

# Revisiting the teaching of perimeter, area and volume at a middle school level with Cabri environments

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**Abstract:** *Several years ago I started cooperating with a middle school teacher to create DGS resources that can be easily used by teachers with their students. Two years ago a joint research project between my Institute of research and her middle school had been started. The aim of this project was to combine my skills as an expert in dynamic geometry and experimental math with the skills of my colleague as a teacher and a technology user. We aimed to respond to her needs for resources involving using Cabri environments to teach some contents of the French middle school syllabus. Our work focused especially on the teaching of the concepts of perimeter, area and volume. This paper describes the files created by the expert, how they were designed with the teacher and how they were used to enhance the more experimental practice of mathematics in the classroom. We will also discuss the format chosen for disseminating this work: this is an important issue because it is important to convince teachers that other ways of teaching are possible with these resources even if they are neither expert in dynamic geometry nor a technology user in their teaching. The theoretical background of this work generated the results I have obtained concerning the experimental process of discovery using Cabri (see [3]).*

## 1. Perimeter

### 1.1. Perimeter of a triangle

#### 1.1.1. The initial question of the teacher

**The teacher** wanted to transmit to her students the idea that a perimeter was not only a formula ; it has also a geometric meaning: the length of a segment. Her demand from the software was to create dynamic figures which show such an approach.

#### 1.1.2. The response of the expert

**The expert** achieved such figures where the given triangle can be unfolded until a position which is a segment having as a length the perimeter of the given triangle. This work led to three models of unfolding a triangle. Each model was recorded under a macro construction in order to be used in other files of the Cabri environment. So, each macro is a new tool giving the unfolded triangle by clicking once on the triangle. The object obtained with such a macro can be folded or unfolded by dragging one or two points. Figure 1 shows below the three models created by the expert.

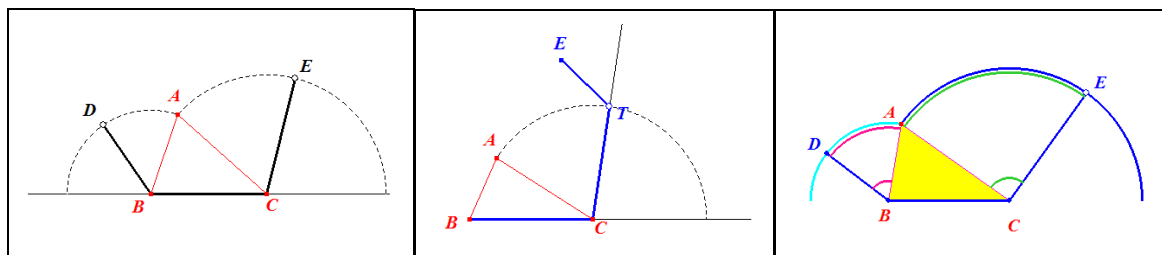


Figure 1

Figure 1 on the left: unfolding ABC gives the black object DB-BC-CE; D and E can be dragged separately along the dotted arcs.

Figure 1 on the centre: unfolding ABC gives the blue object BC-CT-TE; T can be dragged along the dotted arc to unfold ABC.

Figure 1 on the right: unfolding ABC gives the blue object DB-BC-CE; E can be dragged along an arc and commands the unfolding of ABC (this model is different from the first one because the motion of point D is obtained when point E moves).

We do not give the details of the constructions: they only use proportionality and rotations.

### 1.1.3. Designing these files

This work has been achieved by the teacher and the expert. It led to dynamic figures using **hide/show** buttons to enhance its use in the classroom even by clumsy users. The teacher has decided to share these files with her colleagues in Powerpoint documents with a minimum of explanation about the use of the **hide/show** buttons. Thanks to the Cabri plug-ins, it is possible to insert Cabri files in a Powerpoint document: they can be manipulated even if the software is not installed in the computer (the user needs only to install the Cabri plug-ins; they can be downloaded and installed freely). These plug-ins allow everybody to export Cabri files in HTML documents. These can be manipulated under Internet Explorer in the same conditions as the files inserted in Powerpoint documents.

Here, in Figure 2, is the second model designed by the expert-teacher couple. Five **hide/show** buttons are added. The red one hides or shows the red triangle ABC, the grey one hides or shows the blue model BC-CT-TE and the blue one hides or shows the dotted blue segment [BE]. The two other buttons hide or show the perimeter of triangle ABC and the length of segment [BE].

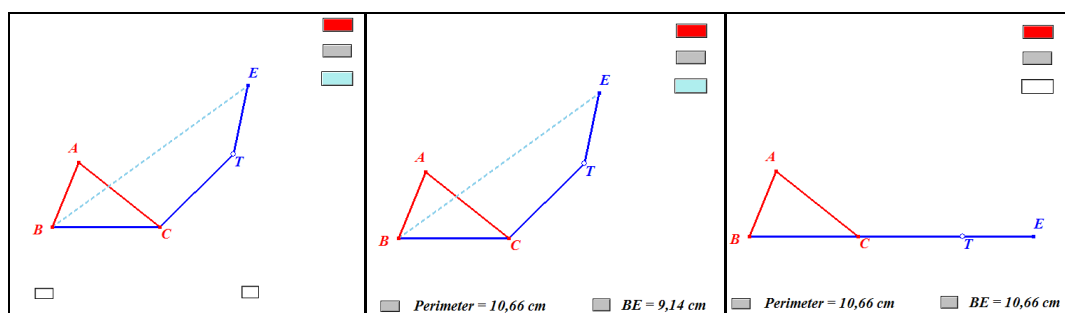


Figure 2

### 1.1.4. A scenario of using such a file

As shown in figure 2 (centre), the numbers displayed (perimeter and BE) are not equal. So, we move point T until the complete unfolding of triangle ABC (when points B, C, T and E are collinear). We can see that at this time these numbers are equal even if we change the shape of triangle ABC. We have conducted two experiments (see [3]) : the first one, generative (generating the conjecture: the perimeter is the length of a segment obtained by unfolding ABC) and the second one, validative (validating the previous conjecture : i.e. verifying it in several particular cases).

## 1.2. Perimeter of a quadrilateral

A similar outcome was achieved for quadrilaterals leading to two unfolding models. The first one needs three points to be dragged independently to unfold a quadrilateral (drag points D, T or E in figure 3 on the left). The second one unfolds a quadrilateral by dragging only one point (drag point

T in figure 3 on the right). The constructions of the expert led also to two macro constructions that could be used later in other Cabri files.

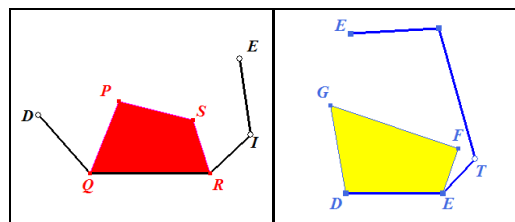


Figure 3

### 1.3. An example of the use of macros of unfolding

#### 1.3.1. An exercise (comparison of perimeters)

The **teacher** wants to give her students a task using paper and pencil: compare the perimeters of a given triangle and a given quadrilateral; she asks for a geometrical solution with the use of the compass and the previous approach. She wants the **expert** to create dynamic figures allowing her to display and visualise such figures with her laptop and a data projector (on a screen or on an interactive white board). The expert creates a figure with Cabri showing the triangle and the quadrilateral, and uses its macros to unfold them. The teacher prints the figure after hiding the nets she obtained with the macros to unfold them. She copies it for the students. **Hide/show** buttons are added to display the perimeters : they must be used for the last verification.

#### 1.3.2. How to use this file (figure 4)

The teacher displays the file provided by the expert and designed by the expert-teacher couple. She unfolds quadrilateral ABCD by dragging point A', and triangle EFG by dragging G' et G'' (figure 4 on the left). She drags triangle EFG to superimpose points G' and C (figure 4 on the right). Then she can visualise the two segments modelling the two perimeters, and let the students see that the red one is longer than the blue one. It is possible to validate this conjecture by displaying (with the hide/show buttons) the perimeters and the length of the segments modelling them.

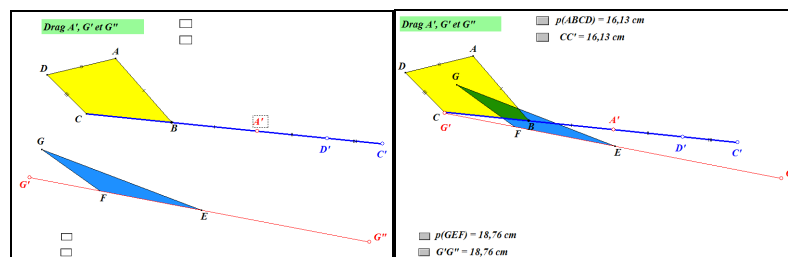


Figure 4

### 1.4. Perimeter of a circle

#### 1.4.1. An approach with Cabri 2 plus files

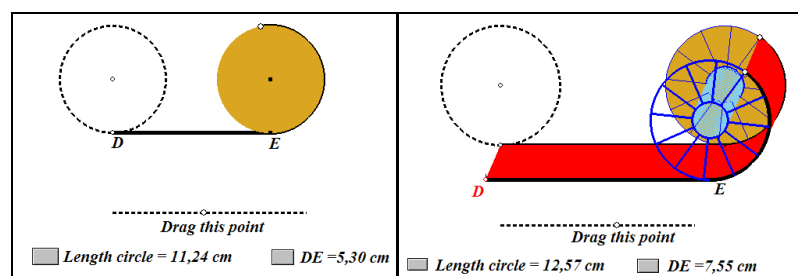
In this paragraph, we show the files created by the expert in order to understand dynamically that the perimeter of a circle is the length of a segment obtained by unfolding such a circle. It is a natural manipulation in our real world which models the length of a circle (we admit that rowling without sliding transmits exactly the lengths).

Figure 5 shows the modelling of the unwinding of a rear by dragging the point on the dotted segment. The right figure is an improvement of the left one: we have used a translation to get this

representation in parallel perspective. The length of the circle and the length of DE can be displayed with **hide/show** buttons

**Figure 5 on the left:** the dotted circle is translated from D to E. The black segment [DE] is the part of the unwound thread. The part which is not unwound is attached to the coloured circle: it is represented by an arc : the second edge of this arc is constructed by transferring the measurement of the difference between the perimeter and DE. We have also created a slider to vary the motion of point E along a horizontal line.

**Figure 5 on the right:** it is the same as the previous one ; we have only used a translation whose vector can be changed by dragging point D. The figure becomes more realistic when we add a regular polygon and the translated polygon (using the previous translation), a blue cylinder with a locus of parallel segments and a red tape with two loci of segments. That allows us to model the unwinding of a tape.



**Figure 5**

#### 1.4.2. How to use such a file with students

As done before with triangles and quadrilaterals, **hide/show** buttons can display the perimeter of the circle and DE. These numbers are equal (same visible digits) when the unwinding is over. The experimentation allows the users either to discover that the perimeter of a circle is the length of a thread obtained by a process of unwinding, or to validate this result.

#### 1.4.3. An approach with a Cabri 3D file (figure 6)

##### **The Cabri 3D file provided by the expert and designed by the expert-teacher couple**

A more realistic figure has been created with Cabri 3D. Here we can model the unwinding of a tape represented by a set of rectangles (where the borders are empty). The unwinding is obtained by dragging only one point. The diameter of the rear (circle) is displayed (it can be changed by changing the position of the centre). The width of the tape can also be changed by dragging a special point. The ratio between the perimeter of the rear (length of the tape) and the diameter has been calculated and displayed in a text beginning with >>>. Only this sign is visible on the screen and the ratio can appear when this sign is dragged to the left.

**How to use this file with the students :** the tape is completely unwound in order to display the same numbers for the perimeter and the length of the tape. Then the teacher asks this question to the students: is it possible to guess the ratio between this length and the diameter ? Students can use their calculators to evaluate this ratio with the data provided by the software. This activity is more interesting with students of the first year of middle school than with older ones. For the first ones, this activity would be a generative experiment to discover number  $\pi$  ; for the others, it would be a validative one to check if the result given by the software is really the right one. At the end of the activity, the teacher displays the text following >>> that can be 3,1415926536 (if the number of digits has been set at its maximum). This number does not change even if the diameter is modified.

This work can lead to the conjecture of the formula giving the perimeter of a circle with a special number called Pi and written  $\pi$  because letter “P” is the first letter of the word “perimeter” in Greek.

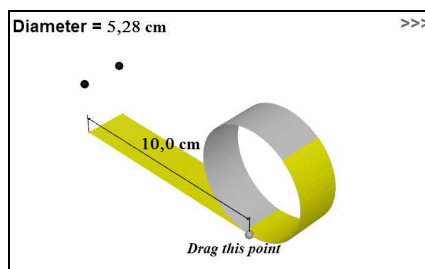


Figure 6

## 2. Area

### 2.1. Area of a disk

**First part of the activity with the designed file** (figure 7): a yellow square is displayed with its area (first picture) ; then a big pink square appears (second picture) when one clicks on the pink **hide/show** button. The students are asked a question : with what number do you multiply the « yellow » area to obtain the « pink » area ? The answer is easily given: number 4. This number appears when one clicks on the  button. The validation of the answer is obtained by clicking the  button: the result of the multiplication of the yellow area by 4 appears ; this is the area of the big square.

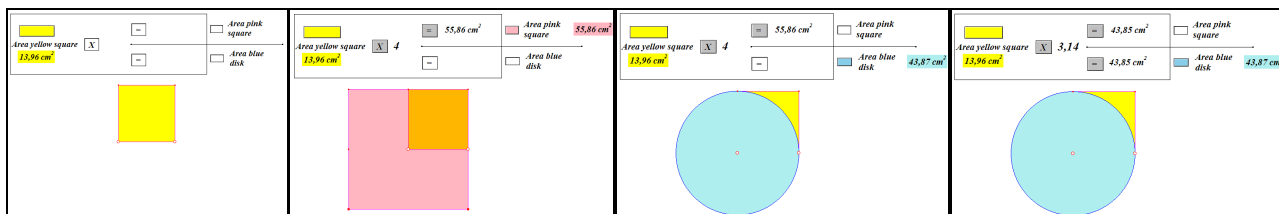


Figure 7

**Second part of the activity with the same file** (figure 7): the blue disk is displayed (third picture) with its area (blue hide/show button). The students are asked a question : with what number do you multiply the « yellow » area to obtain the « blue » area ?

Everybody can check that multiplying by 4 (number already displayed with the  button) gives too big a result (this result is displayed with the inferior **hide/show** button ). After that, we start an experimentation by **trial/error**: it is a natural dichotomy leading to an approximation of  $\pi$ , and the conjecture: the formula for the area of a disk with radius R is proportional to the area of a square having R as a side (fourth picture).

### 2.2. Lateral area of a cylinder with Cabri 3D

#### 2.2.1. Modelling the unfolding of a cylinder (file provided by the expert: figures 8)

We do not give the constructions to obtain such a file where the folding and the unfolding is controlled by a cursor (first and second pictures, see [4]). These constructions could nevertheless represent a different way to approach 3D geometry for teachers and students.

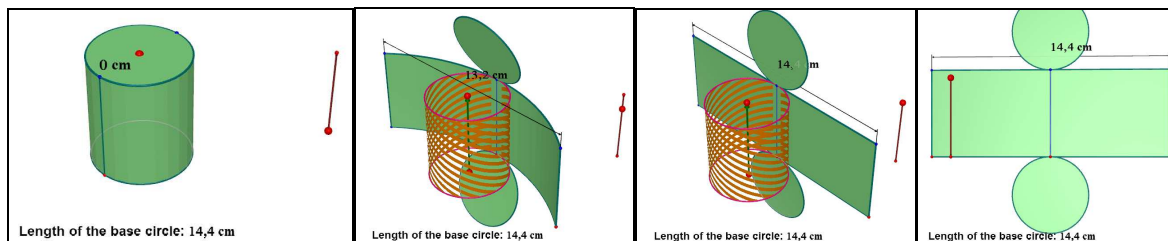


Figure 8

### 2.2.2 How to use this file with the students (figure 8)

The teacher leads the experiment by unfolding completely the cylinder (third picture) : everybody must see that we obtain a rectangle (change the point of view if necessary). We can see also that the length of the rectangle is the same as the perimeter of the base (circle) of the cylinder (fourth picture). This experiment allows the teacher to transmit easily the formula giving the lateral area of the cylinder with respect to the perimeter of the base and the height of the cylinder.

Remark: this model is an approximation of a cylinder with a regular prism where the borders of each side are empty. We can obtain nets of prisms in Cabri 3D, but not at all the one used before.

## 2.3. Area and enlargement

### 2.3.1. The Cabri 2 Plus file created by the expert and designed by the expert-teacher couple

The aim of the activity generated by this file is to help students to conclude that if distances are multiplied by  $k$ , areas are multiplied by  $k^2$ . We designed this file as a game to avoid a symbolic approach which does not really enhance the understanding of new concepts.

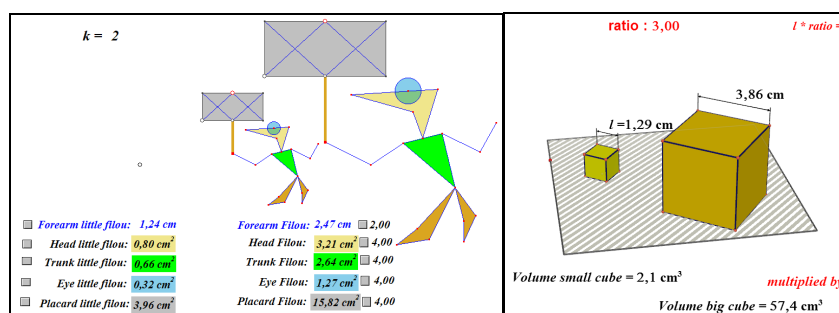


Figure 9

This file (figure 9 on the left) contains little Filou and Filou with their placards. Filou is created in transforming little Filou using an enlargement (coefficient  $k$ : here  $k = 2$  which is a didactical choice). The **hide/show** buttons of the first line display the lengths of the forearms of little Filou and Filou and the ratio of these two measurements (which is indeed 2). In the four other lines, buttons display the areas of some parts of the bodies of little Filou and Filou and the ratio of these measurements (here indeed: 4 !!!).

### 2.3.2. How to use this file with students

The teacher starts with all the hide/show buttons unactivated

**First experiment:** transform little Filou in order to observe how Filou is transformed. The students must conjecture quickly that Filou is an enlargement of little Filou. To validate such a conjecture, the teacher displays the measurements of the forearms and their ratio with the right button. Students

must link this ratio to the number  $k$  displayed. The teacher will change the value of  $k$  in order to see how little Filou is transformed and especially to check that the ratio between the measurements stays equal to  $k$ .

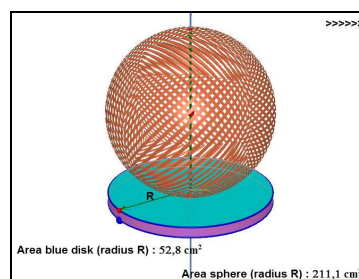
**Second experiment:** activate the other buttons to display the areas of other parts of the two bodies and their ratio all equal to 4. These observations must lead students to a conjecture linking  $k$  to the ratio of areas. It is a generative experiment. Young students conjecture the formula  $2.k$ , but older ones will easily conjecture the formula  $k^2$ . This activity must become validative: the students must try either to validate the conjecture (validated means non rejected, and so true until another experiment, see [7]) or to invalidate it (reject the conjecture even if, often, it is a stage of doubt).

**Third experiment:** change the value of  $k$  from 2 to 3 : it helps students to reject definitely their wrong conjecture and generate the good one. For those who have done the right conjecture this change is a confirmation, it increases the plausibility of this conjecture. It is not forbidden to change  $k$  to 10, 100 and 1000 and to display the ratios of areas evaluated with 100, 10000 and 1000000 even if Filou disappears from the screen.

## 2.4. Area of a sphere

### 2.4.1. The Cabri 3D file provided by the expert (figure 10)

This file represents a sphere and a disk having the same radius. The areas of the disk and the sphere are given by the software. At the top right of the screen, we can see some text beginning with >>>>>>. If this text is dragged to the left side, the ratio between these two areas appears. It is also possible to modify  $R$  by dragging the red point. Dragging the blue point allows us to check that the given sphere is inscribed in a cylinder based on the given disk (height:  $2R$ ).



**Figure 10**

### 2.4.2. Conjecturing the area of a sphere with students

The students are asked by the teacher to compute with their own calculators the ratio of the displayed areas for different values of  $R$  (that can be done by dragging the red point). They must obtain results close to 4: so they can conjecture the good ratio after a debate. The teacher will validate their conjecture by displaying the ratio given by the software. (drag leftside the text >>>>>>). As the students know the formula  $\pi R^2$  (area of a disk having  $R$  as a radius), they will propose the formula  $4\pi R^2$  for the area of the sphere. We can increase the plausibility of the conjecture by displaying the maximum of digits for the number equal to the ratio.

### 2.4.3. Conjecturing without proving deductively

A deductive proof is not possible for middle school or highschool students. Nevertheless the status of this proof (in the meaning of experimental proof : validated in a great number of cases) is an official status for researchers in mathematics. Everybody knows that the last Fermat theorem has been at last proven by Andrew Wiles as a consequence of the proof of the Shimuro-Tajiyama

conjecture. Even when this conjecture was not officially proven, for mathematicians, it was like an evidence: therefore lots of research works started with the starting point such as: let us admit the Shimuro-Tajiyama conjecture.

### 3. Misconceptions about perimeter and area

As the middle school teacher wanted to fix some misconceptions about perimeter and area, she asked the expert to create some files to help her. So, he created two files in order to enhance a debate between the teacher and her students to help them to be aware of their misconceptions. These files can also be used at a higher lever for a dynamical work around functions.

#### 3.1. Triangle with a constant perimeter

3.1.1. File provided by the expert (figure 11 on the left)

A triangle whose shape can be changed by dragging point E has been constructed with the constraint: the perimeter is constant. With **hide/show** buttons, it is possible to display the perimeter, the area and CE. Other buttons can be used to display a system of axis and the curve of the function giving the area with respect to CE and a conic passing through five points of this curve.

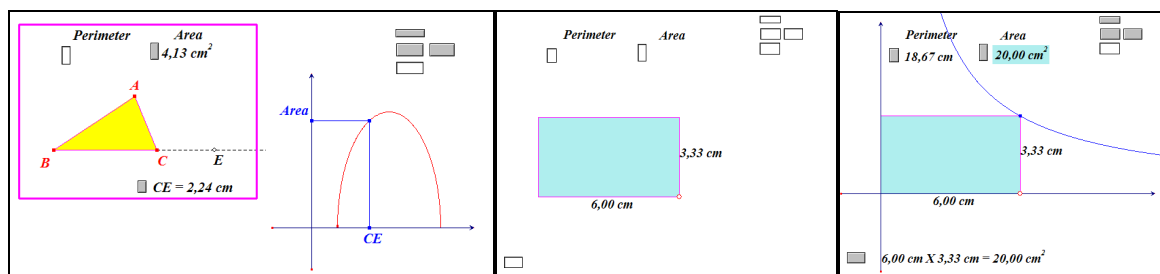


Figure 11

3.1.2. How to use it with students

The teacher changes the shape of the triangle and try to obtain from her students what they think about the perimeter and the area of this dynamic triangle. The aim is to break a systematic link between these two notions. Usually they conjecture that perimeter and area varies in the same way. It is possible to validate the first one about the area and to invalidate the second one about the perimeter in activating the **hide/show** buttons.

3.1.3. How to use it at a higher level

Using the **hide/show** buttons gives the curves of the variations of the area with respect to CE. The shape of this curve cannot suggest a parabola. In fact the last button display the previous conic superimposed to the curve. Cabri says that this conic is an ellipse.

3.2. Rectangle with a constant area (figure 11 on the centre and on the left)

A similar work can be done with a file where the expert has constructed a rectangle with the constraint: the area is constant. It is possible to imagine what has been done with the screenshots displayed in figure 14.

#### 3.3. Helping the teacher to present the solution of a paper and pencil task

The students had to compare the perimeters and areas of all polygons of figure 12 to the perimeter and area of the yellow rectangle. This figure was printed on a piece of paper. The expert created



with the teacher the file presented in figure 12 with buttons to display all perimeters and areas: in this figure, it was also possible by dragging some points to superimpose each polygon to the yellow rectangle in order to generate or validate a conjecture about the comparison of perimeters and areas.

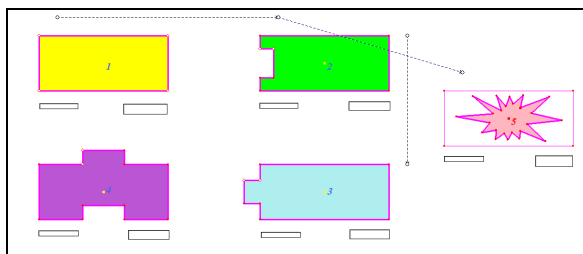


Figure 12

## 4. VOLUME

### 4.1. Volume of a sphere

4.1.1. The Cabri 3D file provided (figure 13 on the left)

This file has been created by the expert and designed by the expert-teacher couple in order to lead an investigation similar to the one led for the area of a sphere (figure 13 on the left). Here are displayed the volume of the sphere ( $V_S$ ) and the volume of the cylinder ( $V_C$ ). Dragging the red text leftwards displays the ratio  $V_C / V_S$ .

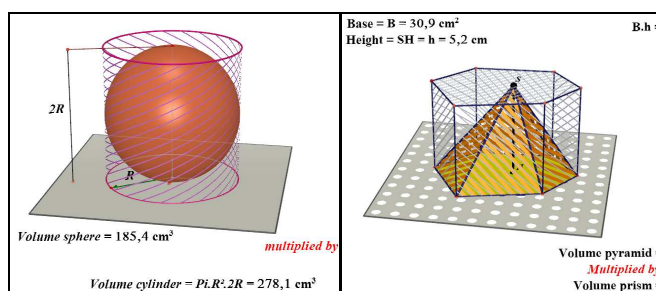


Figure 13

4.1.2. How to use this file

Here the conjecture will be  $V_C = 1,5.V_S$ . Therefore, knowing  $V_C = 2R.\pi R^2$ , students can obtain the « expected » formula  $V_S = 4/3.\pi R^3$ .

### 4.2. Volume of a pyramid

The same technique of investigation can be used with the file presented in figure 13 on the right. It leads to the conjecture: **Volume of a prism equal three times the volume of a pyramid.**

### 4.3. Enlargement (reduction) of a cube

We use here with a Cabri 3D file (figure 9 on the right) the same technique as the one used with Cabri 2 Plus in paragraph 2.3. where little Filou is replaced by a little cube. The data analysed are here volumes instead of areas. The aim to reach is  $k^3$  for the ratio between the volumes when the ratio of the enlargement is  $k$ .

## 5. CONCLUSION

We have shown in this paper how an expert-teacher couple can create and design Cabri 2 Plus and Cabri 3D files to teach dynamically the notions of perimeter, area and volume. Such resources enhance an experimental practice of mathematics (see [1]) where two sorts of experiments, generative or validative can be conducted (generating a conjecture or validating a conjecture). Special techniques of investigation within these technological environments had been used: they must be known by teachers to be passed to their students (see [8]). It is the very important process of instrumentation (where the tool becomes an instrument in the meaning of Rabardel, see [9]) different to the similar process in the paper and pencil environment. Here below, in figure 14, are two screenshots of figures created with Cabri 2 Plus and Cabri 3D leading to conjectures about the volumes of some pyramids, knowing the volume of a cube (see [2] for the Cabri 2 Plus file).

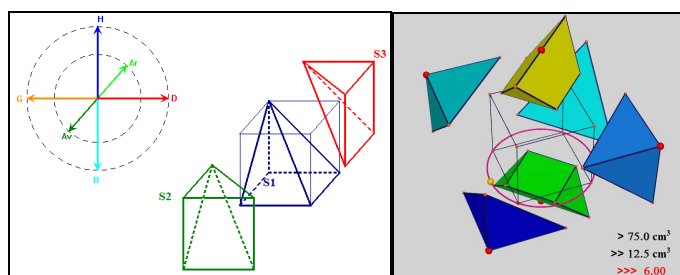


Figure 14

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