area of a parallelogram and a triangle using determinants (Applet 2.)

Obtained new students' knowledge in Linear algebra and analytic geometry is:

- Unit vectors in the coordinate plane,
- Identity 2x2 matrix,
- Vectors in the coordinate plane,
- 2x2 matrix in general,
- Second order determinants.

3x3 determinants

Required previous students' knowledge in 3D Geometry is calculating volumes of cubes, cuboids, and parallelepipeds.

Students know that the volume of a *cube with side 1unit* equals 1 cubic unit (Figure 5.).

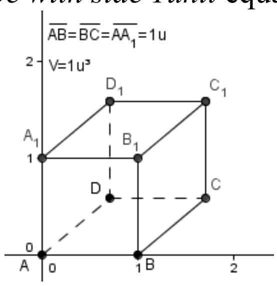

Figure 5. Volume of the unit cube via geometric means and students' previous knowledge

Applying this fact and continuing the previous approach, we define the unit vectors in 3D and the identity 3x3 matrix. This definition is provided geometrically in the 3D space, with the three coordinates of the points (geometric presentation in GeoGebra, Figure 6) and algebraically. The algebraic definition is actually introduction to the notations of 3x3 matrices as squared schemes. In this particular case, these squared schemes are consisted of ones (three elements at the main diagonal) and zeroes (the remaining six elements). Than, we define the determinant as a number, which in this case is equal to the volume of the unit cube, i.e. 1. Thus, the volume of the *unit cube* $ABCDEFGH$ *with side 1,* whose vertices are: $A(0,0,0)$, $B(1,0,0)$, $D(0,1,0)$, $E(0,0,1)$ is:

$$V = 1 = 1 \cdot 1 \cdot 1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \det I_{3\times 3} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

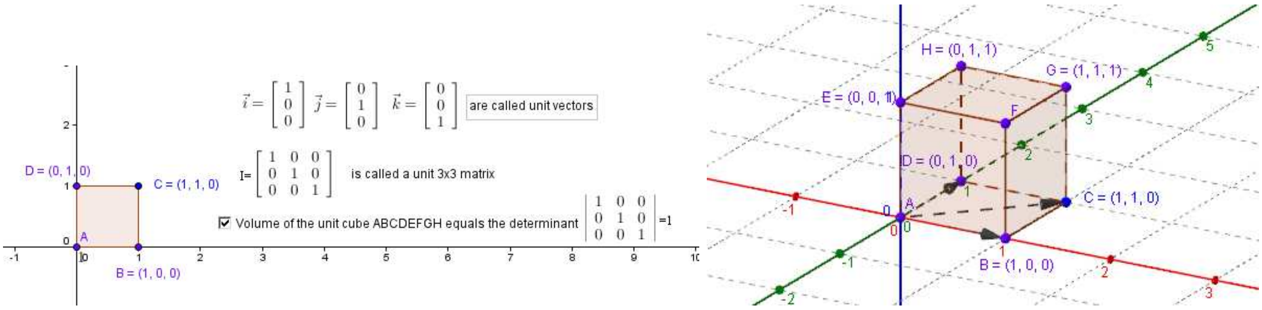

Figure 6. Notation of vertices of a unit cube via algebraic means and introduction to unit vectors and unit 3x3 matrix

Afterwards, the idea is extended to any *cube with side a* and consequently to any 3x3 matrix as a squared scheme consisted of equal real numbers at the main diagonal and zeros (all remaining elements). Than, the 3x3 determinant equals the volume of the corresponding *cube with side a*, whose vertices are given with its coordinates in 3D. Thus,

$$V = a^3 = a \cdot a \cdot a = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = \det M_{3\times 3} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \tag{16}$$

Furthermore, the idea is extended to any *cuboids* with sides *a*, *b* and *c* (whose one vertex is at the origin and three other vertices lie on the axes $A(0,0,0)$, $B(a,0,0)$, $D(0,b,0)$, $E(0,0,c)$). Its volume is:

$$V = abc = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{17}$$

Finally, we define the matrix whose elements are any real numbers and the determinant representing the volume of the *parallelepipeds* with $A(0,0,0)$, $B(a,b,c)$, $D(d,e,f)$, $E(g,h,i)$ (Figure 7. and Applet 3.).

$$V = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{18}$$

The last two Figures in 3D, are designed in GeoGeba 5.0 Beta Version, which is unstable. The release of the new version is expected to be released in August. Then the students by dragging the three vertices of the cube and the cuboids can visually see how changing the elements of the determinant influences their volume.

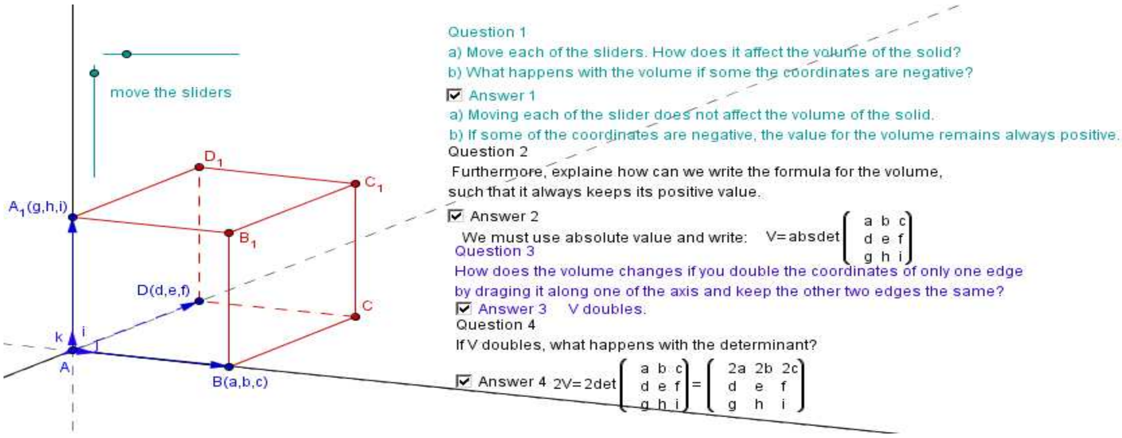

Figure 7. Volume of a parallelepiped using determinants (Applet 3.)

Obtained new students' knowledge in Linear algebra and analytic geometry is:

- Unit vectors in the 3D space,
- Identity 3x3 matrix,
- Vectors in the 3D space,
- 3x3 matrix in general,
- Third order determinant.

2.2 Defining some advanced terms in Linear algebra and analytic geometry

Following the same approach and continuing the concept in constructing the theory from geometric to algebraic aspect, after performing the Applet 3, the multiplication of the determinant by 2 leads towards the *properties of determinants*. In this exact case, that is an operation multiplication of a determinant by a scalar. A clear distinction must be made between this operation and a multiplication of a matrix by a scalar. Furthermore, we can define other operations with matrices, such as addition of matrices as exemplified in the Figure 8 (Applet 4.).

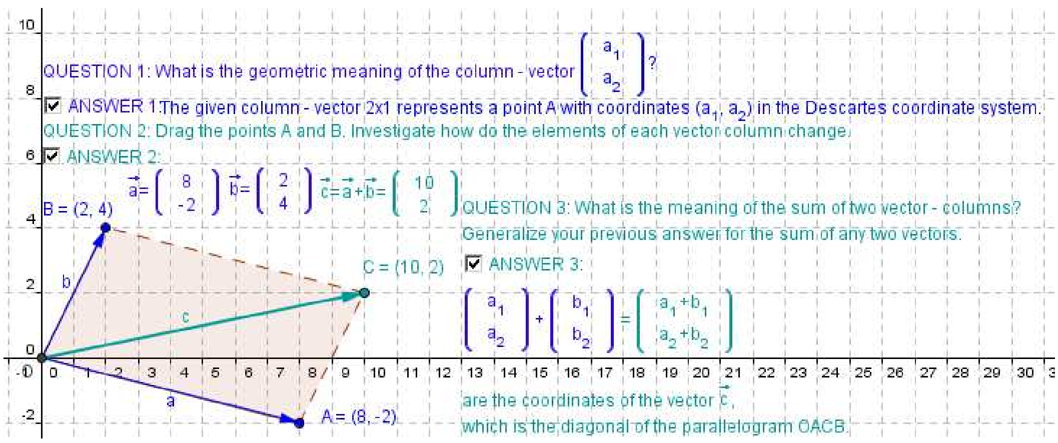

Figure 8. The operation addition of matrices using vector addition (Applet 4.)

Extending the idea for the operation addition of three (n) matrices by addition of three (n) vectors and defining other operations with matrices, for example, subtraction of matrices by using difference of vectors, lead to development of other very important topic. The operations: scalar product, cross product and mixed product of vectors can be defined in 2D, 3D and extended to nD, as well. For the needs of the vector spaces definition, the next paragraph gives details about linear combinations of vectors. It also explicates implementation of linear combinations of vectors in solving systems of linear equations.

Previous students' knowledge includes solving systems of two linear equations in two variables. Besides the substitution method and the method of opposite coefficients for solving these systems (Figure 9.), students also know the graphical method.

$$\begin{cases} 3x + y = -1 \\ -x + 2y = 5 \end{cases} \quad \text{'the given system}$$

$$\begin{cases} y = -1 - 3x \\ -x + 2y = 5 \end{cases} \quad \text{'y in terms of x}$$

$$\begin{cases} y = -1 - 3x \\ -x + 2(-1 - 3x) = 5 \end{cases} \quad \text{'substitution of y}$$

$$\begin{cases} y = -1 - 3x \\ -7x = 7 \end{cases} \quad \text{'calculating x}$$

$$\begin{cases} y = 2 \\ x = -1 \end{cases} \quad \text{'calculating y}$$

$$\begin{cases} 3x + y = -1 \\ -x + 2y = 5 / \cdot 3 \end{cases} \quad \text{'the given system}$$

$$\begin{cases} 3x + y = -1 \\ -3x + 6y = 15 \end{cases} \quad \text{'opposite coeffcients of x}$$

$$\begin{cases} 3x + y = -1 \\ 7y = 14 \end{cases} \quad \text{'addition of the coefficients}$$

$$\begin{cases} 3x + 2 = -1 \\ y = 2 \end{cases} \quad \text{'calculating y}$$

$$\begin{cases} x = -1 \\ y = 2 \end{cases} \quad \text{'calculating x}$$

Figure 9. Substitution method and method of opposite coefficients for solving systems of linear equations

According to the graphical method, the solution values of the variables represent the coordinates of the intersection point of two lines in the coordinate plane (Figure 10. Applet 5.). The dynamic software GeoGebra provides an excellent opportunity, which is a solution of the same system using linear combinations of vectors (Figure 11. Applet 6.). This approach is mostly recommended for university students and is a great support in the transition from the secondary to the tertiary level of the course linear algebra and analytic geometry.

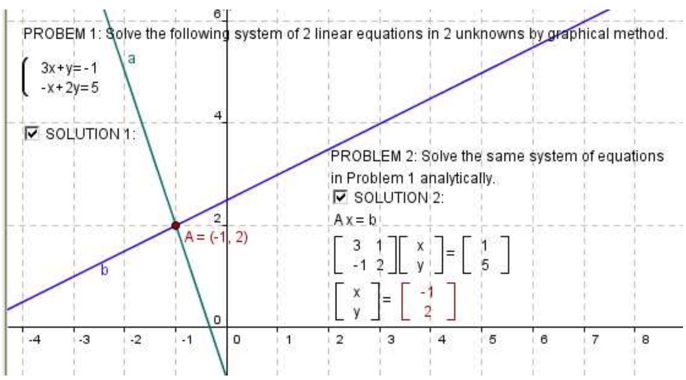

Figure 10. Geometrical solution of systems of two linear equations in 2 variables (Applet 5.)

A confirmation to the previous statement is that the concept of linear combination of vectors is extremely essential to introduction of a more advanced topic such as Vector Spaces. Moreover, the solution of the system may be provided by using wxMaxima too, at any level of the course. In addition, this solution is provided in the last Figure 12.

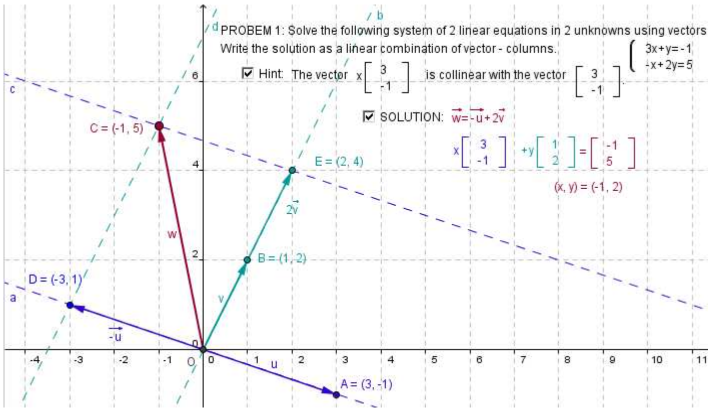

Figure 11. Analytical solution of systems of two linear equations in 2 variables as linear combinations of vectors (Applet 6.)

On one hand side, the CAS wxMaxima offers a possibility for a quick verification of the obtained solution of the system by a single command. On the other hand side, it provides a base for the university student to create more complicated programs.

```
Solve the system of two linear exuations in two variables.
(%i1) linsolve([3*x+y=-1, -x+2*y=5], [x, y]);
(%o1) [x=-1,y=2]
```

Figure 12. Algebraic solution of systems of two linear equations in 2 variables in CAS wxMaxima

The three different approaches (geometric, analytic and algebraic) to the exact same problem can be expected to provide easier understanding of its solution. The idea for the different approaches may be easily applied for solving systems of three linear equations in three variables, as provided by [5] and [6], visualizing the Gauss algorithm. Their connection offers students another perspective and demonstrates that there exists a close relation between different mathematical disciplines.

3. Conclusions and further research work

This paper aims to develop different concepts in teaching/ learning Linear Algebra and Analytic Geometry from those that we are generally used to. Application of plane and solid geometry using the dynamic applets [7], in order to gain specific knew knowledge in Linear Algebra and Analytic Geometry is an innovative method. As I have mentioned in the Introduction, the paper is a part of a larger research (i.e. doctoral project). Although, further separate papers are planned on the topics of expansion of the suggested innovations and wider implementation of the created dynamic worksheets in an authentic classroom atmosphere, this paper has a significant importance. Even if this paper refrains from a deeper observation of the teachers and students' attitudes and achievements when applying the recommended teaching/learning methods, it has many other values. *Detailed description of variety of approaches and their cohesion in solving a single problem is provided.* Namely, the paper observes comparatively how Linear Algebra and Analytic Geometry can be treated by different computer systems. *Mathematical correctness of the topics being introduced by the proposed approaches has been in the focus of the research. The paper covers those terms in Linear Algebra and Analytic Geometry that integrate interdisciplinary approaches*. These specific terms, such as linear combination of vectors or graphical solution of systems of linear equations are taken into consideration in order to illuminate elegant proposals for teaching and learning advanced concepts in Linear Algebra and Analytic Geometry. A valuable contribution of the discussed topic in the paper will promote new questions seeking debate of the experts in the field of integration of the DGSs and CASs.

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[3] Donevska Todorova A. (2011). *Computer Algebra Systems Supporting Teaching/ Learning Linear Algebra*, Proceedings of the International Geogebra Conference for Southeast Europe, January 2011, Novi Sad, Serbia.

[4] Filler A. (2011). *Elementare Lineare Algebra. Linearisieren und Koordinatisieren*, 1st Edition ISBN: 978-3-8274-2412-9 Spektrum Akademischer Verlag.

Internet Recourses

[5] http://www.mathematik.hu-berlin.de/~filler/3D/gaussalg/ga-eben.htm

[6] http://www.mathematik.hu-berlin.de/~filler/3D/gaussalg/ga-vekt.htm

[7] http://www.math.hu-berlin.de/~filler/donevska/applets

Software packages

[GeoGebra] http://www.geogebra.org/cms/

[Maxima] http://maxima.sourceforge.net/