Interactivity in dynamic mathematics environments: what does that mean?

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It is commonly accepted that an important feature of technology and technology based tasks lies in their interactivity and in the possibility of providing feedback to students' actions. The talk will address the notion of interactivity and feedback from a twofold perspective, on the one hand the design of a dynamic mathematics environment and on the other hand the design of tasks based on this environment and of different types of feedback provided to students. It will be shown that the degree of interactivity may greatly vary and that interactivity affects many aspects of the use of such environments. The discussion will be illustrated by the various Cabri technologies

The paper discusses the notion of interactivity in dynamic mathematics environment from a twofold perspective:

- problems faced by software designers when working on the features providing direct manipulation and feedback
- a didactic dimension: what is the nature and the role of feedback to students when they solve problems in a dynamic mathematics environment ?

1. Designing the features of direct manipulation: the case of Cabri 3D

Direct manipulation has proven to be a key feature to facilitate creative user interaction with a computer and has slowly generalized to most of computer platforms. For educational software nevertheless *Direct manipulation* cannot be designed by chance and has to follow some additional principles, one of them is called epistemic fidelity: the representation of mathematical objects have to avoid any contradiction with the abstract object they are supposed to represent; and this has to be true at the graphical level as at the level of their behavior under direct manipulation. Let us elaborate on the difference between interactivity and direct manipulation.

When interacting with a modern computer, the interface is essentially interactive in the sense that the user is "asking" the software to perform something and after the reaction of the computer, s/he asks something again for a next step. The most basic interactivity is offered by so-called "interactive books" giving essentially the possibility to display pages, and by pressing buttons to turn the pages of the books.

This kind of interactivity (unfortunately still widely spread, -especially –through Internet-) is easy to develop and leads to a form of "impoverishment" of user interface, with the generalization of the use of Internet. In contrast authentic direct manipulation software are mainly not driven by the press of buttons, or by the filling of dialogs (or forms) or by typing command lines. They offer an interface where the user is invited to directly act on the mathematical objects. Actually, the action is on the representation of an object or an abstract entity; nevertheless if the implementation is sophisticated enough and if the interaction turns to be of Direct Engagement type (Schneidermann, 1983), eventually the user perceives the representation of the object and the abstract object itself as already noticed by the five main designers of the Star Machine (Smith et al., 1982).

The need for extending the benefit of direct manipulation present in many 2D environments, just followed the first introduction of direct manipulation geometry software of Cabri type. Recall that Cabri actually stands for “Cahier de BRouillon Interactif”, somehow “Interactive Sketchpad”. When tackling with 3D various issues arise about complexity, specificities of spatial rendering, perspective choices, and many others…

Complexity

A very common idea about extending 2D environment to 3D is to think that this could be achieved (somehow) merely in adding an additional coordinate to the internal representation of the objects at the level of their data structure: essentially this would be then a trivial task. Actually as it is well known by mathematicians, 3D objects are “essentially” more complex than 2D objects: in most of the cases, augmenting the dimension leads to some complexity increase, even if eventually in higher dimension the situation might show more regularities. In 3D, many “basic problems” are still open. Let us mention the classification of quadrics. In 2D there are only 3 conics, ellipses, parabolas and hyperbolas. In 3D, i.e. for quadrics, nobody has yet found any really “elegant” classification that would satisfy everybody. Another example could be the conjecture about the probable existence, for any convex polyhedron, of its unfolding as a net of not overlapping connected (convex) polygons. In 2D “to follow”, in a reasonable way, the intersection of 2 conics is not an easy task (actually many of the DGS - dynamic geometry systems- cloned from the main ones, fail in trying to dynamically follow, in rather simple cases, the intersection points of a circle with a straight line!). In 3D nobody knows (yet and apparently) how to dynamically follow the intersections of a quadric with a line and even less the intersection curves of two quadrics.

Perspective issues

In classroom the so-called cavalier perspective has been the most popular way for representing 3D scenes. The main reason for that is because it is easy to create perspective drawings using the rules of the cavalier perspective. Recall that this perspective is governed by the fact that it is a parallel (yet not orthogonal) perspective (the observer is at infinity, and then parallel lines are still parallel in the perspective drawing). In addition there is a plane in which objects are in real size and actually lines perpendicular to this plane are represented as oblique lines in the drawing. This is really different from the “natural” perspective as introduced by the painters at the time of the Renaissance (e.g. Alberti) and which today can be considered as realized by high quality camera lenses.

Below (Fig. 1) is a cube, in “natural perspective” versus the same cube in cavalier perspective: if we would animate the cube to rotate it round its vertical axis of rotational symmetry, the cube in natural perspective would keep its "cubic" shape, but, in cavalier perspective, the cube would actually somehow pulse as its shape would change during the various phases of the rotation. Cavalier perspective is so conflicting the user perception about the cube as a solid object.

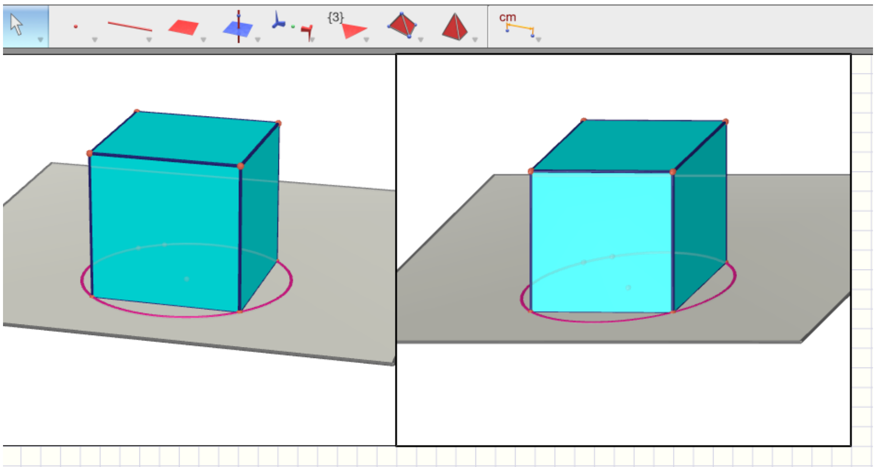

Figure 1 - *On the left a cube in natural perspective, on the right the same cube in cavalier perspective.*

Therefore we assumed that natural perspective favors the appropriation of the figure by students and users more generally. This is the reason why in Cabri 3D, default perspective, is actually a natural perspective. Its characteristics match to the view of an object of approximately 40 cm in size hold (as in the hand of the user) at a distance of 50 cm, this in contrast to some other software that by exaggerating the perspective effect (for some artistic purpose?) apparently do not favor the appropriation just mentioned. In above figures, one can clearly understand how, somehow, cavalier perspective is an attempt to look at an object from two points of view at once: from front and (here) from right. Actually this kind a representation has developed in many cultures, ranging from the ancient Egypt to China and Japan.

Other Specific issues (mouse, infinite object, clipping…)

One of the first things to be considered in order to directly manipulate objects in space is to have a mouse able to drive a point in 3D. Ordinary mice are essentially 2D, though it has existed for a while expensive 3D-mice. Because such 3D devices are not supposed to be available soon in the regular classroom neither at home, we have been looking for various solutions based on an ordinary mouse combined with modifiers (at keyboard level). Actually in Cabri3D is used the metaphor of the old "typewriter": as long the Shift Key is not pressed, the mouse simulated a displacement in some horizontal plane and if the user presses the Shift Key (implying a vertical movement of the carriage on antique typewriters) the mouse movement is interpreted as a movement along a vertical axis. Note that other environments may have done another choice like Archimedes Geo 3D in which pressing the Shift key provokes a move orthogonal to the screen plane. For reasons making sense for users, Cabri 3D does not permit any arbitrary rotation of the scene (as looking at the scene in some upside down way) and the horizontal reference plane always stays horizontal and consequently verticality is preserved.

The room here is short to address all specificities of the rendering of 3D mathematical objects (lines, planes, spheres, cylinders…), some of them being "in nature" infinite. Let us just mention the case of the plane, an infinite object. In textbooks, planes are most of the time represented using a rectangle to display a "portion" of the plane. It is also worth to note that textbooks present only a really limited number of figures. Space geometry textbook display hardly more than 10 different types of 3D situations we could consider as stereotypic. Among them one is the illustration of the famous “théorème du toit”, stating that if 2 planes intersect a plane along 2 parallels lines, their intersection is a third parallel line (Fig. 2).

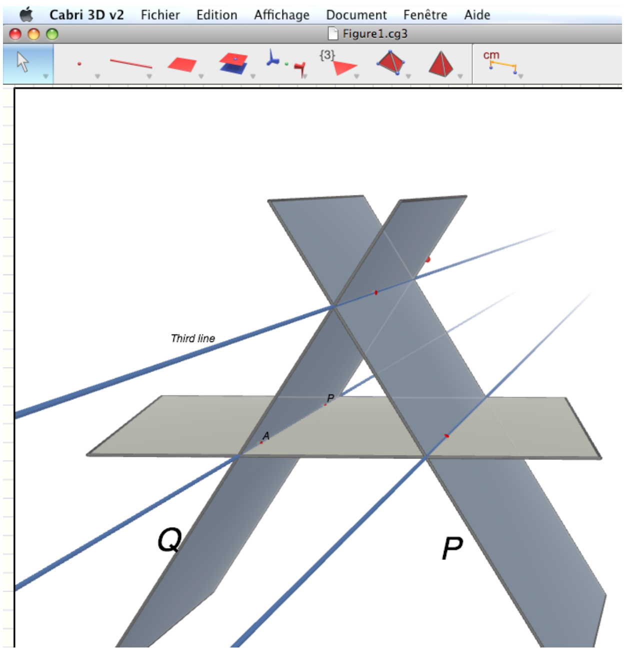

Figure 2 - *Two planes intersecting in a third parallel line*

In textbooks, which display only a static image, one can easily agree about "the good rectangle” to be taken to represent each plane. When things turns to be dynamic there is no “natural” way to “follow” the plane as it evolves. This is the reason why some 3D Geometry SW considering that plane are essentially infinite, does not limit their representations. A plane (up to the special of being viewed as a line, in French “de bout”) covers the whole screen and, in such environments, practically cannot be "seen" and so is not directly represented. We do not consider that this is a good idea for learner. In Cabri after various attempts we decided to represent a plane as a rectangle (in some earlier version as a parallelogram) presenting a certain amount of thickness, in the very same spirit as when Hilbert and Cohn-Vossen designed their 3D figures for their famous “Anschauliche Geometrie” (Hilbert & Cohn Vossen, 1952).

All these choices done on the basis of epistemic and ergonomic reasons must then be confronted to the real use by teachers and students. Let us mention here a thorough study (Hatterman 2011) analyzing the use of Archimedes Geo 3D and Cabri 3D by university students, giving evidence of various aspects of the ways of using these environments for solving different kinds of problems. It is worth mentioning that students solved problems more rapidly in Cabri 3D than in Archimedes Geo 3D (p. 164) without a clear evidence of an effect due to a particular feature in the interface of both software environments. It may be the combination of several aspects of the whole interface, which played a role.

2. A didactic dimension: organizing tasks and feedback by making use of the features of the software

The idea of a technological environment to be explored and interacting with the learner can be linked to the notion of "adidactical milieu" in the theory of didactic situations by Brousseau (1997). In this latter theory, knowledge is constructed by the student as a solution to a problem for which the constructed knowledge item provides an efficient solving strategy. The student solving the problem does not solve it for satisfying the expectations of the teacher but because it is a genuine problem for him, a problem of the same kind as problems encountered in the real life outside classroom. The only difference is that real life problems are not organized by a teacher for promoting learning. But although it is designed by the teacher, an adidactical situation is experienced by the student as a real life problem. In the core of the notion of adidactical situation is the notion of "adidactical milieu". An "adidactical milieu" offers information and means of action to the student and reacts by providing feedback to his/her actions. It can be of material nature as well as of intellectual nature.

We would not claim that dynamic environments like Cabri II Plus and Cabri 3D are "adidactical milieu" but an adidactical milieu can be organized and based on them for at least two main reasons:

- the available tools which allow the user to perform mathematical operations on the representations of the mathematical objects
- the feedback offered by the drag mode that allows the user to check whether his/her constructions are done by using mathematical properties and relations and not simply visually done .

Numerous examples of construction tasks with Cabri I, Cabri II or Cabri II Plus are given in the literature and show how the first solving strategies of students are visual and evolve towards more geometrical constructions through the drag mode playing a double role. The drag mode invalidates purely visual constructions and also provides information about the geometrical behavior of objects (Hoyles and Noss, 1996, p.125; Jones, 1998, pp. 79-82). In these constructions tasks, geometrical knowledge is efficient since it is the only way to build a construction, which is "drag mode proof". As it is possible to configurate the software and to make available a restricted range of default tools or new tools obtained as macro-constructions, the designers of construction tasks can thus promote the use of specific properties by the students and contribute to learning through the organized "adidactic milieu". An eloquent example is given by the task of drawing a perpendicular line to a line without the tool "Perpendicular" but with transformation tools, in particular the "Reflection tool".

Example of an adidactical milieu in Cabri 3D

A more recent example (Mithalal 2010) is given here about a construction task in Cabri 3D. Cabri 3D is used to create an adicatical milieu fostering the move by students from a pure global visualization of a solid object, called iconic visualization by Duval (2005), to an analytical breaking down of a solid object into parts interrelated through geometrical relationships called dimensional deconstruction by Duval (*ibid*.).

Grade 10 students of two classrooms using Cabri 3D for learning 3D geometry (Mithalal 2010) were faced with the following activity at the beginning of the teaching of 3D geometry and after an introduction to the use of Cabri 3D: A cube with a triangular cross section was given on the screen and students must reconstruct the missing vertex. So that it remains a vertex even when the cube is enlarged or dragged (Fig. 3).

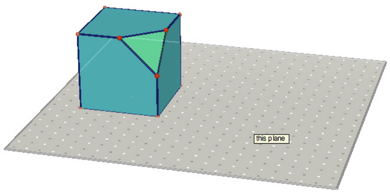

Fig. 3 – *A truncated cube*

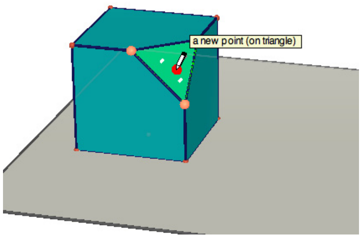

Fig. 4 – *A point visually located*

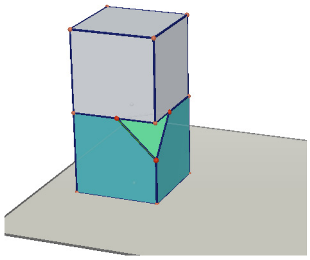

Fig. 5 – *Adjusting a second cube*

In order to foster learning, students were asked to find several solving strategies and tools were withdrawn from Cabri 3D once visual strategies appeared and once it was observed that these strategies did not produce robust vertices against dragging. It was expected that the added constraints would lead students to move to geometric characterizations of the missing vertex.

Producing a vertex by a purely visual strategy is not as easy in Cabri 3D as in paper and pencil environment. If the user attempts to put a point by eye at the desired location, Cabri 3D proposes to create the point on the triangular cross-section (Fig. 4).

Visual strategies must be a little more elaborated and include some geometric component. An example of such a semi-visual strategy consists of creating a cube on the top of the original cube by visually placing its center at the center of the squared top face and putting one of its vertices to a vertex of the original cube and then visually adjusting this second cube so that its bottom face is coinciding with the top face of the original cube (Fig. 5). Of course a vertex reconstructed in this way is not "dragging resistant". It was decided to withdraw the tool cube from Cabri 3D after such this strategy was proposed by students.

Then strategies conceiving the vertex as intersection of straight lines or of planes were expected in a second phase after semi-visual strategies. Students proposing such strategies should be asked to find other strategies without using the tool "Point".

The fine observation of 30 pairs of students shows that few of them resorted in a first phase to visual or semi visual strategies. Some of those students tried to create a tetrahedron based on the triangular cross section with a fourth vertex providing the missing vertex of the cube. Non-iconic visualization clearly underlies such a strategy. Students intended to reconstruct the entire cube as a material entity.

The most prevailing spontaneous strategy was to construct the missing vertex as the intersecting point of the three straight lines supporting the segments adjacent to the cross section (although two lines would be enough). The fact that often three lines and not two were constructed can be interpreted as a strategy inherited from paper and pencil environment mixed with iconic visualization. The three lines allowed students to restore the original representation of the whole cube in paper and pencil environment. Some students then moved to the construction of the vertex as the intersecting point of planes or of a plane and a line. They took advantage of the possibility of using 2D objects in Cabri 3D and extended the intersection strategy. The use of a plane supporting a face moved them away from a purely iconic visualization of the cube and very often when using a plane and a line, the vertex was constructed only with one plane and one line and not with two planes and one line and two lines and one plane. Through the instrumental deconstruction of the cube made possible by Cabri 3D, students moved towards a non-iconic visualization.

Finally it must be stressed that after the tool point was withdrawn, some students constructed the vertex using geometric transformations like point symmetry or translation: the vertex was constructed as the reflected image of another vertex with respect to the center of a cut face of the cube, or it is the image of a vertex in a translation with vector a side of the cube.

Tasks specific to dynamic geometry

Over the twenty years of existence of the Cabri technology, the design of "adidactical milieu" led to experiment new kinds of tasks that cannot exist in paper and pencil environments, in particular:

- "black box" tasks in which dynamic constructions are given to students who must reconstruct them again so that the reconstructions have the same behavior in the drag mode as the original ones ;
- prediction tasks in which students must predict without dragging what will happen if a specific point is dragged.

These tasks of a new kind are based on the same idea that students explore and interact with the environment and that through feedback and the available tools they will develop strategies involving geometrical knowledge. The fact that it is the environment and not the teacher who reacts contributes to make the problem analogous to a genuine problem for students.

It must be mentioned that with the extension of Cabri tools to algebraic and graphing tools, the design of tasks making use of the Cabri features was extended beyond geometry. For example, Falcade (Falcade et al. 2007) designed a milieu for constructing the notion of graph of a function as expressing the covariation of two variables, the first one independent and the second one depending on the first one. Moreno (2006) designed a milieu in which the students had to find the ordinary differential equation a family of dynamic curves, by exploring the variation and the invariant of this family in the drag mode, a kind of black box task in calculus.

3. Needs of the teacher

The adidactical situations are very appropriate when facing students with microworlds but observations coming from research show that students may understand that their solution is wrong when using the drag mode but that they do not know how to find another solving way. The intervention of the teacher is necessary to reformulate the problem without giving the solution so that students can search for another strategy. The situation of the missing wheel (Restrepo, 2008) is a good example of an adidactical situation requiring the intervention of the teacher for avoiding students' repeated adjusting solutions and drawing attention of the students to the importance of constructing geometrically the center of the circle.

Teachers may be reluctant to use such situations in which they cannot predict what students will do and they likely must build the way to intervene on the spot. This also may explain the gap between situations used in research and the everyday teaching practice. From all these observations done on everyday practice and teacher education, it seems that teachers could take more advantage of dynamic mathematics environments if

- they were provided with ready made tasks carefully designed for dynamic mathematics environments and already experienced with students, that the teacher may change and adapt for his/her classroom.
- and they had at their disposal comments about these tasks: what are the learning aims? what are the possible solving strategies and difficulties of students?

It is interesting to not that some tools were made available in Cabri II Plus in this direction of ready made tasks:

- the .env files saving the configuration of the tools provided to the students
- the buttons hide/show that can be used to bring a help to the students or to provide a validation of the students' constructions.

4. Cabri elem, a technology for building resources offering various types of feedback

The most recent Cabri Elem technology confirms the evolution towards taking into account the voice of the teacher in the feedback provided by the environment. Cabri Elem comprises two environments: an author environment in which the author can create a resource meant for the student in all parts of mathematics and a student environment in which the student works on a resource. For the time being, a collection of resources called "1 2 3… Cabri" in Spanish, French, English or Italian has been designed for primary school and beginning of middle school.

Cabri Elem provides dynamic representations of two kinds of objects: of real objects and of mathematical objects. The behavior of both kinds of objects is based on a mathematical model and thus these objects react to the actions of the student providing feedback. In particular real objects may behave beyond the way they do in reality. They become augmented realities and as such can be used to promote new solving strategies. For example, students must count the number of cubes that two right prisms may contain. As in reality students may put cubes one by one in the prisms but they also can just move a prism in the set of cubes and see at once how many cubes are in a layer of the prism (Fig.6 and Fig. 7).

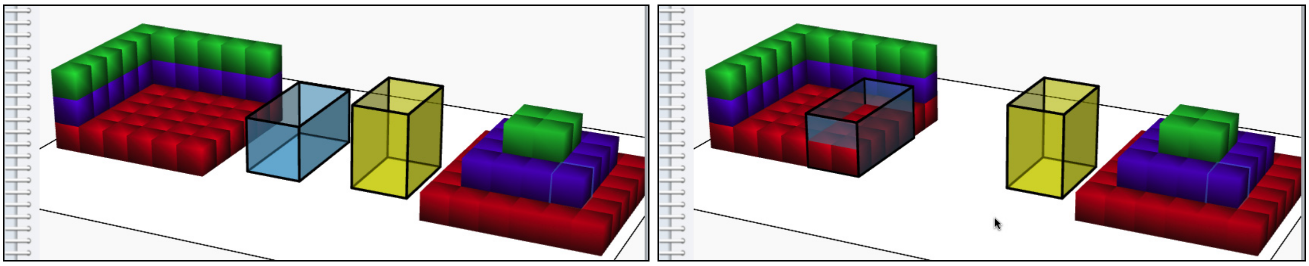

Fig. 6 and 7 – *Counting the number of cubes in a right prism by moving the right prism in the cubes*

Cabri Elem is both a 2D and 3D environment to model the situation of the student working jointly on a 2D notebook and in the 3D environment.

More precisely, a resource is an activity book with several pages that allows the author to create a progression in the questions given to the student in order to foster the change of solving strategies. The book also contains a part meant for the teacher not available for students in which the learning aims, the tasks and the management of the class around the activity book are commented. The student is provided with tools for exploring, building, computing and solving and with feedback of various type:

- feedback linked to the modeled situation embodying mathematics

- feedback depending on the solution of the student, in particular feedback showing to the student the effect of his/her answer (Fig. 8)

- feedback in terms of correct/incorrect.

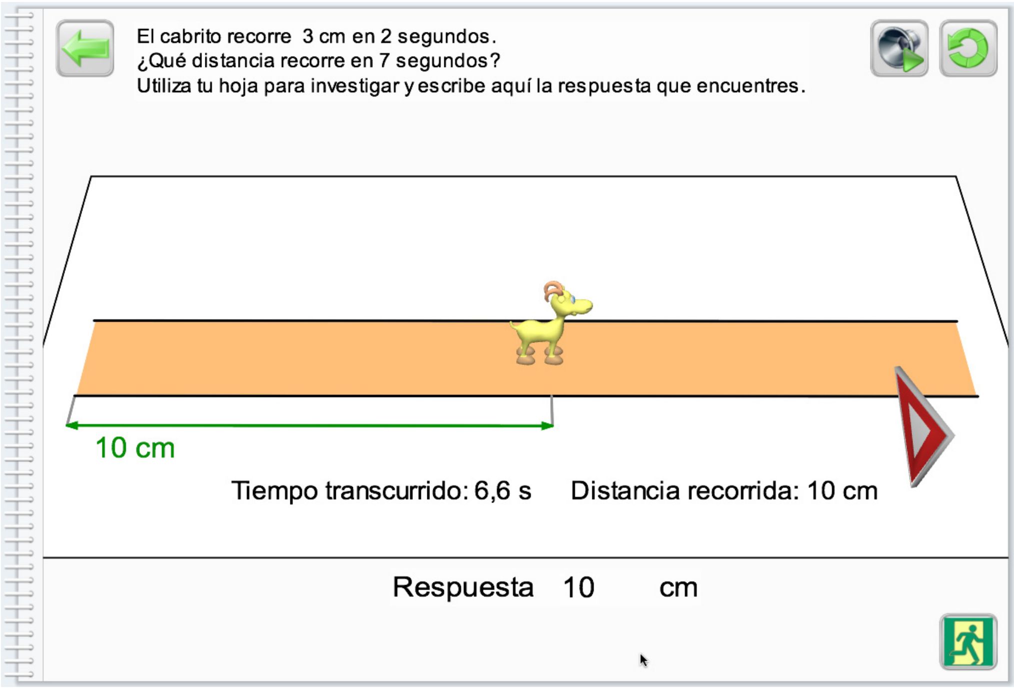

Fig.8 – *The deer is moving on the distance given by the students'answer*

5. Conclusion

In the first years of technology in school, learning was mainly considered as emerging only from the interactions between the student and the machine, rather than between the students and appropriate tasks to be done with the machine. The focus moved onto the teaching environment, and in particular to the role of the teacher. The evolution of the Cabri technology shares similarities with didactical research. Firstly focused on interactions between an open ended environment and students, it took more and more into account the teaching dimension and the role of the teacher and his/her possible feedback to students' solutions.

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