On Some Advantages of Using Mathematics Software

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Abstract: *In this paper, we share some observations we have made in some of our courses. We give some examples that increase students' interest when given an opportunity to use some mathematics software exploring some concepts in the course content. For studying group action, which is one of the standard mathematical topics in algebra, we give some data to students and ask them to produce their own conjectures. Results show that our students are more willing to prove their own conjectures and eagerly try to explain them.*

1. Introduction

Like many universities, our undergraduate students take some selective courses, especially in their last year, to specialized fields of their interest. Then, they are asked to prepare a thesis concerning the content of courses. In the defense of their theses, a measurement and evaluation of students' understanding of the subject is carried out. Two of these selected courses, at both of Karadeniz Technical University and Rize University, are Discrete Groups and Hyperbolic Geometry. We use Jones & Singerman's book [6] for the first and Anderson's book [1] for the second. Until recently, the content of the lessons has been theoretical and practical applications have not been discussed too much. After recognizing this shortcoming, we decided to provide some examples of our research topics that are easy to understand.

2. Course Structure

The history of non-Euclidean geometries is well known. Classic-Euclidean geometry is based on five basic axioms. Many attempts have been made to prove the fifth postulate, known as Playfair's axiom, from the others, but it is understood that the fifth is free from the other four. So, a lot of new geometries were born based on different versions of the fifth postulate. At this point, we talk about the Erlangen program and Klein's very famous description about new geometries. Klein proposed an idea that all these new geometries are just special cases of the projective geometry, as already developed by Poncelet, Möbius, Cayley and others. With every geometry, Klein associated an underlying group of symmetries. For example, lengths, angles and areas are preserved with respect to the Euclidean group of symmetries. In pedagogic terms, the program became transformation geometry. For a geometry and its group, an element of the group is sometimes called a motion of the geometry [2]. In our lessons, after this beginning, we focus on transformation groups (especially $PSL(2,\mathbb{R})$ and $Möb(\mathbb{H})$) and their properties. Our subsections are as follows:

- Topological Groups
- Topological Transformation Groups
- Investigate properties of $PSL(2,\mathbb{R})$ and $Möb(\mathbb{H})$
- Geometric Classification of Aforementioned Groups
- Explanation the fact that basic geometric materials (where we mean vertex, edge, area) are invariant under the action of transformation groups

- Modular Group and Its Subgroups

For learners to have deeper understanding of the "action", in our opinion the fifth and sixth subsections are very important which arose in the defense of thesis. Some concepts in these sections can be supported by using mathematics software instead of giving theorems only. Visualizing abstract mathematical structures such as non-Euclidean is difficult, although not impossible. Standard Euclidean models for non-euclidean geometries, such as hyperbolic and elliptic geometry, have been developed over the last century to provide visual support for those trying to understand the geometries [9].

At this point, we also investigate some group actions in the spirit of the theory of permutation groups and how some graphs arise from these actions. For us the most useful example is the action of a modular group on the extended rational numbers. Whenever we mention it in class, we notice that it catches the attention of the students. In addition, such a study gives an opportunity to consider many elementary methods in several areas of mathematics: number theory, group theory, graph theory, hyperbolic geometry and combinatorics. Of course, the introduction of these concepts was a difficult challenge for the students at the beginning. But we advocate replacing background knowledge with the use of some new technologies in these topics. In our opinion, in this way our students experience the mathematization process——experimentation, conjecture and explanation——more effectively. First, they produce the conjecture by means of empirical verification. Then, they were willing and eager to explain why the conjecture was true by them.

We give two examples as follows.

Example 2.1 We give our students some properties of Farey Graph obtained by the action of a modular group on the extended rational numbers. The following is a theorem from the content:

Theorem 2.1 [4], [7], [8] Let $r/s \in \mathbb{Q}$ and $x/y \in \mathbb{Q}$ be reduced rationals. Then the following three conditions are equivalent:

i. r/s and x/y are adjacent vertices in Farey Graph;
ii. $ry - sx = \pm 1$;
iii. r/s and x/y are adjacent terms in Farey sequence.

Here, for each integer $m \geq 1$, the Farey sequence-F_m of order m consists of all rational numbers x/y with $|y| \leq m$, arranged in increasing order. For example F_4 is:

$$\cdots, -\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1, \frac{5}{4}, \frac{4}{3}, \cdots$$

The proof of this theorem requires much more background than provided in this level of the course. However, to visualize the graph, we have represented the edges of the graph as hyperbolic geodesics in the upper half plane; that is, as Euclidean semi-circles or half-lines perpendicular to $\mathbb{R}$. Students know that two vertices r/s and x/y can be combined by an edge if the condition $ry - sx = \pm 1$ is satisfied. The first task for our students is drawing this shape. Here, we suppose that $0 \leq \frac{r}{s}, \frac{x}{y} \leq 1$. For simplicity we restrict these Farey sequences to $F_m \cap [0,1]$. So, we have:

$$F_1 \cap [0,1]: \quad 0 \overset{\curvearrowright}{,} 1$$

$$F_2 \cap [0,1]: \quad 0 \overset{\curvearrowright}{,} \frac{1}{2} \overset{\curvearrowright}{,} 1$$

$$F_3 \cap [0,1]: \quad 0 \overset{\curvearrowright}{,} \frac{1}{3} \overset{\curvearrowright}{,} \frac{1}{2} \overset{\curvearrowright}{,} \frac{2}{3} \overset{\curvearrowright}{,} 1$$

$$F_4 \cap [0,1]: \quad 0 \overset{\curvearrowright}{,} \frac{1}{4} \overset{\curvearrowright}{,} \frac{1}{3} \overset{\curvearrowright}{,} \frac{1}{2} \overset{\curvearrowright}{,} \frac{2}{3} \overset{\curvearrowright}{,} \frac{3}{4} \overset{\curvearrowright}{,} 1$$

$$F_5 \cap [0,1]: \quad 0 \overset{\curvearrowright}{,} \frac{1}{5} \overset{\curvearrowright}{,} \frac{1}{4} \overset{\curvearrowright}{,} \frac{1}{3} \overset{\curvearrowright}{,} \frac{2}{5} \overset{\curvearrowright}{,} \frac{1}{2} \overset{\curvearrowright}{,} \frac{2}{3} \overset{\curvearrowright}{,} \frac{3}{5} \overset{\curvearrowright}{,} \frac{3}{4} \overset{\curvearrowright}{,} \frac{4}{5} \overset{\curvearrowright}{,} 1$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

To draw the Farey graphs (with respect to the value of m) for the associated F_m - Farey sequence $(m \geq 1)$, we use an algorithm which generates the monotone increasing F_k - Farey sequences $(1 \leq k \leq m)$. The MATLAB code for this algorithm is provided as follows.

Program 2.1

```
function fareygraf(m)
F=0;
format rat
for i=1:m
   for r=0:i
     for s=0:i
        for x=0:i
            for y=0:i
                 if r*y-s*x==1|r*y-s*x==-1
                     if r/s<=1
                        if x/y<=1
                     F=[F;r/s];
                    F=[F;x/y];
                        end
                     end
                 end
              end
        end
    end
        end
F=sort(F);
n=length(F);
syms x y
ezplot('x+0*y=0',[0,1,0,1])
hold on
ezplot('x+0*y=1',[0,1,0,1])
  for i=1:n-1
A=(F(i)+F(i+1))/2;
B=abs(F(i)-A);
```

```
y=strcat('(x-','a)^','2+','y^','2=','b^2');
yg=subs(y,{'a','b'},[A,B]);
ylim([0 0.8])
ezplot(yg,[0,1,0,1])
title('')
hold on
pause(0.005)
  end
end
```

However, since the condition $ry - sx = -1$ is satisfied by the values r, s in the F_k- Farey sequences, we can dispense with this part of the code in the program. Consequently, the following simplified code can be used:

Program 2.2

```
function fareygraf(m)
F=0;
format rat
for i=1:m
       for x=0:i
           for y=0:i
                       if x/y<=1
                         F=[F;x/y];
                       end
          end
      end
F=sort(F);
n=length(F);
syms x y
ezplot('x+0*y=0',[0,1,0,1])
hold on
ezplot('x+0*y=1',[0,1,0,1])
  for i=1:n-1
      A=(F(i)+F(i+1))/2;
      B=abs(F(i)-A);
      y=strcat('(x-','a)^','2+','y^','2=','b^2');
     yg=subs(y,{'a','b'},[A,B]);
       ylim([0 0.8])
       ezplot(yg,[0,1,0,1])
      title('')
      hold on
      pause(0.005)
end
end
```

The goal in the above is to emphasize that the use of extraneous lines of code will slow down a program. Hence, the above algorithm implemented in the program will run faster. In this way, we are speeding up the rendering of the graph. Following this stage, we prepare a MATLAB GUI (Graphical User Interface) of program.

Here the graphical user interface of related Farey graphs for $m=5$ and $m=15$ is given by the following two figures respectively:

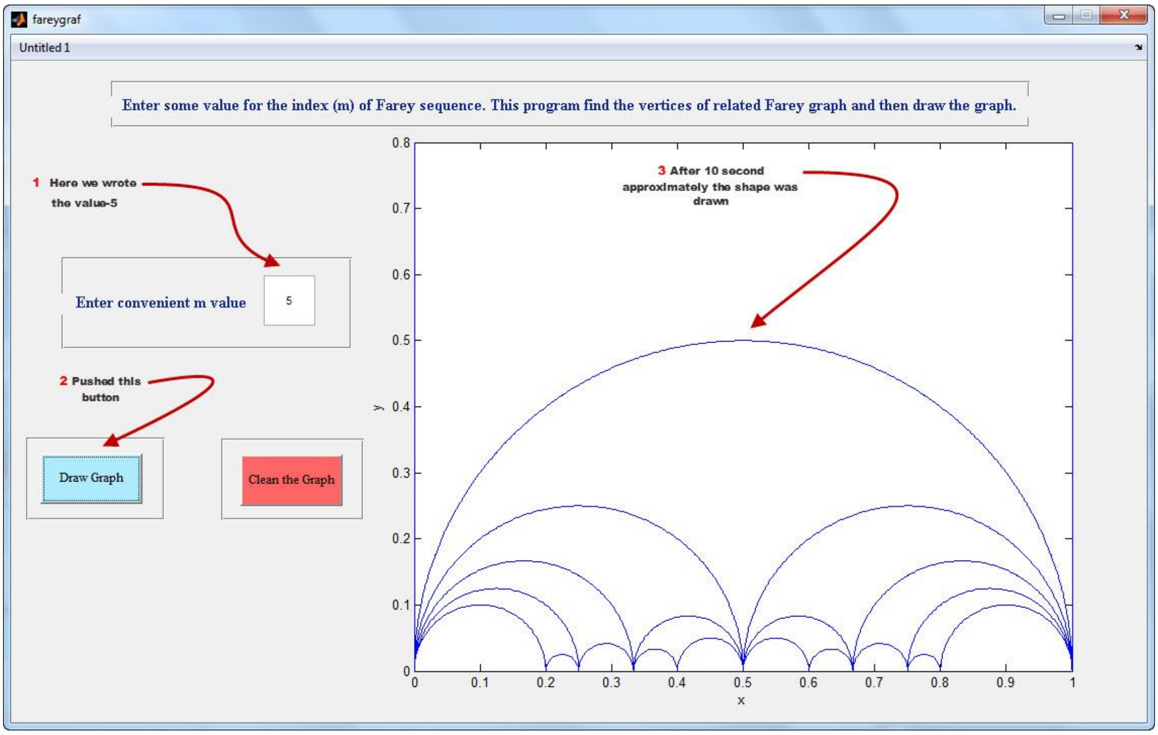

Figure 2.1 Related Farey graph with $m=5$

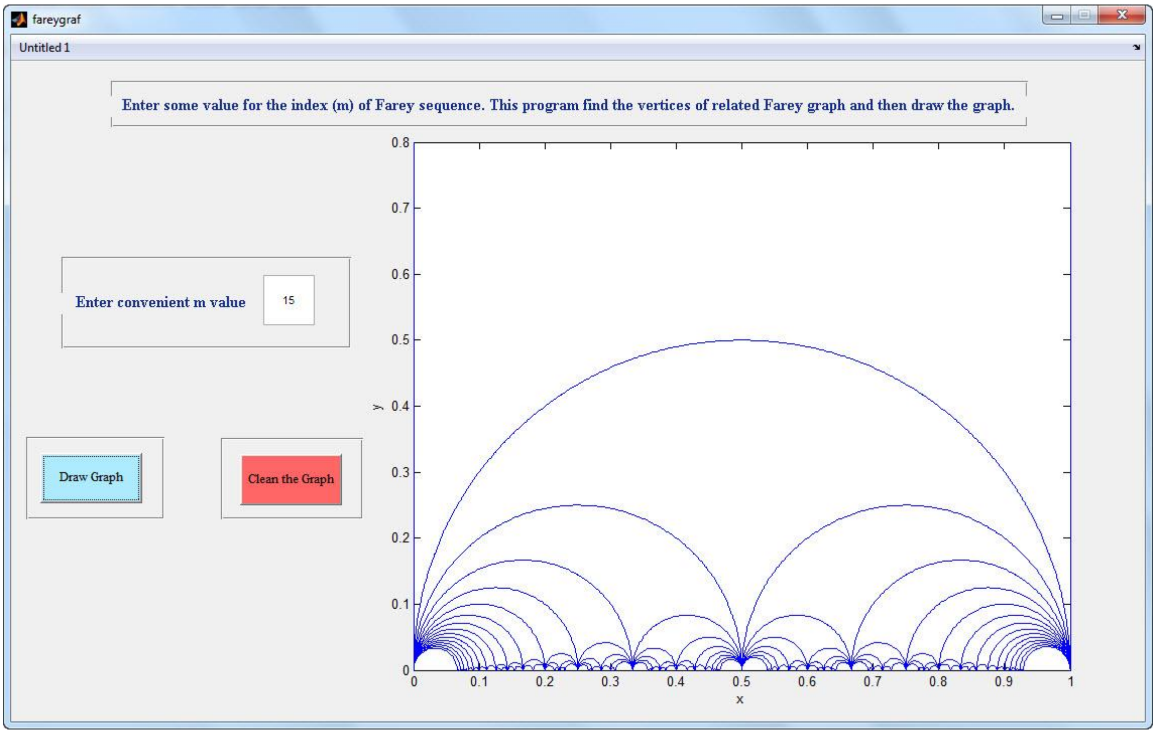

Figure 2.2 Related Farey graph with $m=15$

When the students are prompted to make comments on the figures they produce, they answered the following: “the edges don’t cross to each other”. Then we want them to prove their conjecture and suggest that probably the easiest way to do this is to produce an algorithm for coding in MATLAB. Encouraging our students, we emphasize the importance of computer support for solving combinatorial problems referring, for example, to the four-color problem. It was proven in 1976 by Kenneth Appel and Wolfgang Haken and was the first major theorem to be proved using a computer [5].

On the other hand, we note that theoretical proof of our problem is not too difficult. We also reproduce it from [7] as follows:

Theorem 2.2 No edges of Farey graph cross in the upper half-plane.

Proof. We suppose that two edges cross in $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Because of transitive action, we may assume that one of them is the edge $\operatorname{Re}(z) > 0$ joining 0 and ∞, so other must join rationals v and w where $v < 0 < w$. By Theorem 2.1, v and w are consecutive in some F_m, which is impossible since 0 will intervene.

Let us look at the second example.

Example 2.2 Another problem is the investigation of solution sets of some congruence equations.

As usual, in a graph containing vertex and edge conditions, then we examine circuit conditions. We call these circuits as triangle, quadrilateral, pentagon, etc. (or n-gon more generaly) according to its number of sides. In general, these circuit conditions appears as a congruence equations. For example, in Farey graph, the figure contains a triangle if and only if $u^2 \pm u + 1 \equiv 0 \pmod{n}$ such that $(u,n) = 1$. Other congruence equations in some other graphs are: if $u^2 + 1 \equiv 0 \pmod{n}$ for 2-gon, $2u^2 \pm 2u + 1 \equiv 0 \pmod{n}$ for quadrilateral, $3u^2 \pm 3u + 1 \equiv 0 \pmod{n}$ for hexagon [3]. Here, making calculations easier we choose a prime $-p$ for n. Thus, the question turns to be, which prime numbers provide these equations?

At this point, the task of our students is finding these primes using a computer.

For the congruence equation $u^2 + u + 1 \equiv 0 \pmod{n}$, the program is:

Program 2.3

```
function sonuc=pnorm3(u)
  if u==1
     disp('u must be different from 1');
  else
p=0;
s=u^2+u+1;
f=factor(s)
z=length(f);
       for i=1:z
          if gcd(f(i),u)==1
             if f(i)~=3
            p=[p;f(i)];
             end
```

```
            end
        end
           g=length(p);
            p=p(2:g);
            if length(p)==0
            sonuc='For this value of u there is no prime p';
            elseif length(p)==1
            sonuc='For this value of u there is only one prime p';
            else
        sonuc=p;
            end
  end
```

For the congruence equation $u^2 - u + 1 \equiv 0 \pmod{n}$, the program is:

Program 2.4

```
function sonuc=pnorm4(u)
     if u==1
        sonuc='u must be different from 1';
     else
p=0;
s=u^2-u+1;
f=factor(s);
z=length(f);
        for i=1:z
            if gcd(f(i),u)==1
               if f(i)~=3
             p=[p;f(i)];
               end
            end
        end
           g=length(p);
            p=p(2:g);
            if length(p)==0
            sonuc='For this value of u there is no prime p';
            elseif length(p)==1
            sonuc='For this value of u there is only one prime p';
            else
        sonuc=p;
            end
  end
```

Hence, MATLAB GUI of program is given by the following figure:

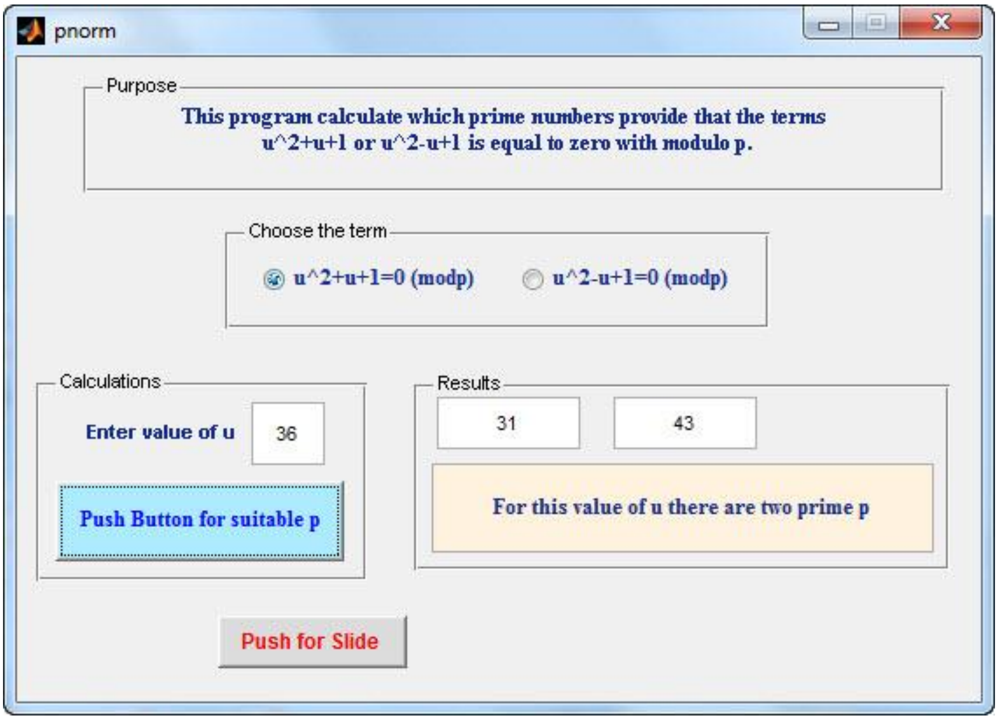

Figure 2.3 Related Figure for Example 2.2

As a result, here, with this GUI, when students entered the appropriate u values and then they see that, prime values p satisfied $p \equiv 1 \pmod 3$. In the above figure the button "Push for Slide" gives them visual actuality for p values for some u.

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