Dragging quadrilaterals into tetrahedra

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Abstract: *In this paper we present the results of an experimental sequence of classroom activities in Euclidean geometry, both plane and space geometry, proposed to high school students. During the activity students discover surprising analogies between quadrilaterals and tetrahedra, by means of analogy and of the dynamic geometry software they are working in.*

1. Introduction

Euclidean geometry allows us to interpret the space we live in and provides instruments often used in several jobs: geodetic lines for sailing and air navigation, the perception of movement of a solid by engineers, geometric vision in micro-surgery, etc. The cultural formation/education of today's citizen is weak without geometry, in particular without space geometry. Space geometry is indeed part of school programs but, in the classroom, it is often relegated to the end of the year and therefore covered superficially at best, if not left out completely. This is because, even though we live in a three dimensional space, teaching/learning three dimensional geometry presents difficulties in graphic representation as well as in mental visualization: it is not easy to draw a three dimensional figure on a plane and it is not easy to imagine the mutual position of objects in space.

In this paper we present the results we found after proposing a sequence of classroom activities in both 2D and 3D Euclidean geometry to high school students. The educational rationale of the proposed sequence of activities is to have students experience 3D geometry in order to enhance their sense of self-efficacy, to help them reach an accurate vision of the discipline, and experience positive emotional stimulation [9]. Fostering a positive attitude in the students towards space geometry is the goal of the project.

The activity has been realized by means of a conceptual tool, analogy, and of an operative one, a particular dynamic geometry software. The chosen topics are quadrilaterals and tetrahedra that share a lot of analogies [8]: we start with quadrilaterals, already familiar to the students, and study tetrahedra using the numerous analogies with quadrilaterals. Thus, space geometry is less hard because students are faced with 3D problems after having already become familiar with the solution of an analogous problem in the plane [12]. Analogy, in fact, creates a significant bridge between the plane and space and makes properties of tetrahedra easier to perceive.

In the sequence of activities we proposed we use the conceptual instrument of analogy, and also an operative instrument, a dynamic geometry software, that is fundamental in the activity of exploring and discovery. We use both Cabri II Plus and Cabri 3D: the former allows to study 2D figures, the latter 3D figures, through a dynamic vision achieved from several points of view, and they help to overcome the well known problems of spatial visualization/representation.

The sequence of activities was carried out in computer laboratories, during which students, by constructing and observing, discover the surprising existing analogies between quadrilaterals and tetrahedra. Geometry teaching in the laboratory is not only reduced to the formal learning of a proof, but it takes into consideration the richness of the geometric reasoning, which is divided into various stages: 1) observation of a figure; 2) development of conjectures; 3) critical exam and final valuation of conjectures with proofs.

The atmosphere of a laboratory is similar to the one of a “Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practising. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together […] to the communication and sharing of knowledge in the classroom, either working in small groups in a collaborative and cooperative way, or by using the methodological instrument of the mathematic discussion, conveniently guided by the teacher” [11]. The teacher, by trials and errors, leads students to solve the problem. In the laboratory then, students, young math researchers, enjoy the taste of the discovery and the joy of reaching the result, improving their self-efficacy [1].

2. The proposed sequence of activities: content and worksheets

The proposed sequence of activities, based on [8], was elaborated by a team of university professors and one high school teacher and is on a topic that is not in school curricula. It is addressed to talented students that already know some plane geometry and, possibly, some space geometry. A simplified version of this sequence of activities, addressed to all high school students, has been elaborated in [5].

In this activity, tetrahedra, instead of being assimilated to triangles --being polygons of the plane respectively polyhedra of the space with the least number of vertices-- are considered in analogy with quadrilaterals [8]. We define quadrilaterals as figures of the plane determined by four vertices, such that any three of them are non-collinear, and with six edges (the four sides and the two diagonals) and four faces (the triangles determined by three vertices of the quadrilateral), and tetrahedra as 3D figures determined by four non coplanar vertices, and with six edges and four faces. Then, a quadrilateral becomes a tetrahedron by “extracting”, from the plane, one of its vertices (Figure 2.1).

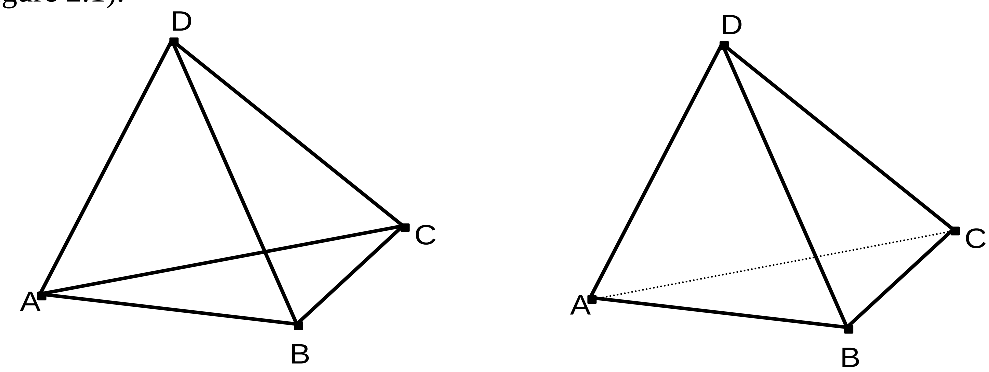


Figure 2.1 Quadrilateral - Tetrahedron

In both figures we introduce analogous definitions (bimedian, centroid, median, axis/axial plane, circumcentre, maltitude/Monge plane, anticentre/Monge point) and prove analogous properties. See [2, 3, 4, 6, 7, 8, 10]. The following table contains the content of the proposed sequence of activities.

Table 2.1 Table of contents

QUADRILATERALS	TETRAHEDRA
Q is a ***convex quadrilateral*** with ***vertices*** *A, B, C, D*. The points A, B, C, D are such that any three	***T*** is a ***tetrahedron*** with ***vertices*** *A, B, C, D*. The points A, B, C, D are non coplanar.

of them are non-collinear. The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called ***edges***. The edges of ***Q*** are the four sides and the two diagonals. Two ***edges*** are said to be ***opposite*** if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD, that is, either two opposite sides or the two diagonals. We call ***faces*** of ***Q*** the triangles determined by three vertices of ***Q***. There are four faces: ABC, BCD, CDA, DAB. A ***vertex*** and a ***face*** are said to be ***opposite*** if the vertex does not belong to the face. For each vertex there is one and only one opposite face.	 The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called ***edges***. Two ***edges*** are said to be ***opposite*** if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD. We call ***faces*** of ***T*** the triangles determined by three vertices of ***T***. There are four faces: ABC, BCD, CDA, DAB. A ***vertex*** and a ***face*** are said to be ***opposite*** if the vertex does not belong to the face. For each vertex there is one and only one opposite face.
The segment joining the midpoints of two opposite edges of ***Q*** is called ***bimedian*** of ***Q***. ***Q*** has three bimedians, two relative to a pair of opposite sides and one relative to the diagonals. ***Theorem 1. The three bimedians of a quadrilateral all pass through one point.*** The point G common to the three bimedians of ***Q*** is called the *centroid* of ***Q***. ***Theorem 2. The centroid bisects each bimedian.***	The segment joining the midpoints of two opposite edges of ***T*** is called ***bimedian*** of ***T***. ***T*** has three bimedians. ***Theorem 1. The three bimedians of a tetrahedron all pass through one point.*** The point G common to the three bimedians of ***T*** is called the *centroid* of ***T***. ***Theorem 2. The centroid bisects each bimedian.***
The segment joining a vertex of ***Q*** with the centroid of the opposite face is called ***median*** of ***Q***. ***Q*** has four medians. ***Theorem 3. The four medians of a quadrilateral meet in its centroid.*** ***Theorem 4. The centroid of a quadrilateral divides each median in the ratio 1:3, the longer segment being on the side of the vertex of Q.*** ***Theorem 5. The quadrilateral of the***	The segment joining a vertex of ***T*** with the centroid of the opposite face is called ***median*** of ***T***. ***T*** has four medians. ***Theorem 3. The four medians of a tetrahedron meet in its centroid (Commandino's Theorem).*** ***Theorem 4. The centroid of a tetrahedron divides each median in the ratio 1:3, the longer segment being on the side of the vertex of T.*** ***Theorem 5. The tetrahedron of the centroids***

centroids of the faces of a quadrilateral Q is the image of Q with dilatation ratio –1/3 and center the centroid of Q.	***of the faces of a tetrahedron T is the image of T with dilatation ratio –1/3 and center the centroid of T.***
The line that is perpendicular to an edge of ***Q*** in its midpoint is called ***axis*** of the edge. ***Q*** has six axes. ***Theorem 6. The axes of the edges of a quadrilateral meet in a point if and only id the quadrilateral is cyclic.*** The common point to the axes of a cyclic quadrilateral ***Q***, i.e. the centre of the circle circumscribed to ***Q***, is called ***circumcenter*** of ***Q***.	The perpendicular plane to an edge of ***T*** in its midpoint is called ***axial plane*** of the edge. ***T*** has six axial planes. ***Theorem 6. The axial planes of the edges of a tetrahedron meet in a point.*** The common point to the axial planes of a tetrahedron ***T***, i.e. the centre of the sphere circumscribed to ***T***, is called ***circumcenter*** of ***T***.
The line that is perpendicular to an edge of a quadrilateral ***Q*** and passes through the midpoint of the opposite edge is called ***maltitude*** of ***Q***. ***Q*** has six maltitudes. ***Theorem 7. The maltitudes of a cyclic quadrilateral are concurrent.*** The common point to the six maltitudes of a cyclic quadrilateral ***Q*** is called ***anticenter*** of ***Q***. ***Theorem 8. In a cyclic quadrilateral the anticenter is symmetric to the circumcenter with respect to the centroid.*** ***Theorem 9. In a cyclic quadrilateral the anticenter, the circumcenter and the centroid are collinear.*** The line containing the anticenter, the circumcenter and the centroid of a cyclic quadrilateral ***Q*** is called ***Euler line*** of ***Q***.	The plane that is perpendicular to an edge of a tetrahedron ***T*** and passes through the midpoint of the opposite edge is called ***Monge plane*** of ***T***. ***T*** has six Monge planes. ***Theorem 7. The Monge planes of a tetrahedron are concurrent. (Monge Theorem).*** The common point to the six Monge planes of a tetrahedron ***T*** is called ***Monge point*** of ***T***. ***Theorem 8. In a tetrahedron the Monge point is symmetric to the circumcenter with respect to the centroid.*** ***Theorem 9. In a tetrahedron the Monge point, the circumcenter and the centroid are collinear.*** The line containing the Monge point, the circumcenter and the centroid of a tetrahedron ***T*** is called ***Euler line*** of ***T***.

The sequence of activity is organized in worksheets, five on 2D geometry and five on 3D geometry. The teaching/learning strategy that we used in the worksheets follows the scheme:

Explore and verify, using Cabri – Conjecture – Prove

e.g., by observing and exploring a figure, perceive the relations between objects; then, by manipulating the figure, experimentally verify the hypothesis; once they are confirmed, formulate a conjecture and finally prove it. The activity is proposed in the plane first and then, with the use of the existing analogy, in the space.
The worksheets have been organized so as to offer an immediate correlation between quadrilaterals

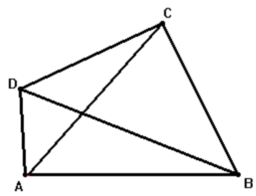
and tetrahedra: for any Worksheet Q, related to a property of quadrilaterals, there is a Worksheet T, related to the corresponding property of tetrahedra. In fact, there is one pair of worksheets per each row of the "Table of contents".
The activity is strictly led, both in the discovery phase and the proving phase. We have chosen to do so in order to get the same results in the plane as well as in the space (so as to point out the analogy), and to get the job done in a set amount of time.
The first worksheet introduces the objects and the procedures that will be used next. In particular, in Worksheet 1T, it is showed how to "transform" a quadrilateral into a tetrahedron. This operation has then been repeated in several other pairs of the following worksheets.

Worksheet 1Q

Quadrilaterals: Old figures and new definitions

1. Open *Cabri II Plus*.
2. By using the instrument ***Polygon*** draw a **convex quadrilateral** of **vertices** A, B, C, D, that we call ***Q***.
3. Draw its diagonals with the instrument ***Segment***.

Definition. We call **edges** of a quadrilateral the six segments that join two vertices.
The edges of ***Q*** are: ……………………………….
Then ***Q*** has 6 edges: the four sides and the two diagonals.



4. Save the figure in a file and call it *Quadrilateral*.

Definition. Two ***edges*** are said to be ***opposite*** if they do not have common vertices.
Opposite edges of ***Q*** are: …………………………………………...
Then ***Q*** has 3 pairs of opposite edges: the 2 pairs of opposite sides and the pair of the diagonals.
Definition. We call ***faces*** of ***Q*** the triangles determined by three vertices of ***Q***.
Then ***Q*** has ….... faces: …………………………………………….
Definition. A ***vertex*** and a ***face*** are said to be ***opposite*** if the vertex does not belong to the face.
For example, the vertex A and the face BCD are opposite.
The face opposite to the vertex B is ……………………
The vertex opposite to the face ABC is …………………

Worksheet 1T

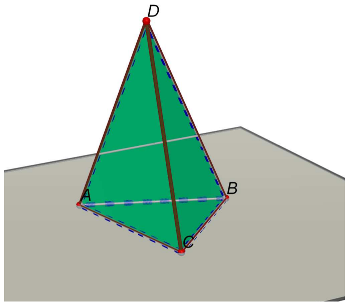
Tetrahedra

1. Open *Cabri 3D* and, with F1, visualize the *Tool Help* window, if not open.
2. Select the origin point of the coordinate system and delete it.
3. In the base plane construct a convex quadrilateral as follows:
   - With the instrument ***Point*** draw the 4 vertices such that any three of them are non-collinear and call them A, B, C, D;
   - with the instrument ***Segment*** draw the 6 edges;
   - with the instrument ***Triangle*** draw the 4 faces.
4. Save the figure in a file and call it *Quadrilateral in the space*.
5. With the instrument ***Redefinition*** of the menu ***Manipulation*** "extract" the vertex D from the plane as follows: click on D and release it, keep pressing ⇧ (*Upper-case*) on the keyboard and move the mouse up without clicking. From now on the four points A, B, C, D are not coplanar

anymore.

6. With the instrument ***Manipulation*** rotate the figure, by pressing the right click of the mouse and moving the mouse itself (as suggested in the *Tool Help* window).

The figure you have obtained is a **Tetrahedron**, that we call ***T*** and the 4 vertices, the 6 edges and the 4 faces of the quadrilateral are now the **vertices**, the **edges** and the **faces** of ***T***.



Attention. If the figure, when you rotate it, does not look nice, you can move the starting vertices; in particular, if the tetrahedron appear "way too squeezed on the plane" we suggest you to move the point D such that his projection on the plane is close to or inside the face ABC (the projection of D is indicated when you move it). Finally, in order to have a better view of the figure you can change the **colour** or the **curve radius** (right click of the mouse) of the segments and of the faces, and, eventually, even the style (***Edit/Preferences/...***).

7. Save this figure in a file and call it *Tetrahedron*.

Definition. Two ***edges*** of a tetrahedron are said to be ***opposite*** if they do not have common vertices.

Opposite edges of ***T*** are: …………………………………………...

Definition. A **v*ertex*** and a ***face*** of a tetrahedron are said to be ***opposite*** if the vertex does not belong to the face.

For example the vertex A and the face ……… are opposite, the vertex opposite to the face ABC is … .

Suggestion: rotate the figure so to see the "hidden" objects.

Consider two opposite edges (AB and CD for example). You have already seen that the vertex D does not belong to the plane of A, B, C. Then you can conclude that two opposite edges of a tetrahedron belong to two lines that are:

- ☐ intersecting
- ☐ parallel
- ☐ skew

Here it follows, for example, Worksheet 2 (that corresponds to the second row of the table of contents). In the assignment, after introducing the concept of bimedian of a quadrilateral and of a tetrahedron, students study a property of their concurrency. Note that each one of the two worksheets consists of two parts: the first part (Part I) gives tips to students on how to discover and form conjectures on properties, whereas the second part (Part II) contains the statement of the theorem and a guided route to prove it. The same scheme is repeated in the other worksheets.

Worksheet 2Q – Part I

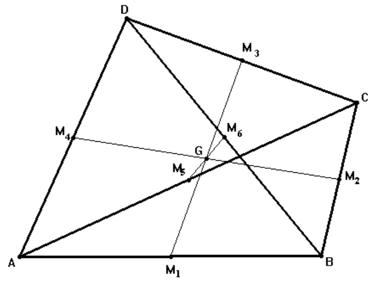
The bimedian of a quadrilateral

Definition. We call **bimedian** of a quadrilateral ***Q*** the segment joining the midpoints of two opposite edges of ***Q***.

1. Open, with *Cabri II*, the file saved with the name *Quadrilateral*.
2. With the instrument ***Label*** call M_1, M_2, M_3, M_4, M_5, M_6 the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively.

There are three bimedians, two relative to pairs of opposite sides and one relative to the diagonals, i.e.: ……………………….

Observation. The points M_1 and M_3 are distinct because they belong to opposite edges of the quadrilateral. For the same reason M_2 and M_4 are distinct. Instead, the points M_5 and M_6 are not always distinct. In fact, since M_5 and M_6 are midpoints of the diagonals of the quadrilateral, they coincide if and only if the diagonals bisect each other, i.e. **if and only if** the quadrilateral is a ………………………



3. With the instrument ***Segment*** draw the bimedians M_1M_3 and M_2M_4. With the instrument ***Intersection Point(s)*** draw their meeting point and call it G.
4. Draw the bimedian M_5M_6 and with the instrument ***Member?*** verify if G belongs to it. Using the mouse drag some of the vertices of the quadrilateral. Does the property still holds?

   ☐ YES ☐ NO
5. Draw the two segments in which any of the three bimedians is divided by the point G and find their measures with the instrument ***Distance or Length***. What do you observe? Using the mouse drag some of the vertices of the quadrilateral. Does the property still holds?

   ☐ YES ☐ NO

Considering what you have discovered about the three bimedians and the point G, you can state the following

Conjecture:

……………………………………………………………………………………………..

Worksheet 2Q – Part II

<u>The bimedians of a quadrilateral</u>

In the previous worksheet you have discovered the following property that we will now prove:

Theorem 1Q

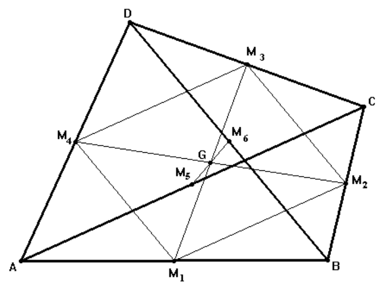
The three bimedians of a quadrilateral all pass through one point that bisects each bimedian.

Proof. Consider a convex quadrilateral ***Q*** with vertices A, B, C, D. Let M_1, M_2, M_3, M_4, M_5, M_6 be the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively.

The bimedians M_1M_3 and M_2M_4 of ***Q*** meet in a point G. Consider the quadrilateral $M_1M_2M_3M_4$.

Prove that the segments M_1M_2 and M_3M_4 are parallel to each other.

(Hint: consider the triangles ABC and ADC and prove that the segments M_1M_2 and M_3M_4 are parallel to AC).



…………………………………………………………………

………………………………………………………………………………

- Analogously you can prove that the segments M_2M_3 and M_1M_4 are both parallel to BD and therefore they are parallel to each other.

  Then the quadrilateral $M_1M_2M_3M_4$ is a ………………… because ………………………… and the point G, being the common point to its diagonals, divides them in ………… parts.
- Consider now the third bimedian M_5M_6 and the bimedian M_2M_4. Prove that they meet in their midpoint.

  ……………………………………………………………………………………..

…………………………………………………………………………………………

- But the midpoint of the bimedian M_2M_4 is …, then … is the midpoint of M_5M_6 as well.

Thus we can conclude that the three bimedians of ***Q*** all pass through G that bisect them.

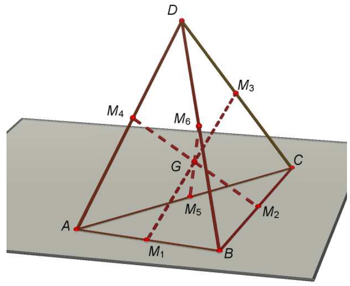
Definition. The common point G to the three bimedians of a quadrilateral is called **centroid** of the quadrilateral.

Worksheet 2T – Part I

The bimedians of a tetrahedron

Definition. We call **bimedian** of a tetrahedron the segment joining the midpoints of two opposite edges of the tetrahedron.

1. Open, with *Cabri 3D*, the file saved with the name *Quadrilateral in the space*. In analogy with what we have done in Worksheet 2Q (Part I), draw the midpoints M_1, M_2, M_3, M_4, M_5, M_6 of the edges AB, BC, CD, DA, AC, BD, respectively. Draw the three bimedians of the quadrilateral and dash them by choosing, with the right click of the mouse, *curve style/Dash-line style*. With the instrument ***Intersection Point(s)*** draw their meeting point and call it G.



2. Extract the vertex D from the plane, by using the instrument ***Redefinition*** (click on D and release, keep on pressing ⇧ (Upper Case) and move the mouse up without clicking). Remember that you can move the vertices if the tetrahedron appear "way too squeezed on the plane". In order to have a better view of the figure you can make transparent the faces of the tetrahedron (***Manipulation***, select the face, right click, *Surface Style/Empty*).

The three segments M_1M_3, M_2M_4 and M_5M_6 are the **bimedians** of the tetrahedron.

3. By rotating the figure you can observe that the bimedians keep meeting in G. With the mouse drag some of the vertices of the tetrahedron. Does the property still hold?

☐ YES ☐ NO

4. Find the distance of G from the endpoints of any bimedian by using the instrument ***Distance***. What do you observe?

....

5. Drag with the mouse some vertex. Does the property still hold?

☐ YES ☐ NO

Considering what you have discovered about the three bimedians and the point G, you can state the following

Conjecture:

…………………………………………………………………………………………
……………………………………………………………………………………….....

Worksheet 2T – Part II

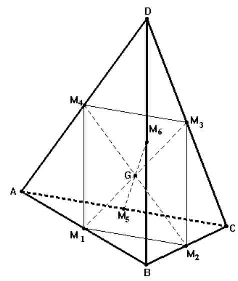
The bimedians of a tetrahedron

In the previous worksheet you have discovered the following property that we will now prove:

Theorem 1T

The three bimedians of a tetrahedron all pass through one point that bisects each bimedian.

Proof. Consider a tetrahedron with vertices A, B, C, D. Let M_1, M_2, M_3, M_4, M_5, M_6 be the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively.



- Prove that M_1M_2 and M_3M_4 are parallel to each other.

………………………………………………………………………
………………………………………………………………………

- Prove that the four points M_1, M_2, M_3, M_4 lie on the same plane.

………………………………………………………………………
………………………………………………………………………
……………………………………………………………………

- Moreover the segments M_2M_3 and M_1M_4 are both parallel to BD. Therefore the quadrilateral $M_1M_2M_3M_4$ is a …………………………, because ……………………………………….., and the point G, common point of the bimedians M_1M_3 and M_2M_4, is the midpoint of both of them because ………………….. .
- In the same way you can prove that the bimedians M_1M_3 and M_5M_6 meet in their midpoint. But the midpoint of M_1M_3 is G, then G is the midpoint of M_5M_6 as well.

Thus we can conclude that the three bimedians all pass through G that bisects them.

Definition. The common point G of the three bimedians of a tetrahedron is called **centroid** of the tetrahedron.

Similar worksheets guide the students along the entire activity. Nevertheless, from the third worksheet the proofs of the properties of tetrahedra are not as guided as the analogous properties of quadrilaterals: this is because the proof is often very similar to the one students already did in the plane.

At the end of each pair of worksheets students fill in a table, called Table of the Analogies. At the end of the entire activity, it will look like the Table of contents. This table enables students to have an overview of what they have done so far (you can e-mail the authors for a copy of the worksheets)

3. Modalities of the experimentation

The experimentation of the proposed sequence of activity was done at the Department of Mathematics and Computer Science of the University of Catania, between March and May 2010, and it was led by the Professors that wrote the worksheets. Twenty-three high school students got involved, 7 of them were attending the second year of their studies, whereas 7 of them were from the third year along with 9 coming from the fourth year; students of the last year were not invited to participate because busy in preparing finals. The students belonged to 6 different schools located within the Province of Catania: 5 Scientific Lyceums and 1 Classic Lyceum. The schools themselves selected their own best students; the selected students decided, on a voluntary base, whether or not enrolling for the course. The class met 6 times, once a week, for 2.5 hours in a computer laboratory, and each student had a computer to work with.

At the first meeting an assessment of the geometry knowledge of the students was done, and the output was that none of them had ever done 3D geometry in high school. Then students were provided with the basic concepts of space geometry, and after that the activity started with the hand out of the first worksheet. Each meeting focused on carrying out two worksheets (Q and T) and the "table of analogies" according to the relative worksheets.

At the end of each “lesson” the students' assignments were collected and copies were given back to them the following week. At the beginning of each session, except the first one, besides reviewing

what they had previously studied during the last encounter, they could also check what kind of faults and errors the teachers had conveniently corrected.
At the end of the course each student took an oral final exam. Two students missed out two classes and therefore were not allowed to take finals, whereas two of them did not want to take finals although they did not miss out any lessons. All the other students performed well at the oral exam with the professors in charge.

4. Results of the experimentation

We present here an analysis of the students' assignments: we distinguish the worksheets on the exploration of the figure till the conjecture (Worksheets - Part I) from those on the proof of the property (Worksheets - Part II).

4.1 Results on Worksheets Part I: Exploration and formulation of the conjectures

All the students, with the guided exploration, are able to perceive the properties they were looking for. Difficulties arise, instead, when formulating conjectures. This is probably due to the fact that students are not well trained in writing about mathematics, in fact, this inability decreases along with practice. Percentage of mistake in conjecture's formulation is 40% in Worksheet 2Q – Part I and lower in the following ones, while in Worksheet 4Q – Part I, in which students are asked to research a concurrency property on the axes of a quadrilateral, there are no mistakes at all.
Recurrent examples of wrong formulation of the conjecture in Worksheet 2Q – Part I are: "The point G is the midpoint of the bimedians of a quadrilateral"; "The point G lies on the three bimedians and it is their midpoint". Note that both formulations are well anchored to the exploring activity (that leads to determine point G) and they do not reach the degree of a general form: "*The three bimedians of a quadrilateral all pass through one point that bisects each of them*". Later on during the activity students, after being helped to individuate the mistake, become able to formulate conjectures with higher degree of abstraction and generality.

4.2 Results on Worksheets Part II: Proof of the properties

At the point of having to prove the properties some difficulties arose. In some Worksheets (for example in Worksheet 2) the later elaboration of the proofs in space are sensibly clearer than those previously done in the planar case. In other worksheets this improvement does not take place. In Worksheet 3 this is probably due to the fact that the sheet contains no hints on the proof of the property (Commandino's Theorem). Several students were not able to reproduce without imperfections the analogous proof that they had elaborated in the plane for the medians of a quadrilateral.
In general, the mistakes that we have found in the proofs underline inability to follow a reasoning in depth (for example students do not worry to prove that some lines are coplanar, before saying that they intersect) and difficulties in visualizing objects (for example the mutual position of lines in the space).

5. Final conclusions

In order to make an overall evaluation of the whole activity, we submitted a questionnaire to the students and told them to fill it in anonymously. They handed it back in at their final exam. We had asked them to express their opinions about the topic they had developed during the course, as well as about the use of Cabri II Plus and Cabri 3D on the activity performed, and how, according to them, the whole activity was conducted, and their attitude towards geometry after the course.
Here we give full details of some of their responses.

5.1 Topic of the course

We asked the students to express their opinions on the topic of the course, and particularly how they liked it, how hard it was, etc.

"*The topic of the course turned out to be suitable to the entire class, including those of the second year and the ones of the forth. It was adequately difficult*";

"*The topic of the course was extremely interesting, because it was completely new to me. It was not very difficult and the worksheets were rather exhaustive. I worked by myself without great difficulties*";

"*I enjoyed the topic of the course, because space geometry is not quite covered at school. I did not find the course to be so hard anyway*";

"*The topic of the course was easy, nice and exiting*";

"*At first I thought that the course would turn out to be boring, and then I changed my mind when I found out about the existing analogies of properties among geometric figures, apparently so different from each other, like quadrilaterals and tetrahedra*";

"*I enjoyed the analogy between quadrilaterals and tetrahedra of the two columns* (Table of the analogies)";

"*Some proofs were hard: they needed more hints*";

"*Sometimes we came across some difficulties that it was possible to overcome thanks to our teachers' help*".

5.2 Use of Cabri II Plus and Cabri 3D

We asked the students to express opinions on the use of Cabri II Plus and Cabri 3D during the activity and in general about their use for the study of geometry.

"*Both Cabri II Plus and Cabri 3D were useful to understand geometry and to immediately and 'manually' experiment some concepts that otherwise would have not be clear*";

"*These software are user-friendly and very practical too*";

"*Thanks to Cabri II Plus and Cabri 3D it was easier for me to handle with figures, especially those in the space, also because we could rotate them. Moreover, the computer made drawing figures a lot faster and the entire activity more fun*";

"*Cabri II Plus and Cabri 3D were so essential for the carrying out of the course. I believe that without them the worksheets would have been a lot harder and boring*";

"*With Cabri you cannot prove a property. The good thing about it though is the sense of awareness provided by it of a geometric situation*".

5.3 Development of the activity

We asked the students to express their opinions on the way the activity was conducted and in particular on the worksheets.

"*The course was very well planned; the worksheets were always very easy to understand. I did like the fact that we started from just observing a property (and elaborating the conjecture) ending up with the mathematical proof of what had been observed*";

"*The method of alternating 'guided working sheets', where we were supposed to find out properties and elaborate conjectures, with 'proving working sheets' enabled me to play an active role in the whole process of learning, which is a completely new way for me to learn; at school usually you listen to somebody talking and that is it*";

"*Sometimes it was a quite fast-paced type of class environment, and some of the worksheets provided gave me just a few tips for the actual achievement of final proofs*";

"*The entire activity was carried out with no pressure at all, which is always a good thing when dealing with such topics*";

"*The worksheets were good guidelines and made us understand how 'to do' geometry*";
"*The worksheets were good for formulating the conjecture; maybe they should have contained more tips for proving the properties*";
"The review part of a previous lesson at each session worked out very well for all of us".

5.3 Attitude towards geometry

We asked the students whether their attitude towards geometry had changed after the activity, and in particular towards 3D geometry.

"*I found out that space geometry is a lot easier than I thought*";
"*I observed that [space geometry] uses always the same plane geometry theorems*";
"*I did not have a very deep understanding of 3D geometry other than what I studied in middle school, so my approach to the discipline got definitely improved. I particularly enjoyed learning how to analyze problems in the space, which is probably the most important thing I've learned*";
"A*fter this course my attitude towards geometry is more mature, so I guess my viewpoint has also changed and I liked this challenge, it's more like a 'game now'*";
"*After the course, I must admit that I'm much more fascinated by geometry. Besides, the course made me understand concepts and definitions that were too fast for me to pick up and not very clear in my mind*".

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