

New Content, Methods, and Approaches Promoted by the Integration of Technology in Secondary Mathematics

ANTONIO R. QUESADA

Department of Mathematics

The University of Akron, Akron, Ohio 44325-4002

aquesada@uakron.edu

Abstract. *In this presentation we review relevant changes that the integration of technology in general and of hand-held graphing technology (HHGT) without CAS, in particular, facilitates at the secondary and basic college mathematics levels. We will show how numerous topics and concepts, traditionally reserved for upper level mathematics courses, are now accessible at lower levels. Likewise, the different representations and data types that HHGT provides, and lately the possibility of recognizing in one platform a variable defined in any other, makes possible the introduction of different approaches to solve a variety of problems, as well as some new approaches inherent to the new technology. Finally, more than ever, technology makes it possible to implement the use of the “explore-discover-test-conjecture-prove” model at every level, providing the possibility of increasing the inquiry-based approach in the way we teach and learn mathematics. This approach may help students become independent learners, in a time when it is becoming increasingly important to do so, since new discoveries and the access to data is increasing exponentially. The implications of these potential changes in content, approaches to problem solving, and methods affect most aspects of the teaching and learning process. Given the importance and scope of the applications now accessible, it is important to ask: does our curriculum responds to these new topics at this level? What criteria can be used to decide what topics should be added and which ones should be removed? Does the preparation of our future mathematics teachers respond to these new realities? Do our current textbooks take into consideration these possible changes?*

Introduction

The continuous improvement of hand-held graphing technologies (HHGT) in the past twenty-five years and the increased accessibility by students to these devices, make it possible to bring about relevant changes to the teaching and learning of mathematics as compared with what has been done traditionally. These changes extend from the amount and depth of the content now accessible at lower levels, to the quantity and relevance of the applications that students can be exposed to, as well as to new approaches used to solve problems, and the new tools available. The availability of different data types and multiple representations have made it possible to expand the study of functions before calculus by adding from the onset numerical and graphical views to the traditional analytical approach. Moreover, these capabilities not only facilitate the introduction of traditionally advanced topics at lower levels, but also allow bridging over cumbersome calculations to focus on the analysis of the results rather than the algorithmic processes needed. Thus, we see at the precalculus level, that the study of families of continuous functions can now include properties and applications traditionally studied in calculus; at the same time that every family of continuous functions being studied is accompanied by real data sets that can be modeled via regression using functions from the family. Similarly, the ability to perform matrix algebra and elementary row operations has made possible the introduction of modern matrix applications in secondary.

We remark that the images included in this paper have been produced with a TI-Nspire calculator. When four images are displayed in one figure, we will refer to them going left to right and top to bottom. In addition to a larger memory and speed, this latest generation of calculators incorporates dynamic geometry and spreadsheet capabilities to the previous generation of HHGT. Moreover, any variable defined in a particular platform used in a problem, is recognized anywhere in that problem.

Goals of this presentation

In this paper we present a brief selection of examples to illustrate in part how the different representations and data types that modern HHGT (without computer algebra system –CAS- capabilities) provide:

- I. Increase the depth and breadth of content included at the secondary and basic college mathematics levels, with relevant concepts traditionally reserved for upper levels, in different areas like calculus, geometry, discrete mathematics etc.
- II. Provide new approaches to solve problems.
- III. To properly study some of these new approaches, it is important to informally address some collateral ideas such as common shapes of the graphs of different families of continuous functions (in particular expected shapes of polynomial graphs), rate of growth within and between families of functions, etcetera.
- IV. Enhance the way we teach and learn mathematics.

I. Changes on content

The integration of HHGT allows the informal introduction of key concepts essential to the understanding of calculus such as:

1. Studying continuous functions as families. Each family with a root or parent function from which the rest of the functions in the family can be obtained via transformations like $f(x)+a$, $f(x+a)$, $-f(x)$, $a \cdot f(x)$, $f(ax)$, $|f(x)|$, and $f|x|$.
2. In addition to the properties traditionally considered for every family of continuous functions, the integration of technology allows the inclusion of:
 - a) finding the range of all functions studied,
 - b) determining irrational zeros, allowing to completely factor polynomials over the reals,
 - c) finding local extrema, with intervals where the function is increasing or decreasing,
 - d) solving optimization problems,
 - e) exploring the local behavior of functions, hence their continuity at a point,
 - f) exploring the end behavior of functions.
3. Accompanying each family of continuous functions studied, with data sets from real life examples that can be modeled using a function from the family. Thus, linear and nonlinear regression are now included in most precalculus books as part of the study of these functions.
4. Collecting data from simple experiments on motion, heat, light eta, using simple devices attached to the HHGT, and modeling this data using appropriate functions.

Dynamic geometry software (DGS), included in the latest generations of HHGT, has totally changed the way Euclidean geometry can be studied. With a minimal introduction, students may dynamically explore and discover relevant properties rather than being told about them. An immediate consequence has been that some important results from modern Euclidean geometry are beginning to be studied by secondary students in the form of

interesting applications. It is encouraging to learn that students at this level are already producing some very interesting new results, (Quesada, 2009). Moreover, DGS is becoming an invaluable tool to introduce concepts such as the slope of a line or the instantaneous rate of change, where invariant or relational graph properties can be easily observed by moving a point on a graph. In fact, the use of DGS contributed positively in an experimental study that produced significant results on enhancing students' understanding of the epsilon-delta definition of the limit of a function, (Quesada, Einsporn, & Wiggins, 2008).

In discrete mathematics, the capability to perform matrix operations that HHGT provides has facilitated that important matrix applications such as networks, cryptography, transformations, traffic flow, Leontiev input-output model etcetera, are finding their way to secondary mathematics. At the same time that being able to perform elementary row operations and to obtain directly the *ref* and the *rref* of a matrix allows for a more in-depth study of linear systems of equations.

II-III On new problems solving approaches and collateral ideas needed

Previous to the introduction of HHGT, continuous functions were initially studied mostly algebraically; we needed to wait until calculus to learn how to graph most continuous functions. Once HHGT makes the graph and the table accessible, students can input a function and attain a graph for it containing all the important features (zeros, local extrema, asymptotes...), that is, obtaining what was coined by Demana and Waits (1998) as the complete graph of the function. However, HHGT presents the graph of the function in a default window or in the last window used; thus, in general, the graph obtained will not be the complete graph. Therefore, often, the user needs to redefine the window settings in order to obtain the complete graph sought. In addition to gathering some information from the algebraic equation of the function, a technique often used in this case consists in tracing the function numerically or the graph initially obtained, even if it does not show in the window, and reading the coordinates displayed to get a sense of the window dimensions needed. But how can a student obtain a complete graph of a function if he does not know beforehand what to expect? For many continuous functions a general idea of how the graph of the parent function looks like suffices. In others, like in polynomial functions, one must supplement the information traditionally provided at this level so that the student has an idea of what to expect. A simple activity on discovering the possible shape of a polynomial of degree n lets the students know that the number of "turning points" is in general $n - 1 - 2k$, $k \in \mathbb{N} \cup \{0\}$. The fundamental theorem of algebra can be intuitively introduced in the same way. Similarly, not knowing the number and type of asymptotes that a rational function has might easily contribute to a student missing a branch of the function. We have found that, once these ideas are informally introduced, students are able to obtain the complete graphs of most functions used before calculus. Once the students can obtain the complete graph of functions, they can easily get most of the important properties, mentioned previously, of the function. But, is it enough to be able to obtain the complete graph of algebraic and transcendental functions in order to solve equations involving them? In many cases the answer is no. Without an intuitive knowledge of the number of solutions one can expect, or being unfamiliar with the rate of growth within and between families of continuous functions, students are bound to easily miss some solutions. In our work with pre- and in-service secondary teachers (Quesada, & Dunlap, 2011) we have found that when we show them the graph of $10x^5 = 2^x$ in the standard window, with dimensions $[-10,10]x[-10,10]$, a large percentage of them ignore the faster growth rate of the exponential function and miss the

hidden root. It is interesting to notice that this audience has completed the calculus sequence and therefore has been exposed to L' Hospital Rule. Fortunately, it is enough to use the table to observe the values that the quotient or the difference of two functions take for increasing values of the independent variable, to decide if one will overtake the other and therefore we should expect additional intersections between their graphs. This simple mechanism allows secondary students to solve algebraic and transcendental equations and inequalities simply by graphing and locating the intersection points, and then finding their coordinates on the screen, or by using a numerical zoom-in. Alternately, they can solve $f(x) = g(x)$ by finding the zeros of the difference function $h(x) = f(x) - g(x)$. The examples that follow illustrate these ideas.

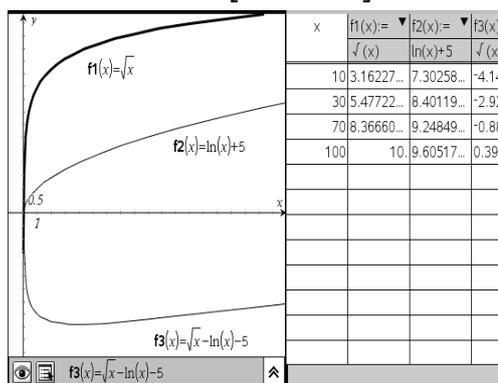
Example 1. Solve graphically and numerically the following:

- a) $\ln x - 5 < \sqrt{x}$ b) Solve $5 - 3x^2 = -0.5x^3 + 3x^2 + x - 3$.

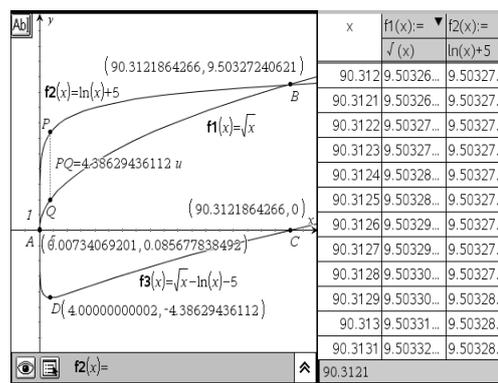
Solution. a) In Figure 1.1 we see the graphs of the given functions together with their difference function in the window $[-4,16]x[-7,7]$. The table in Figure 1.2 confirms that for $x=100$ the graph of the radical has grown larger than the graph of the logarithmic function, and that the difference function becomes positive. This allows adjusting the window, as seen in Table 2.1, so that the hidden intersections, as well as the corresponding zero of the difference function are displayed. Students can observe how the distance between the two functions, given by the length of \overline{PQ} , corresponds to the y -coordinate of the difference function.

The table of Figure 2.2 illustrates one step of the numerical zoom in process to find the solution, showing that the difference function changes signs at $x=90.3121$.

Figure 3.1-3.4 show the graphs of the functions in the second equation, first in the window $[-5,5]x[-10,10]$. Students unfamiliar with the fundamental theorem of algebra and unaware that the cubic grows faster than the quadratic will miss the third intersection point. However, testing a few values in the spreadsheet (Figure 5.3) allows us to easily notice that the difference function changes signs between 10 and fifteen, or that the growing quotient function becomes larger than one in that interval. The new windows in Figure 3.2-3.4 with domain $[-6,16]$ allows us to see the three intersects, or the three zeros of the difference function, using the ranges $[-750,150]$ and $[-13,5]$ respectively.



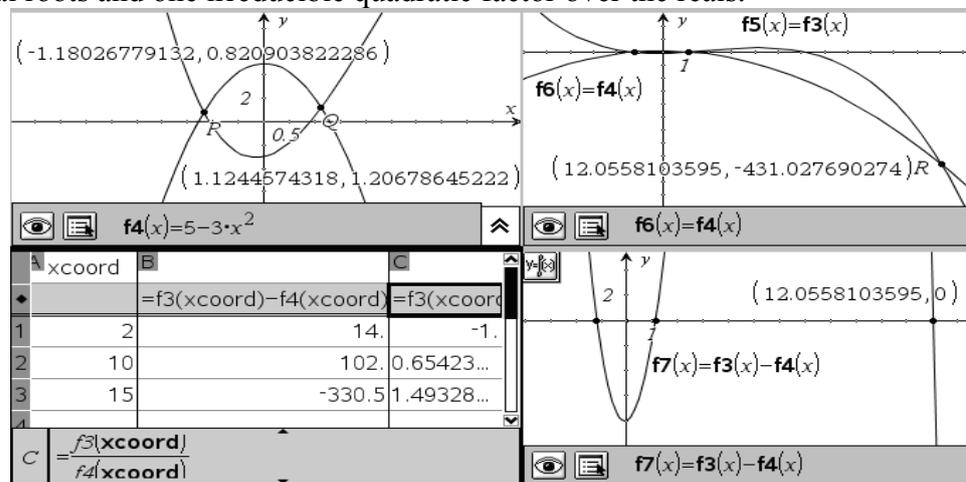
Figures 1.1-1.2. The graph of $y = \ln x - 5$, & $y = \sqrt{x}$ in the initial window $[-4,16]x[-7,7]$.



Figures 2.1-2.2. The graphs of $y = \ln x - 5$, & $y = \sqrt{x}$ in the window $[-10,100]x[-10,14]$.

We remark that in general, students find it easier to locate the zeros of the difference function, since often the appropriate window dimensions are easier to find (Figure 3.4).

In our second example we illustrate how factorization of polynomials can be extended to include irrational roots and one irreducible quadratic factor over the reals.



Figures 3.1-3.4. Graphs of $f(x) = 5 - 3x^2$ & $g(x) = -0.5x^3 + 3x^2 + x - 3$ in the windows $[-5,5]x[-10,10]$, $[-6,16]x[-750,150]$, and of $f(x)-g(x)$ on $[-6,16]x[-13,5]$

Example 2. Factor the polynomial

$$p(x) = 4x^5 + 17x^4 - 19x^3 - 13x^2 + 32x - 15$$

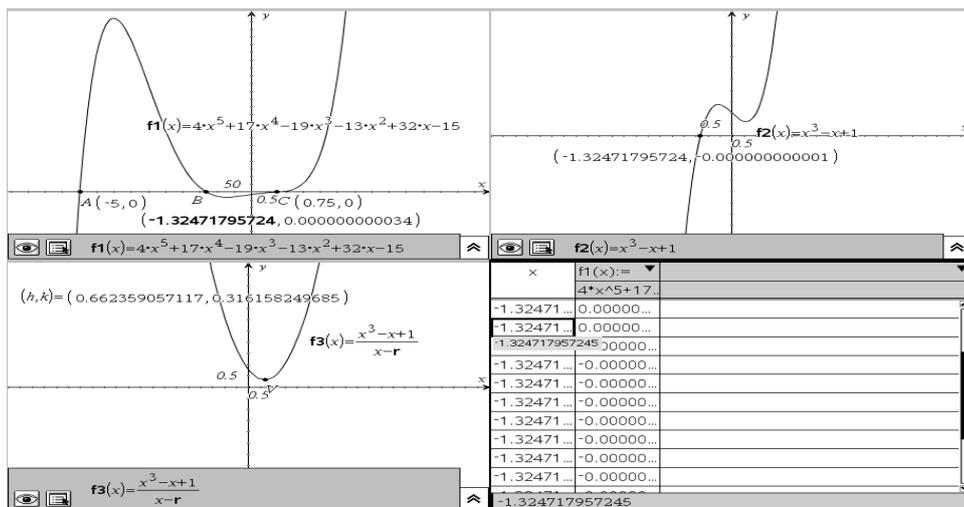
over the fields \mathbb{Q} , \mathbb{R} and \mathbb{C}

Solution. The theorem of possible rational zeros in this example yields the set

$$S = \{\pm 1, \pm 1/3, \pm 1/5, \pm 1/15, \pm 2, \pm 2/3, \pm 2/5, \pm 2/15, \pm 4, \pm 4/3, \pm 4/5, \pm 4/15\}.$$

By simple observation we see that the graph of the polynomial in the interval $[-10, 10]$ shows only three possible rational roots. Calculating the observed zeros on the graph, we confirm that two are rational $\{-5, 3/4\}$ and one is irrational. In Figure 4.1 we present the graph in the window $[-7, 7]x[-370, 1230]$ that displays three real zeros. However, we don't know if we have the complete graph, since there may be other irrational roots beyond our window. The second graph corresponds to the cubic factor $q(x) = p(x)/[(x+5)(x-3/4)]$, and confirms that the graph of $p(x)$ was the complete graph since $q(x) = x^3 - x - 1$ has one real and two imaginary roots. But, can we determine the imaginary roots of $p(x)$?

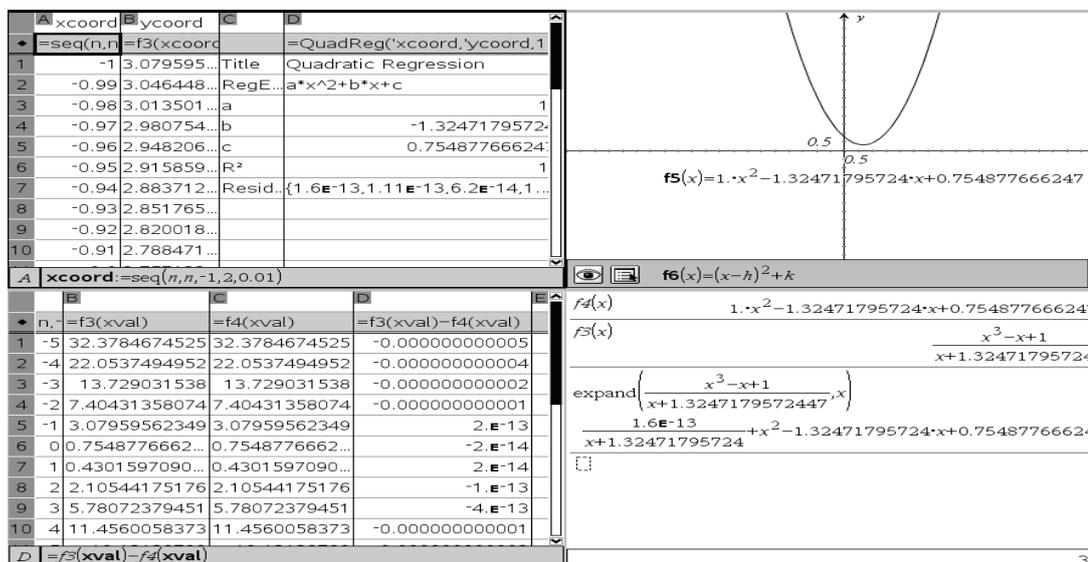
We show next two different ways for determining the quadratic factor. First, to improve the precision, we extended the numerical approximation of the irrational root obtaining $r = -1.324717957245$ in the table shown, and then we got the graph of Figure 4.3 that corresponds to the quadratic factor sought as $q(x)/(x-r)$. Next we determine graphically the vertex of the parabola and store its coordinates in h and k . The solutions sought will be the roots of $(x-h)^2 + k = 0$.



Figures 4.1-4.4. Factoring a polynomial over the reals.

The second approach consists in generating a sequence of points from the graph of $q(x)/(x-r)$ and then fitting a quadratic regression on these points. Figure 5.1 shows that we generated the sequence of 300 x -values using the command $seq(n, n, -1, 2, 0.01)$, and then evaluated $q(x)/(x-r)$ on them. The quadratic regression of the set of 300 points thus obtained has $r^2 = 1$.

The graphs of both quadratic functions obtained are graphically indistinguishable (Figure 5.2). To compare numerically the quadratics resulting from both approaches, we evaluated their difference function on some numerical values (Figure 5.3), and also expanded them algebraically (Figure 5.4). In both cases the differences found are negligible, proving that both methods give excellent results.



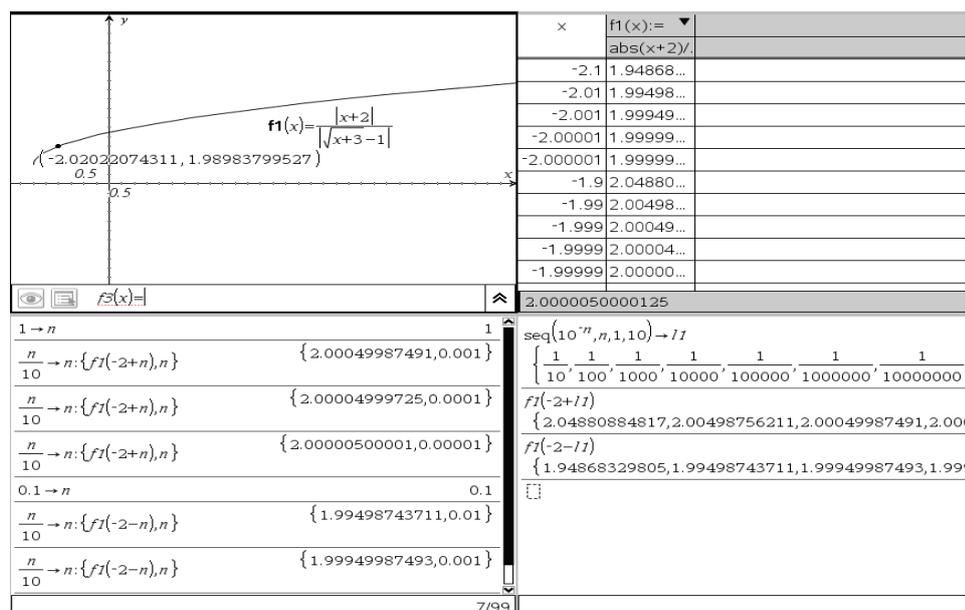
Figures 5.1-5.4. Getting an irreducible quadratic factor.

Our next example illustrates that it is quite accessible for students at the precalculus level to get introduced to the concept of limits, admittedly one of the more difficult concepts of calculus.

Example 3. Analyze the local behavior of the function $f(x) = \frac{|x+2|}{|\sqrt{x+3}-1|}$ at $x = -2$, and the

global behavior of the function $g(x) = \frac{x^3 - x}{x-1}$.

Solution. Through the years many of our mathematics majors have commented that they gained a deeper understanding of the concept of limit when they studied advanced calculus. Initially, I thought that the added mathematical maturity gained through the study of other mathematical courses was a clear justification to these comments. Later, I realized that the approach used in the study of this concept in the different courses may have also had contributed to a better understanding. In Figure 6.1 we did trace on the graph to get an idea of the behavior of the function at the given value. However, it is numerically that we get a better understanding. We study the local behavior of the given function by using the sequences $\{-2 \pm 10^{-n}\}$ that approach arbitrarily -2 from both sides. Thus, in figure 6.2 we have used the table, while in figure 6.3 we have illustrated the use of recursion, and in Figure and 6.4 we generate the sequences in the domain before evaluating the function at them. We have found that precalculus students have no problem understanding the use of the table to obtain the limit of a function at a given value.



Figures 6.1-6.4. Local behavior of $f(x)$ at $x=-2$.

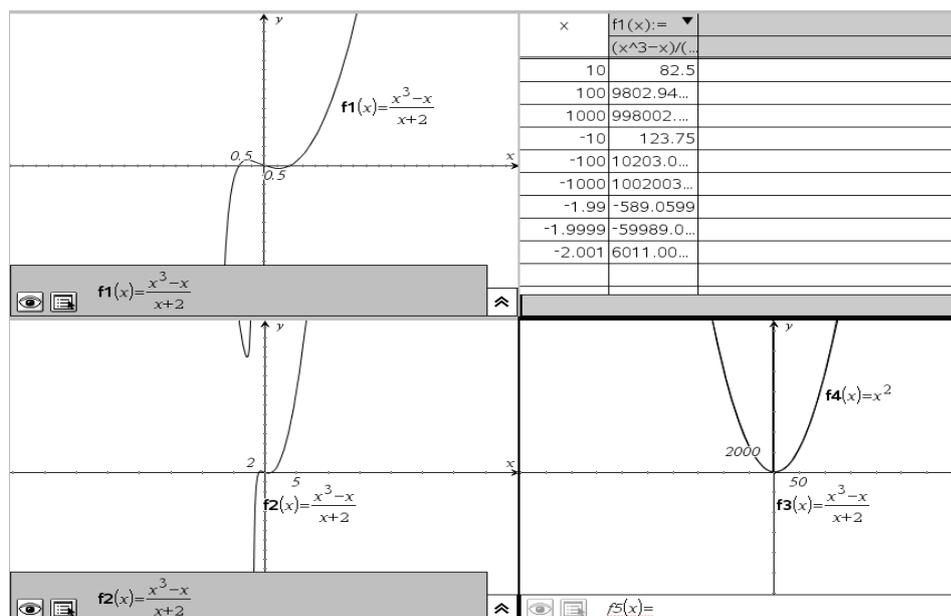


Figure 7.1-7.4. Local versus local behavior of a rational function

In Figure 7.1 we see the incomplete graph of the function $g(x)$ in the window $[-10, 10] \times [-7, 7]$. The table (Figure 7.2) shows the values of the function as x grows unbounded in both directions. Rather than using recursion or lists as in the previous example we find of more interest to present the graph in the windows $[-40, 40] \times [-30, 30]$ and $[-500, 500] \times [-15000, 15000]$. The second graph shows the branch missing from the first; while the last graph is of particular interest to show the students how as we look at the graph from a higher and higher view point, the local behavior vanishes and we see only the global behavior. Our students tend to be impressed when they see that in this last window the given graph cannot be distinguished from the graph of $y = x^2$.

The numerical approach to get the limit of a function using HHGT is so easy to use and understand as compared with all the different algebraic techniques used in calculus, that one wonders why is not an established approach in all calculus books?

Our last example is included to underline the fact that once technology in general and DGS in particular are included in the curriculum, beautiful mathematical results that have been archived for years because of the prior time constraints involved in their study, suddenly become accessible. Case in point is some nineteenth century Euclidean geometry results that have begin to appear as part of some discovery activities using DGS in Euclidean geometry.

Figure 8.1 displays the nine point circle of a triangle, the incircle that is contained and tangent to it, and the Euler line that connects key centers of the triangle. Figure 8.2 displays the Simson's line of a triangle for a point P on its circumcenter, and the deltoid that is the locus of that line when P moves on the circle. The fact that secondary students have access to these results via guided discovery activities is representative of the impressive possibilities that technology offers to empower our students at every level.

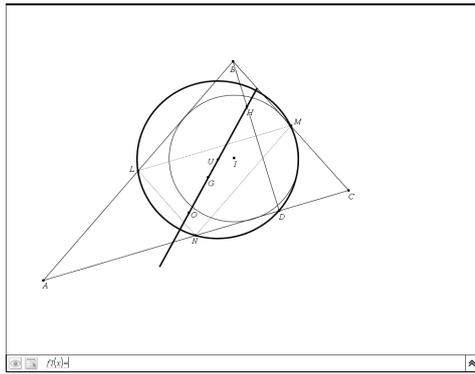


Figure 8.1 The nine point circle

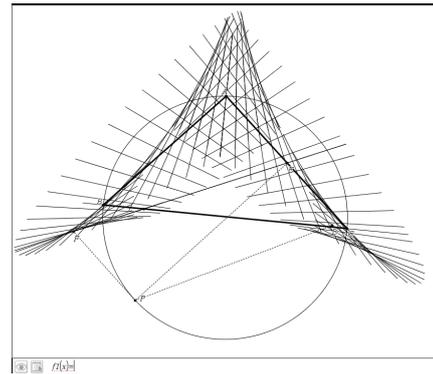


Figure 8.2 The locus of Simson's line

In the selection of examples that we have presented, we omitted one of the best known, due to limitations on the size of this paper. It addresses solving an optimization problem using multiple representations. One particularly new approach consists of creating a geometrical model using DGS in which the independent and dependent variables can take different values by moving a point of the model. Hundreds of pairs of values are then collected and a best fit model is then obtained using regression. The optimal value sought can then be read directly from this model. The interested reader is encouraged to examine (Quesada & Edwards, 2005).

IV. Enhancing the way we teach and learn mathematics.

Some of the best known contributions in the enhancement of the didactics of mathematics by HHGT are:

1. The simultaneous introduction of the algebraic, graphical, and numerical aspects of functions makes it possible for students to
 - i. support graphically and/or numerically analytical solutions of different problems,
 - ii. support analytically graphical and/or numerical solutions, whenever possible (Demana & Waits, 1998). The ability to double-check the answer(s) to problems is particularly valued by students at every level (Quesada, 1994a, 1994b, 2003).
2. The introduction of the numerical representation, makes it possible to study some properties of functions in the way they were originally developed and are better understood, i.e., via approximations.
3. The graphical representation allows students to visualize from the beginning the graphs of functions and their properties. Therefore, students can associate the goals of algebraic procedures with their graphical interpretation. Thus solving the equation $f(x)=a$ consists graphically in finding the points of intersection of the graph of $f(x)$ and the horizontal line $y=a$.
4. Technology facilitates the active involvement of the student in the learning process. The model “explore-discover-conjecture-test-proof,” traditionally used in research, should become an integral part of every mathematics classroom. In general, machines are much faster and precise than humans, therefore, the extent to which algorithms may or may not be taught, should be determined by learning theories rather than tradition.
5. Technology enables us to easily revisit problems from different perspectives based upon the depth of the students' mathematical knowledge.

6. The toolbox we provide the students can now include the use of nontraditional tools such as lists, sequences, regression, and recursion to solve different problems.

We include next some examples on how DGS graphs or applets obtained from them, can be used to present in class or to be discovered by the students, some fundamental concepts.

Figure 8.1 displays the epsilon-delta definition. The value of ε can be changed directly or dynamically by grabbing the point labeled $L + \varepsilon$ and moving it up and down. Students can see that the interval (B, C) and the distances AB and AC change accordingly, and that the smaller of the two is δ . In the experiment afore mentioned (Quesada, et al., 2008), after working with this type of pictures, students answered basic conceptual questions significantly better, and were capable of determining algebraically and graphically the smallest δ associated with a given ε , by obtaining the x -coordinates of R and Q .

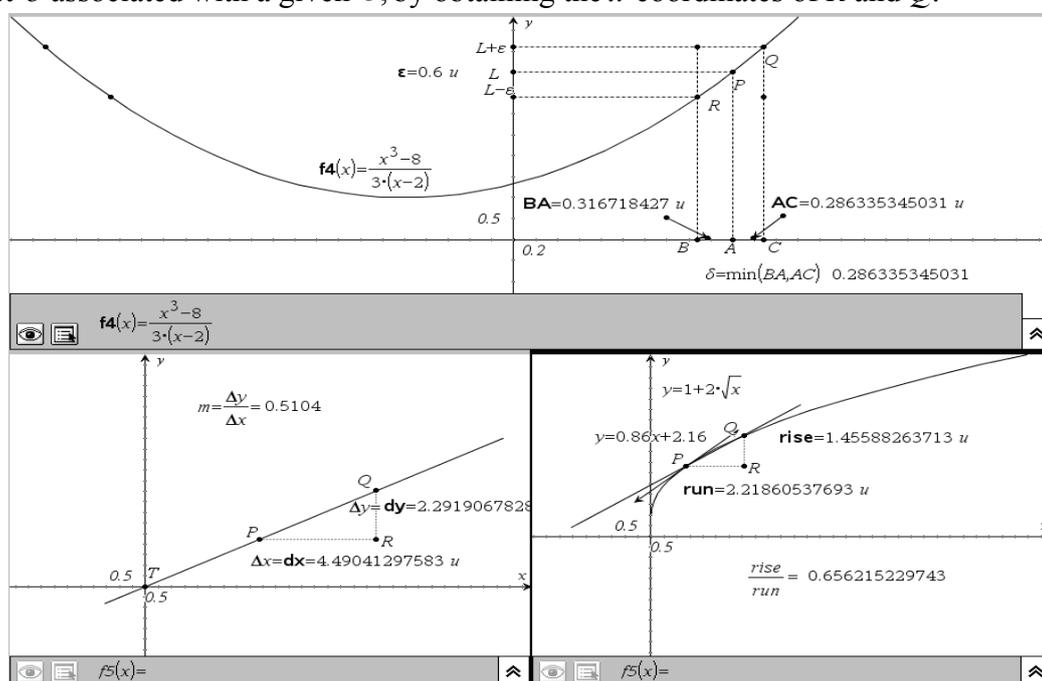


Figure 8.1-8.3. Using DGS to illustrate dynamically the epsilon-delta definition, the slope of a line, and the instantaneous rate of change.

Similarly, by moving the point Q up and down the line in Figure 8.2, students can observe that the rise and run change but their quotient remains constant, which leads to the definition of slope. Likewise, one can introduce the instantaneous rate of change of a function $f(x)$ by moving point Q toward point P on the graph of the function in Figure 8.3. It can be observed that the secant PQ approaches arbitrarily the tangent at P , and therefore the slope of the secant approaches the slope of the tangent.

We conclude with some remarks on concerns and open questions associated with the integration of HHGT.

One cannot ignore the fact that with every new technology a new set of misconceptions and student errors appear. It is important that teachers become aware of common student errors and misconceptions associated with the graphing technology. Some well known sources of difficulties are:

- a) The syntax needed to input a function. A good aphorism to share with the students is: “when in doubt use parenthesis.”
- b) Careless input of data. A bad input yields bad output even if the student knows what he or she is doing. Hence one should always double check the input.
- c) Because of the way computers draw graphs (by connecting pixels), discontinuities are rarely displayed, as a result students tend to think that “every function is continuous.” In many calculators, particularly the older ones, when graphing a rational function the pixels immediately before and after an asymptote are connected, leading the student to wrongly think that an asymptote is actually drawn.
- d) It is not uncommon that some students exceed the precision of the machine obtaining the corresponding truncation errors.
- e) As seen in the examples, the use of inadequate window dimensions will lead to erroneous conclusions.

The selection of examples presented gives an idea of the potential impact that HHGT may have on the content, methods, and assessment we use. Considering the relevance of the applications now accessible and the fact that the secondary curriculum is normally crowded we may ask:

- a) What criteria can be used to decide which topics should be added and which ones should be removed?
- b) Does the content of our secondary curriculum include these new topics?
- c) Are we teaching topics obsolete in light of the new technology?
- d) Does the preparation of our future mathematics teachers respond to these new realities?
- e) Do our current textbooks take into consideration these possible changes?
- f) Do the corresponding mathematics courses taught in our universities respond to these new realities?

With respect to the first question, we agree with Manoucherhri (1999) when she says “... the relevance of a topic for a particular grade level is no longer determined based upon its historical value in school mathematics. Relevance is, however, determined based upon the extent to which the topic empowers students to understand related concepts in greater depth and detail.”

We believe that any new topic should be assessed based on the relevance of the concepts that it addresses and/or facilitates as evidenced by i) its usefulness, as defined by the variety and importance of applications that the topic can be used for, and ii) its accessibility, with an appropriate degree of complexity of the concepts and calculations needed for the proposed level (Quesada, 1999).

The answers to the remaining questions will differ in different countries. Certainly, there have been some drastic changes in the last twenty years; however, initial research addressing some of these questions (Quesada & Renker, 2008b), (Quesada & Dunlap, 2008), points to plenty of room for improvement. Let us keep in mind that we have not mentioned the impact that CAS, already available in HHGT, is already having, and that the pace of technology is not going to slow down.

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