Technology Has Shaped Up Mathematics Communities

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Abstract

There is no doubt that technological tools have greatly impacted our mathematics teaching, learning and research in recent years. The exciting innovative ways of presenting learning, teaching and research materials on the internet have prompted educators and researchers to rethink the importance of taking global views to solve local problems. In this paper, we use several examples to demonstrate how some abstract mathematical concepts can be conveyed to students through graphical approach. The urgency of conducting collaborative research because of the existence of a solution is simply not adequate when computational tools are available nowadays. Furthermore, we emphasize that technology can be implemented effectively to enhance pre-service teachers' content knowledge. Finally, we urge all software and hardware developers to work together to make the learning tools more uniformly accessible. After all, those starting to use technological tools should be concentrating on how mathematics content can be learned instead of worrying too much about what software syntax or hardware keystrokes they should be using. Only when interest in mathematics is genuinely cultivated, can one truly appreciate mathematics and discover exciting mathematical theories.

1 Introduction

Lacking visualization and motivation on abstract mathematical concepts are typical reasons why students lose interest in mathematics. It was an eye opening experience when a Computer Algebra System (CAS) allowed us to do mathematics graphically and symbolically in the early 1990s; many of us were enthusiastic about how technology can impact our teaching, learning and research in mathematics. We start with some examples in Section 1.1 to demonstrate how a CAS can assist our teaching and learning. In Section 2, we use examples in integration theory to show why we need to connect theoretical and computational mathematics; existence of a solution is simply not adequate in the current technological era. In Section 3, we pick an example to show how a Dynamic Geometry System (DGS) can be an effective tool for learning and research, which allows us to make conjectures before analytical solutions can be realized. Requiring the

ability of making conjectures and analyzing their findings from learners is crucial for them to continue expanding their passion in mathematics. Consequently, requiring common content knowledge from pre-service teachers is just as important, so they can motivate more students to be interested in mathematics in their early ages. Finally, we have seen many mathematical computational tools, either software or hardware devices, evolve over the years. However, it is still non-trivial for beginners to use a computational tool. Therefore, we urge all software and hardware developers to further improve their products to make learning mathematics more enjoyable and accessible.

1.1   Visualization Makes a Difference

For example, it is non-trivial to prove that

$$f(x) = \sum_{k=1}^{\infty} a^k \cos b^k \pi x,$$

where a and b satisfying the relationships of $0 < a < 1, b \in Z^+$ and $ab > 1 + \frac{3}{2}\pi = 5.712388981$, is nowhere differentiable but continuous everywhere (see [2]). It is definitively stimulating to students when they see the graphs of the respective partial sum, $\sum_{k=1}^{10} \left(\frac{1}{2}\right)^k \cos 12^k \pi x$ and $\sum_{k=1}^{40} \left(\frac{1}{2}\right)^k \cos 12^k \pi x$, and

- ask why these two respective graphs are very close (readers can use a graphical tool to experiment with these two graphs), and

- conjecture that the graph of the infinite series $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \cos 12^k \pi x$ should not be too far away from that of $\sum_{k=1}^{40} \left(\frac{1}{2}\right)^k \cos 12^k \pi x$ on one hand, and

- it should simulate the graph of a function that is nowhere differentiable but continuous everywhere. We graph $\sum_{k=1}^{40} \left(\frac{1}{2}\right)^k \cos 12^k \pi x$ below in **Figure 1**. This will be a good application for students studying the infinite series of functions and the concept of uniform convergence.
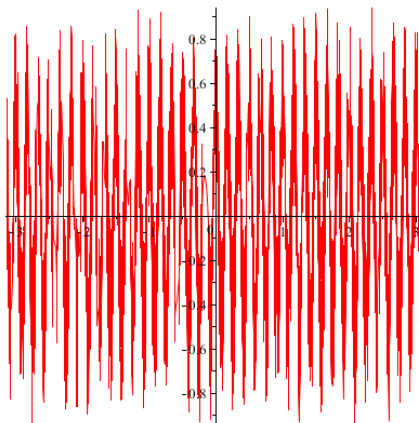

Figure 1. Simulating a nowhere differentiable function

Let's compare the following two statements about quantifiers when students start learning mathematical proofs:

1. *For every number $\varepsilon > 0$ and for every number $x \in [0,1]$ there exists a positive integer N such that for every integer $n \geq N$ we have*

$$\frac{nx}{1+n^2x^2} < \varepsilon.$$

2. *For every number $\varepsilon > 0$ there exists a positive integer N such that for every number $x \in [0,1]$ and for every integer $n \geq N$ we have $\dfrac{nx}{e^{n^2x}} < \varepsilon.$*

For beginners, they have difficulty in distinguishing the differences between these two statements. However, using graphical approaches may assist students in understanding these two concepts. Most students would have studied multivariable calculus before learning these concepts. If we examine these two statements closely, the first one is equivalent to the concept of pointwise convergence (when each $x \in [0,1]$ is given first). For instance, if we choose $f_n(x) = \frac{nx}{1+n^2x^2}$, we invite readers to use any graphing utility to experiment why $\{f_n(x)\}$ converges to 0 pointwisely as $n \to \infty$ (but not uniformly to 0). In the meantime, the second statement is equivalent to the concept of uniform convergence. If we choose $f_n(x) = \dfrac{nx}{e^{n^2x}}$, we invite readers to experiment why $\{f_n(x)\}$ converges to 0 uniformly as $n \to \infty$

Once students understand what is meant by uniform convergence, pointwise convergence, and series of functions using partial sums, the Fourier series of a step function and the Gibbs phenomenon will be a good application to reinforce the relationship between the uniform convergence and the continuity of the limit function using a graphical approach (see [11]). For instance, we consider the function

$$f(x) = \begin{cases} 1 & if \quad x \in (0,\pi) \cup (2\pi, 3\pi) \\ -1 & if \quad x \in (-\pi, 0) \cup (\pi, 2\pi). \end{cases}$$

Let $(S_N f)(x)$ be the partial sum of the Fourier series for f, or $(S_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n \cos(nx) + b_n \sin(nx)]$, where a_n and b_n are the Fourier coefficients for f. Graphing the function f and $(S_N f)(x)$ was not possible in the past when graphing tools were not available. Now we can show the graphs of f and $S_{20}f(x)$ in $(-\pi, 3\pi)$ below in **Figure 2**. We see that the convergence of $\{(S_N f)(x)\}$ to f, when $N \to \infty$, is not uniform otherwise f would have been a continuous function. It is interesting to note that the Gibbs phenomenon was explained after the concept of uniform convergence was developed. Hence, we may naturally conjecture that graphical and computational tools will enable us to discover even more mathematical findings.
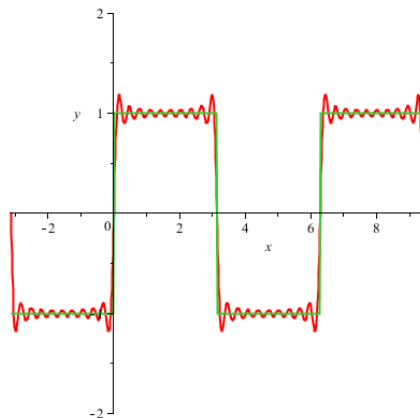

Figure 2. Gibbs phenomenon

2 Connecting Theoretical and Computational Mathematics

Students will desire to estimate a solution and will not be satisfied with only the existence of a solution in current research environment where computational tools are available. So there is the need to conduct collaborative research among experts from different areas to spark new discoveries and pass the knowledge to the students. In the meantime, researchers in mathematics shall consider making theories more accessible to wider audience starting from the undergraduate-level. One way to accomplish this is to give more motivation on abstract definitions or theories and give computation algorithms for students to explore with available computational tools.

2.1 Beyond Riemann and Lebesgue Integrals

Let's take a short tour on Riemann integral in an undergraduate calculus course. Recall the definition of Riemann sum we learned in a calculus class. In short, we say a bounded real-value function f on $[a, b]$ is Riemann integrable if the following identity is true and the limit is a finite number:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^n f(t_i)(x_i - x_{i-1}),$$

where $\{x_0, x_1, ... x_n\}$ defines is a partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i], i = 1, 2, ... n$.

We remark that a computational tool would allow us to compute the Riemann sum for a given n (of course within the computation capability of the computation tool). Such approximation gives immediate feedback or conjecture if an integral would exist or not. To establish if a function is Riemann integrable in an advanced calculus class, one can define upper and lower Riemann sums respectively first, which we refer readers to any textbooks in this area for more details. In view of the lower Riemann partial sum is an increasing sequence and the upper Riemann partial sum is a decreasing sequence, we say a function is Riemann integrable if the limits of these two respective partial sums are equal to each other. However, we found this Riemann definition to be unsatisfactory because of the constant mesh $\frac{b-a}{n}$ should be required to approach to 0 as $n \to \infty$. For example, let's take a look at a bounded function that is not Riemann integrable: We define the Dirac function $f : [0, 1] \to R$ by $f(x) = 1$ if x is a rational number in $[0, 1]$ and $f(x) = 0$ if x is an irrational number in $[0, 1]$. Intuitively, those countably many points with height of f equaling to 1 should not affect the final answer of our integration because we have uncountable points achieving the height of $f(x) = 0$. In other words, we should expect the true value $\int_0^1 f(x)dx = 0$ in this case. Furthermore, we have many unbounded functions that are obviously not Riemann integrable by Riemann definition such as $f(x) = \frac{1}{\sqrt{x}}$ in $(0, 1]$ and $f(0) = 0$, that is why we call $\int_0^1 \frac{1}{\sqrt{x}}dx$ improper Riemann integrable. With the help of the Fundamental Theorem of Calculus (FTC), we define

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}dx = 2.$$

Thus, the Lebesgue integration was born in early 1900, which allows us to handle the imperfect situation above and arrive at the right answers for the integrations as we would have expected.

But now the Fundamental Theorem of Calculus is imperfect under Lebesgue definition. For example the FTC under Lebesgue will read as follows:

Theorem Let $F : [a, b] \to R$ be continuous. and F' be Lebesgue integrable on $[a, b]$. Then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

This is odd because the integration of F' should depend only on $F(b)$ and $F(a)$. Requiring F' to be Lebesgue integrable is not necessary and reflects the deficiency of Lebesgue definition. For function such as $F(x) = x^2 \cos\left(\frac{\pi}{x^2}\right)$ if $x \neq 0$ and $F(0) = 0$, we know that $F'(x) = 2x \cos\left(\frac{\pi}{x^2}\right) + \frac{2\pi \sin\left(\frac{\pi}{x^2}\right)}{x}$ if $x \neq 0$ and $F'(0) = 0$. It is known that $F'(x)$ is not Lebesgue integrable in $[0, 1]$, and it follows from the *Fundamental Theorem of Calculus* that $\int_0^1 F'(x)dx = F(1) - F(0) = -1$. So we have a situation where we know that $\int_a^b F'(x)dx$ should exist and the correct answer should be $F(b) - F(a)$, and yet this is not the case under Lebesgue theory. In addition, the Lebesgue definition can not handle many functions that are highly oscillating and unbounded at one point. For example, if $f(x) = \frac{1}{x} \sin \frac{1}{x}$ if $x \neq 0$, and $f(0) = 0$, then $\int_0^1 f(x)dx$ does not exist in Lebesgue sense. Similarly, if $f(x, y) = \frac{1}{xy} \sin \frac{1}{xy}$ if $(x, y) \neq (0, 0)$, and $f(x, y) = 0$ if $x = 0$ or $y = 0$, then $\int \int_{[0,1] \times [0,1]} f(x, y)dA$ is not Lebesgue integrable. There are two questions, should these two integrals exist under a different definition? This describes why Henstock and Kurzweil independently came up with the following idea (in 1960s) using uneven or locally fine partitions.

2.2 Motivation of Henstock-Kurzweil Integral

Avoiding singularities are typical techniques when doing numerical integrations (see [1],[6],[8]). We shall see how we link theoretical approaches to computational schemes below. For more detailed explorations, we invite readers to read [10], [14]. We adopt the terminologies used in [4],[5]. Let δ be a positive function defined on A. A partition $P = \{(A_1, x_1), ..., (A_n, x_n)\}$ is called δ-fine if $A_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, for $i = 1, 2, ...n$. We first give the definition of Henstock-Kurzweil integration on one dimension (1D).

Definition 1 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on $[a, b]$ if for every $\epsilon > 0$ there is a positive function δ on $[a, b]$ such that*

$$\left| \sum_{i=1}^n f(x_i) |A_i| - I \right| < \epsilon \tag{1}$$

for each δ-fine partition P of A, where $|A_i|$ denotes the length of $A_i, i = 1, 2, ..., n$. In such case, we write $\int_a^b f \, dx$ or simply $\int_a^b f$.

Remarks:

1. Some of the problems we mentioned such as improper Riemann integrals and the deficiency for FTC under Lebesgue definition are solved, which we will not get into the

details. In view of the fact that the HK-integration definition involves the Riemann sum, one naturally would ask how to approximate a solution if we know the existence of an integration.

2. We shall see the following computation methods (see equations (2) and (4)) are to take care of one or finitely many singular points in 1D, where a function is unbounded or highly oscillatory near the singular point. The key is to avoid the singular points by wrapping the singular point by shrinking intervals in one dimension, or wrapping the singular points by shrinking rectangular boxes in two dimensions (2D).

3. On one hand, shrinking the intervals or rectangular boxes, which contain singularities, are consistent with theoretical ideas of using locally fine partitions (see [4],[5],[7]). On the other hand, we should see the computational quadratures in one or two dimensional below are to avoid singular points, which are what one will do in numerical integration (see [1],[6],[8]).

First, for each positive integer m, we make use of finite sum formula, $\sum_{k=1}^{n} k^m$, to define a locally fine partition. For example, if we use $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ to define the matrix $a_{nk} = \frac{2(b-a)k}{n(n+1)}$, we see $\sum_{k=1}^{n} a_{nk} = b - a$. Suppose we choose $a = 0, b = 1$, then the uneven width a_{nk} for each subinterval when $n = 10$ can be seen as follows:

$$\begin{bmatrix} \frac{1}{55} & \frac{2}{55} & \frac{3}{55} & \frac{4}{55} & \frac{1}{11} & \frac{6}{55} & \frac{7}{55} & \frac{8}{55} & \frac{9}{55} & \frac{2}{11} \end{bmatrix}.$$

Similarly, if we pick $a_{nk} = \frac{6(b-a)k^2}{n(n+1)(2n+1)}, a = 0, b = 1$, then the uneven widths for each subinterval when $n = 10$ is

$$\begin{bmatrix} \frac{1}{385} & \frac{4}{385} & \frac{9}{385} & \frac{16}{385} & \frac{5}{77} & \frac{16}{385} & \frac{7}{55} & \frac{64}{385} & \frac{81}{385} & \frac{20}{77} \end{bmatrix}.$$

Notice that we have chosen the widths of subintervals unevenly which allowing us to shrink the width of the interval containing singular point. This is the key to HK-integration theory.

We consider the following quadrature in one dimension

$$Q_n(f) = \frac{1}{2} a_{n1} f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk}) \right) \tag{2}$$

where $u_{nk} = a + \sum_{i=1}^{k} a_{ni}$, and $u_{n,0} = a$.

Example 2 *We define $f(x) = \ln(3 - 3\cos x)$ if $x \neq 0$ and $f(0) = 0$. We shall use the matrix $a_{nk} = \frac{6(b-a)k^2}{m(m+1)(2m+1)}$ and apply equation (2) to estimate $\int_0^1 f(x)dx$ accordingly. We note that f is HK- integrable over $[0, 1]$. In order for us to increase the number of partitions on n when using quadrature in (2) and achieve better computation accuracy, we apply the Richardson extrapolation quadrature by incorporating $Q_n(f)$. In other words, we define*

$$R_n(f) = \frac{1}{2} a_{n1} f(u_{n1}) + \frac{1}{3} \left[4Q_n(f) - Q_{n/2}(f) \right].$$

When using Digits=20 for computation and with the help of Maple 13, we find the approximations for $\int_0^1 f(x)dx$ as follow:

$$R_{700}(f) = -1.6224537082520067239,$$
$$R_{800}(f) = -1.6224535349823094413,$$
$$R_{900}(f) = -1.6224534278058672640,$$
$$R_{1000}(f) = -1.6224533578168565515,$$

We note that Maple 13 gives U.S. $\int_0^1 f(x)dx$ = -1.622453156+0.*I, which we have no idea how the complex portion 0.*I comes about. When using WolframAlpha Integrals (see [9]), we get -1.62245.

Naturally, the 1D Definition can be extended analogously to 2D as follows (see [4],[5],[7]):

Definition 3 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on an interval $A = [a,b] \times [c,d]$ if for every $\epsilon > 0$ there is a positive function δ on $[a,b] \times [c,d]$ such that*

$$\left| \sum_{i=1}^{n} f(x_i) |A_i| - I \right| < \epsilon \tag{3}$$

for each δ-fine partition P of A, where $|A_i|$ denotes the area of A_i, $i = 1, 2, ..., n$. In such case, we write $\int_A f \, dA$ or simply $\int f$.

Accordingly, the quadrature (2) can be extended to two dimensions as follows:

$$Q_{nm}^2(f) = \sum_{l=2}^{m} \sum_{k=2}^{n} \frac{a_{nk}b_{ml}}{4} \left(f(u_{n,k-1}, v_{m,l-1}) + f(u_{nk}, v_{m,l-1}) + \right. \tag{4}$$
$$\left. f(u_{n,k-1}, v_{ml}) + f(u_{nk}, v_{ml}) \right)$$
$$+ \sum_{k=2}^{n} \frac{a_{nk}b_{m1}}{4} \left(f(u_{n,k-1}, v_{m1}) + f(u_{nk}, v_{m1}) \right)$$
$$+ \sum_{l=2}^{m} \frac{a_{n1}b_{ml}}{4} \left(f(u_{n,1}, v_{m,l-1}) + f(u_{n1}, v_{ml}) \right)$$
$$+ \frac{a_{n1}b_{m1}}{4} f(u_{n1}, v_{m1}),$$

in which $u_{n0} = a, u_{nk} = a + \sum_{p=1}^{k} a_{np}$ for $k = 1, 2, ..., n$, $v_{m0} = c$, and $v_{ml} = c + \sum_{q=1}^{l} b_{mq}$ for $l = 1, 2, ..., m$; note that $u_{nn} = b$ and $v_{mm} = d$.

Remarks:

1. We note that the two dimensional quadrature above allows us to approximate an integral when singularities are along $x = a$ and $y = c$ respectively. The first expression in the quadrature, $\sum_{l=2}^{m} \sum_{k=2}^{n} \frac{a_{nk}b_{ml}}{4} (f(u_{n,k-1}, v_{m,l-1}) + f(u_{nk}, v_{m,l-1}) + f(u_{n,k-1}, v_{ml}) + f(u_{nk}, v_{ml}))$, is a two dimensional Riemann sum using uneven partitions when f does not touch the singularities.

2. The expression $\sum_{k=2}^{n} \frac{a_{nk}b_{m1}}{4}(f(u_{n,k-1}, v_{m1}) + f(u_{nk}, v_{m1}))$ is the Riemann sum for $(x, y) \in [u_{n1}, b] \times (c, v_{m1}]$. In other words, it is to take care of singularities when they are in $[u_{n1}, b] \times \{c\}$. Similarly, the expression $\sum_{l=2}^{m} \frac{a_{n1}b_{ml}}{4}(f(u_{n,1}, v_{m,l-1}) + f(u_{n1}, v_{ml}))$ is the Riemann sum for $(x, y) \in (a, u_{n1}] \times [v_{m1}, d]$ or it is to take care of singularities when they are in $\{a\} \times [v_{m1}, d]$. Finally, $\frac{a_{n1}b_{m1}}{4} f(u_{n1}, v_{m1})$ is an approximation for the rectangle $[a, u_{n1}] \times [c, v_{m1}]$

3. We also note that, in general, we choose $m = n$ and $a_{nk} = b_{nl}$ as our two dimensional computation to avoid each sub-rectangle A_n in the Definition 3 becoming too thin which may cause round-off errors when m or n gets large. In other words, the shape of each sub-rectangle is kept as a square.

For computing more sophisticated examples using our 2D quadrature (4), we refer readers to [10], [14]. We use the following example to demonstrate how we may use (4) to detect a double integral that might not exist.

Example 4 *We define $f(x, y) = \frac{xy}{(x^2+y^2)^2}$ if $(x, y) \in (0, 1] \times (0, 1]$ and $f(x, y) = 0$ if $x = 0$ or $y = 0$. We can show theoretically that the double integral, $\int \int_{[0,1] \times [0,1]} f(x, y) dA$, does not exist. Here we use $a_{nk} = \frac{6(b-a)k^2}{n(n+1)(2n+1)}$, $b_{ml} = \frac{6(d-c)l^2}{m(m+1)(2m+1)}$, $m = n$ in $Q_{nm}^2(f)$, and we obtain the following information with Maple 13:*

$$Q_{100}^2(f) = 7.5493300830237376437,$$
$$Q_{200}^2(f) = 8.5854677103194479888,$$
$$Q_{300}^2(f) = 9.1924464266105399032,$$
$$Q_{400}^2(f) = 9.6233554239577289740.$$

We conjecture that the double integral $\int \int_{[0,1] \times [0,1]} f(x, y) dA$ diverges (slowly). The divergence is slow can be explained because the following limits of the tails are all approaching to 0, which can be seen below when Maple 13 is used:

$$\lim_{n \to \infty} \frac{a_{n1}b_{n1}}{4} f(u_{n1}, v_{n1})$$
$$= \lim_{k \to \infty} \frac{a_{nk}b_{nl}}{4}(f(u_{n,k-1}, v_{n1}) + f(u_{nk}, v_{n1}))$$
$$= \lim_{l \to \infty} \frac{a_{nk}b_{ml}}{4}(f(u_{n,1}, v_{n,l-1}) + f(u_{n1}, v_{nl}))$$
$$= 0.$$

In summary, the two dimensional quadrature $Q_{nm}^2(f)$ can assist us to conjecture whether or not a double integral exists.

Remark: We further note that since $\int \int_{[0,1] \times [0,1]} f(x, y) dA$ diverges, it is clear that $\int \int_{[-1,1] \times [-1,1]} f(x, y) dA$ does not exist. Therefore, we cannot apply Fubini's Theorem in this case (using iterated integrals to compute the double integral). However, many computational engines such as Maple or Mathematica are using iterated integration algorithms to calculate the double integrals; for example, in the case of $\int \int_{[-1,1] \times [-1,1]} f(x, y) dA$, both Maple and Mathematica give 0, which is clearly wrong.

Discussions: The quadrature (4) is no more than a trapezoidal rule but using uneven partitions, for a fixed n, we see $a_{n,k} < a_{n,k+1}$ when $a_{nk} = \frac{2(b-a)k}{n(n+1)}$. In other words, the width of each subinterval is smaller on the left compared to that of on the right. This is to take wrap the singularity at $x = a$ by using shrinking intervals. We can use this quadrature (4) to take care of all improper Riemann integrals where f is monotone and has singularity at $x = a$ (see [10]).

1. The 1D quadrature (2) can be adapted to estimate HK integrals where integrand is highly oscillating at one point, such as $f(x) = \left(\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)$ if $x \in (0,1]$ and $f(0) = 0$, by countably many extensions. Similarly, the 2D quadrature (4) can be used to handle functions such as $f(x,y) = \left(\frac{1}{xy}\right)\sin\left(\frac{1}{xy}\right)$ if $x \in (0,1] \times (0,1]$, and $f(x,y) = 0$ if $x = 0$ or $y = 0$, (see [10]).

2. The 2D quadrature (4) can also be used to handle integrals involving the Divergence Theorem. For example, we consider the vector field $\mathbf{F}(x,y) = \begin{cases} (x^2\cos(\frac{\pi}{x^2 y}), 0) & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$ over $A = [0,1] \times [0,1]$. Then $\operatorname{div}\mathbf{F} = \frac{1}{xy}\left(2\pi\sin\frac{\pi}{x^2 y} + 2x^2 y\cos\frac{\pi}{x^2 y}\right)$ if $(x,y) \neq (0,0)$, and $\operatorname{div}\mathbf{F} = \mathbf{0}$ otherwise. We note that $\operatorname{div}\mathbf{F}$ has singularities along $x = 0$ and $y = 0$, $\operatorname{div}\mathbf{F}$ is Henstock integrable (but not Lebesgue integrable) over A, and we have

$$\int\int_A \operatorname{div}\mathbf{F}\,dA = \int_{\partial A}\mathbf{F}\cdot\mathbf{n}\,, \tag{5}$$

see [7]. On one hand, direct calculation shows that $\int_{\partial A}\mathbf{F}\cdot\mathbf{n} = \int_0^1 \cos\frac{\pi}{y}dy = \int_0^1 \cos\frac{\pi}{y}\,dy = -.116770364$ (with the help of Maple 13), but Maple (13) could not evaluate $\int\int_A \operatorname{div}\mathbf{F}\,dA$ directly. In this case, we may apply our 2D quadrature (4) to approximate $\int\int_A \operatorname{div}\mathbf{F}\,dA$ directly.

3 Technology Continue Shaping up Mathematics Communities

3.1 Seeing is just Beginning

In this section, we discuss the impact of technological tools on our teaching and learning. The mathematics education area is where technology immediately plays an important role in increasing content knowledge. We understand that problems and discrepancies exist in the programs of mathematics education from country to country, and we direct readers to [15] and [13]. We emphasize the existing problems in this field as follows:

1. Creativity does not come from a drill or rote type of learning. The Tiger Mother scenario can be seen in many communities, and we see students getting burned out after preparing for high school and university entrance examinations respectively in many Eastern countries. On the other hand, students from the Western countries lose interest in mathematics as early as middle school, when they run into unqualified math teachers. Regardless whether students are from the East or the West, mathematics is ultimately not their passion.

2. One has to worry that pupils around the world are less and less interested in mathematics. The traditional education system mainly focused on the development of students' procedural knowledge. However, educators are now recognizing the importance of development of students' conceptual understanding through use of rich application problems (see [13]). More and more children have access to internet and naturally they like to play games on the internet. What if our mathematics curricula are designed in a way to allow them to explore mathematics content knowledge while they are having fun? A successful curriculum should be designed to inspire students' interests in mathematics and sciences by connecting real-life problems to classrooms.

3. Many Mathematics Education Ph.D. programs in the U.S. are not producing high calibre qualified future math teachers. Many programs were created simply providing the needs for teachers' certification in local areas, and universities could not fill a position whose area is in Mathematics Education. This creates a wave for people wanting to pursue a higher degree in Mathematics Education and yet the lack of content knowledge in the curriculum presents a serious crisis in this area.

One natural approach is to use DGS to stimulate the motivation of how one problem can be solved before using CAS to prove if the conjecture is right. We select the following problem to demonstrate why the development of conceptual knowledge is preferable than that of procedural knowledge when technological tools are adopted. The problem we shall consider below can be explored by middle schools students, where they can think of way of approximating an area. To do so, teachers need to have knowledge of linking various disciplines when solving a problem. In addition, teachers need to be proficient of using computational tools to obtain the final answer. The problem is finding the area in an enclosed region. Our task is to estimate the area of the following **Figure 3(a)** while we are given that the **Figure 3(a)** is symmetric to the line segment EF. In view of this, it suffices to find the area for **Figure 3(b)** first.
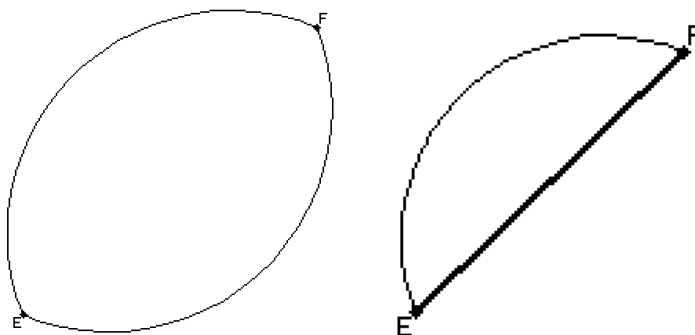

Figure 3(a) and **Figure 3(b)**. Area enclosed by the curves

We would like to add that either **Figure 3(a)** or **Figure 3(b)** was drawn randomly with a DGS, many dynamic geometry software packages are capable of recovering the equations of the curves we draw. In this case, we assume that it is given that **Figure 3(b)** is the union of the equation of part of the circle, $C(t)$, and the line segment EF. In other words, we have

$$C(t) = [1.328006 \cdot \cos(t) - 1.11, 1.328006 \cdot \sin(t) + 0.05], \text{ where } t \in [1.177631935, 3.5147],$$
$$E = (-2.346\,6, -0.434\,07), \text{ and } F = (-0.601\,22, 1.276\,7).$$

We note the followings:

1. We can write the line equation for EF as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2.3466 \\ -0.43407 \end{bmatrix} + t \begin{bmatrix} 1.7454 \\ 1.7108 \end{bmatrix}$, where t is a real number, or we can write it as

$$y = 0.98017 \cdot x + 1.867$$

2. The difficulty is if we use regular horizontal or vertical partitions, we will end up partition the area into at least two sub regions, which makes integration more difficult. For those people who knows how to apply Green's Theorem right away, this is not a difficult problem. However, this problem is intended for those who just finished the study of Riemann integration in the first or second semester of calculus course.

3. The common sense approach with DGS for those people who only know definition of integrations is to partition EF into several subintervals and draw rectangular boxes that are perpendicular to the line EF, and when we increase the number of subintervals along EF, we will get closer estimate for the area. We demonstrate this below by using the Java Sketchpad (see Supplementary Material, thanks to the design by Scott Steketee).

   (a) We subdivide the line segment EF into several subintervals.

   (b) We form the rectangles that are perpendicular to EF and use the Riemann Sum idea to approximate this area. As shown below, when we use $n = 50$, the approximated area using the leftend Riemann sum is about 1.41.
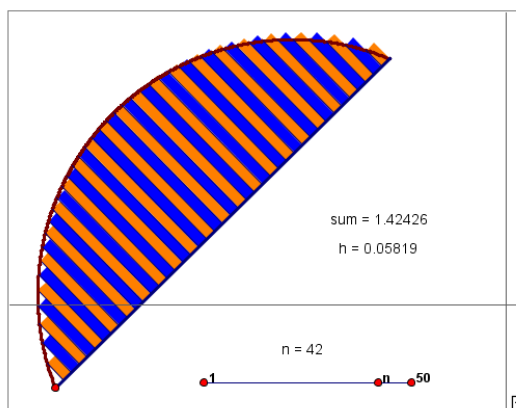

sum = 1.42426

h = 0.05819

n = 42

1   n 50

Figure 4. Natural way of partitions

3.2   Verifying Our Conjectures

In view of how we use partitions with rectangles to approximate the enclosed area shown in **Figure 4**, we naturally should choose a new basis $\{c_1, c_2\}$ where $c_1 \in \overleftrightarrow{EF}$ and c_2 is perpendicular to c_1. For completeness, we describe the techniques from [12]. We introduce some basic concepts in Linear Algebra. Let $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and $\mathcal{B}_2 = \{\mathbf{c}_1, \mathbf{c}_2\}$ be another basis for $\mathbb{R}^2$, where $\mathbf{c}_1 = (\cos\theta, \sin\theta)$ is a unit direction vector of the line L and $\mathbf{c}_2 = (-\sin\theta, \cos\theta)$ is a unit vector perpendicular to $\mathbf{c}_1$ with $\theta = \tan^{-1} m$.

We first express $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ as a vector relative to the basis $\mathcal{B}_2$. In other words, we need to discover $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ so that $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = p(t)\mathbf{c}_1 + q(t)\mathbf{c}_2$, or

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$= \begin{bmatrix} x(t)\cos\theta + y(t)\sin\theta \\ -x(t)\sin\theta + y(t)\cos\theta \end{bmatrix}. \tag{7}$$

Step 1. The integral of $[x(t), y(t)]$ with respect to a line $L : y = mx$ through the origin is

$$A = \int_{t_1}^{t_2} q(t)p'(t)dt \tag{8}$$

$$= \int_{t_1}^{t_2} (-x(t)\sin\theta + y(t)\cos\theta)(x'(t)\cos\theta + y'(t)\sin\theta)\, dt$$

$$= \frac{1}{1+m^2} \int_{t_1}^{t_2} (-x(t)m + y(t))(x'(t) + y'(t)m)\, dt$$

Step 2. Next, integrate a parametric curve $\mathbf{w}(t) = [x(t), y(t)]$, $t_1 \le t \le t_2$, with respect to a line $L : y = mx + b$ (let $\theta = \tan^{-1} m$). By shifting the bounded region vertically by b, we can then apply the same concept above to find the coordinate vector $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ of $\begin{bmatrix} x(t) - 0 \\ y(t) - b \end{bmatrix}$ relative to $\mathcal{B}_2$ below.

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x(t) - 0 \\ y(t) - b \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} x(t)\cos\theta + (y(t) - b)\sin\theta \\ -x(t)\sin\theta + (y(t) - b)\cos\theta \end{bmatrix}.$$

Therefore, the integral of $\mathbf{w}(t) = [x(t), y(t)]$, $t_1 \le t \le t_2$, with respect to the line $L : y = mx + b$ (let $\theta = \tan^{-1} m$) is

$$A = \int_{t_1}^{t_2} q(t)p'(t)dt \tag{10}$$

$$= \int_{t_1}^{t_2} \left(-x(t)\sin\theta + (y(t) - b)\cos\theta\right)\left(x'(t)\cos\theta + y'(t)\sin\theta\right)dt$$

$$= \frac{1}{1 + \tan^2\theta}\int_{t_1}^{t_2}\left(-x(t)\tan\theta + y(t) - b\right)\left(x'(t) + y'(t)\tan\theta\right)dt$$

$$= \frac{1}{1 + m^2}\int_{t_1}^{t_2}\left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt.$$

We remark that since the curve $\mathbf{w}(t)$ travels in the direction from $t = t_1$ to $t = t_2$, the value in equation (10) is positive if the curve is traversed clockwise; otherwise it is negative. If we apply the formula on the given example above of the parametric curve $C(t) = [1.328006 \cdot \cos(t) - 1.11, 1.328006 \cdot \sin(t) + 0.05]$, where $t \in [1.177631935, 3.5147]$, and $y = 0.980\,17 \cdot x + 1.\,867$, we get the following result

$$A = \frac{1}{1 + m^2}\int_{t_1}^{t_2}\left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt = -1.\,423\,8.$$

The area is negative since we use counterclockwise direction. To restate what we have just proved:

Theorem 5 *Let C be the smooth curve $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \leq t \leq t_2$. Let R be the region bounded by C, the line $y = mx + b$, and the perpendiculars to the line from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$. Then the (signed) area of R is given by*

$$\frac{1}{1 + m^2}\int_{t_1}^{t_2}\left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt. \tag{11}$$

When students are exposed to vector calculus concepts, we can reinforce that the above observation is a special case of Green's Theorem. We link Theorem 5 with the concept of Green's Theorem as follows:

Let C be the smooth curve $\mathbf{w}(t) = [x(t), y(t)]$, where $t_1 \leq t \leq t_2$. Let R be the region bounded by C, the line $y = mx + b$, and the perpendiculars to the line from $(x(t_1), y(t_1))$ to $(x(t_2), y(t_2))$. We denote the counterclockwise boundary curve of R by ∂R. If P and Q are scalar fields with continuous. partial derivatives satisfying $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then

$$\int_{\partial R} P\,dx + Q\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA \tag{12}$$

$$= \frac{-1}{1 + m^2}\int_{t_1}^{t_2}\left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right)dt.$$

Note. If we write If $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ is a vector field, we note that $\operatorname{curl}\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} =$

$\begin{bmatrix} 0 & 0 & \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \end{bmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$. Therefore, $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. In other words,

we can write the Green's Theorem in the vector form below:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA. \tag{13}$$

$$= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\, dx + Q\, dy$$

$$= \frac{-1}{1+m^2} \int_{t_1}^{t_2} \left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right) dt.$$

For convenience, one may choose $P(x,y) = -\frac{y}{2}$ and $Q(x,y) = \frac{x}{2}$. We leave it to the reader to verify that if we apply Green's Theorem to find the area bounded by the curve C and EF is same as we have found by using formula (11).

Remarks:

1. Many math educators outside the U.S. will be astonished that most future middle school math teachers in the U.S. do not need to finish calculus courses, and yet it is ironic that the concept of Riemann sum can be understood by a middle school or even an elementary kid. The animation in the Supplementary Material section was shown to a fifth grader (10 years old) student in the U.S. without prior knowledge of what the animations is all about, he replied that the animation shows that 'more rectangles will fit the area better'.

2. It is necessary to integrate the knowledge of Linear Algebra with multivariable calculus, because such knowledge is needed when studying Differential Geometry. We note that formula (14) can be explored before students learn the concepts of Green's or Divergence Theorems, which come in almost near the end of a calculus course.

3. Author presented this problem to a group of Japanese high school teachers who posses master degree in mathematics. Without the presence of a computational software, it is commendable that some teachers immediately think from algebra point of view, by finding the distance from the point on the circle to the slanted line to form the height of a sub-rectangle, before forming the area of each sub-rectangles.

4   A Glimpse of the Future

4.1   Be a Global Mathematics Educator

Many examples from textbooks are non-realistic: why are most of the answers whole numbers? We need to connect more of our daily life experiences to mathematics. Instead of memorizing formulae for testing purposes, we need time for students to do projects. Students need time to entice real-life scenarios to mathematics, and make conjectures and generalize their observations. The impact of evolving technological tools on math and science is immense; it is natural that the National Science Foundation in the U.S. encourages proposals from the Science, Technology, Engineering and Mathematics (STEM) area. However, any similar agencies around the world should note that to increase learners content knowledge in the area of STEM, it is only reasonable for mathematicians, who have strong backgrounds in mathematics, to work with people in mathematics education to come up with content that lead to more students interested in the STEM area.

We believe that CAS and DGS will continue to impact us in teaching, learning and research in mathematics. More specifically, we have seen examples in this paper that visualization plays a pivotal role in stimulating students' interests in mathematics from an early age. We should carefully select and design a wide range of real-life problems, where we may adopt problems for students at different stages of learning. An example can be explored in middle school, high school and again in university level. Students will expand their level of mathematics knowledge when they move from one level to the next. Consequently, when a mathematical problem is encountered in various stages, students can deeply enhance their conceptual understandings and their abilities to solve related real-life problems.

Educators in the East should not only be interested in knowing how to retain or improve their standings in international math and sciences competitions, but also think about how to seriously inspire more students to become interested in math and sciences from early ages. On the other hand, educators in the U.S. should no longer ignore the consequences of the lack of mathematics content knowledge in pre-service schools trainings. Therefore, one way of enhancing mathematics education quality in the East and the West is to continue discussing the innovative use of technology in teaching, learning and research in mathematics.

4.2   Creating Online Math Content

We have adopted various specific software packages in CAS and DGS when exploring examples mentioned in this paper. We encourage readers to experiment with mathematics concepts behind these examples by using their favorite software packages. It will be wonderful if we can all concentrate on communicating mathematics among ourselves by using standard mathematics equations and etc., without worrying about how to manipulate a certain computational tool. Indeed, making a tool easier to use is still somewhat difficult for developers-why not unify all the menu bars and keystrokes? Static math and science content are available in various forms, such as html, video clips over the internet (see examples in [3] for teaching and learning purposes). Users of software packages or hand-held computation devices have wondered why it takes so long to develop a system where users can communicate mathematics content with standardized mathematics symbols over the internet. The computation engines such as CAS and DGS can be stored in a server, giving users an option of computing over the server while companies charge the usage of our computations, just like how the usage of cell phones is charged. It is time for all parties, mathematics educators and software and hardware developers, to work together on a platform where learners can not only read static mathematics but also experiment with dynamic mathematics over the internet. Only when commercial companies work together with mathematics educators to truly develop user-friendly tools, will we able to make mathematics **f**un, **a**ccessible, **c**hallenging and **t**heoretical at the same time.

Supplementary Material

- http://www.radford.edu/wyang/atcm2011

References

[1] Davis and Rabinowitz (1983), *Method of Numerical Integration*, 2nd ed., Academic Press, New York, ISBN978-0122063602.

[2] Körner, T.W., *Fourier Analysis*, page 38-41, Cambridge University Press, 1988.

[3] Khan Academy, http://www.khanacademy.org/.

[4] P. Y. Lee (1989), *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, ISBN 9971-50-892-3.

[5] P. Y. Lee and R. Výborný (2000), *Integral: An Easy Approach after Kurzweil and Henstock*, Cambridge University Press, Cambridge UK, ISBN 0 521 77968 5.

[6] R. K. Miller (1971), *On Ignoring the Singularity in Numerical Quadrature,* Mathematics of Computation, Volume 25, Number 115, July.

[7] W.F. Pfeffer (1993), *The Riemann Approach to Integration: Local Geometric Theory,* Cambridge University Press, ISBN 0-521-44035-1.

[8] P. Rabinowitz and I. H. Sloan (1984), *Product Integration in the Presence of Singularity*, Siam J. Numer. Anal. Vol 21, No. 1, February.

[9] WolframAlpha Integrals, http://www.wolframalpha.com/examples/Integrals.html.

[10] W.-C. Yang, P.-Y. Lee and X. Ding (2009), *Numerical Integration On Some Special Henstock-Kurzweil Integrals*, the Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 3, Vol. 3.

[11] W.-C. Yang and J. Lewin, *Exploring Mathemaics with Scientific Notebook*, page 91, Springer Verlag, 1998. ISBN 981-3083-88-3.

[12] W.-C. and M.-L. Lo, *Finding signed Areas and Volumes inspired by Technology,* the Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 1, Vol. 2.

[13] W.-C. Yang and G.M. Agida (2010), *U.S.e of Dynamic Course Core to Develop Teachers' Subject Matter Knowledge of Mathematics*, the Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 2, Vol. 4.

[14] W.-C. Yang and X. Ding (2010), *Fundamental Theorem of Calculus and Computations on Some Special Henstock-Kurzweil Integrals*, the Proceedings of ATCM 2010, Mathematics and Technology, LLC, ISBN 978-0-9821164-1-8 (hard copy).

[15] W.-C. Yang (1995), *Examples in Advanced CalculU.S. and Scientific Workplace*, the Electronic Proceedings of the Electronic Proceedings of the International Collegiate Technology Conference in Mathematics (ICTCM8).ICTCM8, http://archives.math.utk.edu/ICTCM/EP-8/C88/pdf/paper.pdf.

[16] W.-C. Yang (2009), *Content Knowledge, Creativity and Technology*, the Proceedings of ATCM 2009, Mathematics and Technology, LLC, ISBN 978-0-9821164-2-5 (hard copy).