Fundamental Theorem of Calculus and Computations on Some Special Henstock-Kurzweil Integrals

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics and Statistics
Radford University
Radford, VA 24142
USA

DING, Xiaofeng
dingxiaofeng@hhu.edu.cn
College of Computer and Information Engineering
Hohai University
Nanjing 210098
People's Republic of China

Abstract

The constructive definition usually begins with a function f, then by the process of using Riemann sums and limits, we arrive the definition of the integral of $f, \int_a^b f$. On the other hand, a descriptive definition starts with a primitive F satisfying certain condition(s) such as $F' = f$ and F is absolutely continuous if f is Lebesgue integrable, and F is generalized absolutely continuous if f is Henstock-Kurzweil integrable. For descriptive integrals, the deficiency is that we need to know primitive F for which $F' = f$ and satisfying some properties. For constructive integration, we proposed in [8] using an uneven partition to get a broader family functions which includes some improper Riemann integrals. In this paper, we describe how we can make use of the Fundamental Theorem of Calculus and the constructive definition to reach a description definition for some improper Riemann integrable functions that are monotonic or highly oscillating with singularity on one end.

1 Introduction

If f is continuous over a closed interval, $[a, b]$, the Riemann integral $\int_a^b f$ makes sense and in some cases we are able to find an explicit formula for the expression $F(x) = \int_a^x f(t)dt$; in such case we call F the primitive of f, we can check easily that the condition $F'(x) = f(x)$ holds. However, in almost every case we are unable to produce a simple "closed form" expression for $F(x)$ by the process of taking limits of sums. We recall that the constructive definition usually begins with a function f, then by the process of using Riemann sums and limits, we arrive the definition of the integral of $f, \int_a^b f$. On the other hand, a descriptive definition starts with a primitive F satisfying certain condition(s) such as $F' = f$. For example, F is uniformly continuous over a closed interval if f is Riemann integrable over the same interval. On the other hand, F is absolutely continuous if f is Lebesgue integrable and F is generalized absolutely continuous if f is Henstock-Kurzweil (HK) integrable. For descriptive integrals, the challenge is that we need to know primitive F such that $F' = f$ (and satisfying some properties). In this paper, we first describe we can construct the primitive function from a function f that is Riemann integrable. Next, we extend this idea to two types of improper Riemann integrable functions: One is the family of monotonic functions with singularity at one end, which normally are computed as 'improper' Riemann integrals, but now they are direct results from Henstock-Kurzweil definition. Second, we will take care of functions that are highly oscillatory with a singularity at one end. In [8], we described how these two types of HK integrable functions can be computed by introducing an uneven partition. In this paper, we start with a constructive definition by using uneven partitions and reach a description definition for these two types of improper Riemann integrable functions It is known that Fundamental Theorem of Calculus (FTC) should be valid when a function F is differentiable on (a, b) and we have

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{1}$$

However, the Lebesgue integration requires F' to be integrable over $[a, b]$. Essentially in this paper, we make use of the computation methods described in [8] and show that there is a family of computational functions that are HK but not Lebesgue integrable. In summary, the functions we defined by using the Riemann sum with our computation scheme(s) allows us to reconstruct the primitive $F(x) = \int_a^x f$, and the primitive F is generalized uniformly continuous in a closed interval.

We first recall the following definitions.

Definition 1 *Let F be defined on a set $A \subset \mathbb{R}$. We say that F is uniformly continuous on A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$, then $|F(x) - F(y)| < \varepsilon$.*

Definition 2 *For a sequence $\{F_n : A \to R\}$, we say that the sequence $\{F_n\}$ converges uniformly on A to F if for every $\varepsilon > 0$ there exists a positive integer N such that $|F_n(x) - F(x)| < \varepsilon$ for every $n \geq N$ and for every $x \in A$.*

Definition 3 *The sequence of functions $\{F_n : A \to R\}$ is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists an N, $|F_{n+k}(x) - F_n(x)| < \varepsilon$ for every positive integer $n \geq N$, every positive integer k, and for every $x \in A$.*

It is known from the Weierstrass Uniform Convergence Criterion that a sequence $\{F_n\}$ converges uniformly on A to F if and only if it is uniformly Cauchy. For computation purpose, checking if a sequence is uniformly Cauchy is more convenient than checking if a sequence is uniformly convergence.

1.1 Graphical Approach to The Riemann Sums and Fundamental Theorem of Calculus

We summarize how we may use the graphs of Riemann sums as described in [9] to provide an intuitive approach to the First Form of The Fundamental Theorem of Calculus.

Suppose that f is continuous on an open interval I. Suppose that a is any number in I and that the function F is defined by

$$F(x) = \int_a^x f(t)dt$$

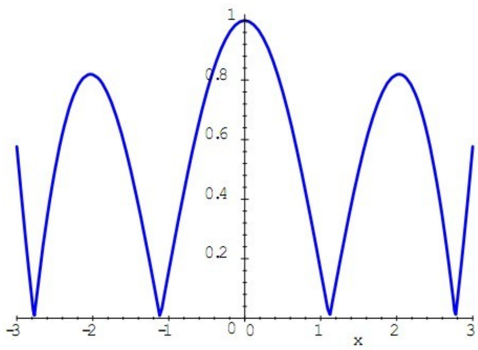


Figure 1: The graph of f

for every $x \in I$. *Then* $F'(x) = f(x)$ *for every number* $x \in I$.

We use the function $f(x) = |1 - x \sin x|$ with $x \in [-3, 3]$ for illustration. We note that $\int_a^x f(t)dt$ does not posses a closed form and note that f fails to be differentiable at several points, which can be seen in Figure 1.

We use midpoint sums sum to approximate $\int_0^x f(t)dt$. If $M_f\,(x, n)$ is the midpoint approximation to $\int_0^x f\,(t)\,dt$ taken over a regular partition of $[0, x]$ into n subdivisions then

$$M_f(x, n) = \sum_{j=1}^{n} \left(\frac{x}{n}\right) f\left(\frac{2jx - x}{2n}\right).$$

As we know, if x is any number then

$$\lim_{n\to\infty} M_f(x, n) = \int_0^x f(t)dt.$$

We shall see how the graph of $\int_0^x f(t)dt$ looks like for each number x by making use of the graphs of $M_f(x, n)$, $n = 1, 2, \ldots$. The graphs of $M_f(x, 5)$ and $M_f(x, 15)$ can be seen in Figure 2(a); while the graphs of $M_f(x, 25)$ and $M_f(x, 35)$ can be seen in Figure 2(b), respectively. Since f is continuous everywhere over the interval $[-3, 3]$, the sequence $M_f(x, n)$ is continuous everywhere for each n, and it easy to check that $M_f(x, n)$ converges uniformly to $\int_0^x f(t)dt$ as $n \to \infty$ in $[-3, 3]$. In other words, we have

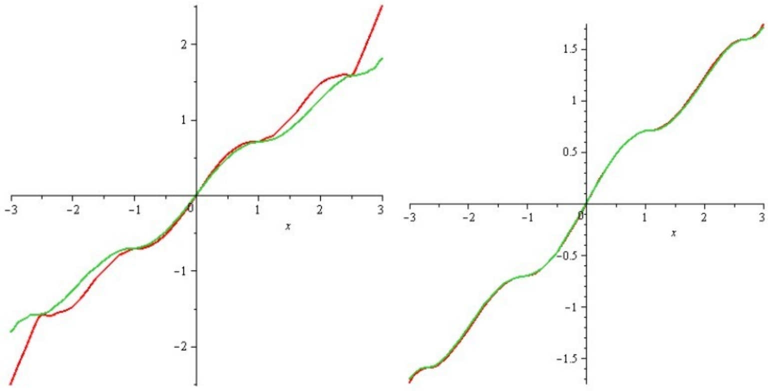
$$\lim_{n\to\infty} M_f(x, n) = \int_0^x f(t)dt. \tag{2}$$

Consequently, we expect the graph of $\int_0^x f(t)dt$ will resemble the one in Figure 2(b).

We leave it to readers to verify graphically that

$$\lim_{n\to\infty} \left(\frac{d}{dx} M_f(x, n)\right) = f(x) \tag{3}$$

Exercise: Use a computational tool to verify that $\lim_{n\to\infty} M_f(1, n) = \int_0^1 f(t)dt = 1 + \cos 1 - \sin 1$.



(a) (b)

Figure 2: The graphs of $M_f(x,n)$. (a): The graphs of $M_f(x,5)$ and $M_f(x,15)$; (b): The graphs of $M_f(x,25)$ and $M_f(x,35)$.

1.2 Extend the FTC to some Henstock Integrals

First, we introduce some preliminary notations and definitions. Let $A=[a,b]$, we say $P=\{(A_1,x_1),...,(A_n,x_n)\}$ is a partition of A if $A_1,...,A_n$ are non-overlapping subintervals, $x_i \in A_i$, for $i=1,2,...,n$, and $\cup_{i=1}^n A_i = A$.

Let δ be a positive function defined on A. A partition $P=\{(A_1,x_1),...,(A_n,x_n)\}$ is called δ-fine if $A_i \subset (x_i-\delta(x_i), x_i+\delta(x_i))$, for $i=1,2,...n$. We first give the definition of Henstock-Kurzweil integration on one dimension.

Definition 4 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on $[a,b]$ if for every $\epsilon>0$ there is a positive function δ on $[a,b]$ such that*

$$\left|\sum_{i=1}^{n} f(x_i)\,|A_i| - I\right| < \epsilon \tag{4}$$

for each δ-fine partition P of A, where $|A_i|$ denotes the length of $A_i, i=1,2,...,n$. In such case, we write $\int_a^b f dx$ or simply $\int_a^b f$.

Alternatively, we can use the following

Definition 5 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on an interval $A=[a,b]$ if there is a sequence of delta functions $\delta_n(x)$ such that for every $\delta_n(x)$-fine division D_n, the Riemann sums $\sum_{i=1}^n f(x_i)\,|A_i| \to I$ as $n\to\infty$.*

The Henstock integral and our quadratures can be stated in higher dimensions as discussed in [7]. However, we describe how we apply the FTC on some HK-integrals in one dimension in this paper. First we use the following definition to introduce the uneven partitions on an interval.

Definition 6 *A matrix A with positive a_{nk} is called uniformly regular if the following conditions are satisfied:*
(i) $\lim_{n\to\infty} a_{nk} = 0$ *uniformly over k.*
(ii) $\sum_{k=1}^{n} a_{nk} = 1$.

We introduce the following quadratures in 1-D:

The closed type quadrature

$$Q_n^1(f) = \sum_{k=1}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right) \tag{5}$$

or

$$Q_n^2(f) = \frac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right) \tag{6}$$

where $u_{nk} = a + \sum_{i=1}^{k} a_{ni}$, and $u_{n,0} = a$.

Remarks:
(1) By looking the $a_{nk} = \dfrac{2(b-a)k}{n(n+1)}$, we have $a_{n1} < a_{n2} < ... < a_{nn}$, $\sum_{k=1}^{n} a_{nk} = b - a$ for each n, which is the basis of our choice of uneven partitions.
(2) Both quadratures are similar to the trapezoidal rule except we are using uneven partitions, which are the essence of the HK-integration.
(3) In the closed type quadrature, if the Eq. (5) contains the singularity at the end point at $x = a$, or $x = b$, we set $f(a) = 0$ or $f(b) = 0$ respectively.
(4) In Eq. (6), we consider the integral value of f over the first interval $[a, u_{n1}]$. In other words, we ignore the singularity at $x = a$. We shall see we apply this quadrature for functions that are monotonic and have singularities near the end point.

2 Improper Riemann Integrals

2.1 Monotone functions

We summarize the computation quadrature mentioned from [8] for completeness. Suppose we denote $Q_n(f)$ as either closed or open quadrature described above in the interval $[a, b]$. We recall the following theorem from [8] without proof, which says that there is no improper Riemann integrals under HK-definition.

Theorem 7 *If f is Riemann integrable over $[c, b]$ for each $c \in (a, b]$ and improper Riemann integrable over the interval $[a, b]$ or $\lim_{n\to\infty} Q_n^1(f)$ exists. Then f is HK-integrable over $[a, b]$, and we have*

$$\int_a^b f = \lim_{n\to\infty} Q_n^1(f) = \lim_{n\to\infty} \left(\sum_{k=2}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right) \right), \tag{7}$$

where $Q_n^1(f)$ is the quadrature applied on the interval $[a, b]$.
In particular, if a function f is monotonic over $(a, b]$ and has a singularity at $x = a$, Eq. (7) can be replaced by

$$\int_a^b f = \lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty} \left(\frac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right) \right). \tag{8}$$

The following theorem is useful when we use a uniformly regular matrix a_{nk} for computing a monotonic and improper Riemann but HK-integrable function.

Theorem 8 *If f is Riemann integrable over $[c,b]$ for each $c \in (a,b]$. If for each $\epsilon > 0$, there exists a regular matrix a_{nk} such that $|f(u_{n1})a_{n1}| < \epsilon$. Then f is HK integrable over $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty}\left(\frac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n}\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)\right). \tag{9}$$

Proof. f is Riemann integrable on $[c,b]$ for each $c \in (a,b]$, so f is HK-integrable on $[c,b]$ for each $c \in (a,b]$.

Let $L = \int_c^b f$ and Let $\varepsilon > 0$. By theorem 8 in [8], there exists a positive function δ_1 on $(a,b]$ and $\mathbb{P}$ is a $\delta_1 - fine$ partition in $(c,b]$, then $|f(\mathbb{P}) - \int_c^b f| < \varepsilon$. Choose $\eta > 0$ so that $|\int_c^b f - L| < \varepsilon$ for all $c \in (a, a+\eta)$, define a positive function δ on $[a,b]$ by

$$\delta(x) = \begin{cases} min\{\delta_1(b), x-a\} & x \in (c,b] \\ min\{\eta, \frac{\varepsilon}{1+f(c)}\} & x = a \end{cases}, \tag{10}$$

Suppose that $\mathbb{P}$ is a $\delta - fine$ partition of $[a,b]$ and $(c,[a,c])$ can be subinterval belongs to $\mathbb{P}$ where $a < x < a+\eta$. Let $\mathbb{P}_a = \mathbb{P} - \{(c,[a,c])\}$ and compute

$$\big|f(\mathbb{P}) - L\big| \le \big|f(\mathbb{P}_a) - \int_c^b f\big| + \big|\int_c^b f - L\big| + |f(c)|\eta < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

We see that f is HK-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty}\left(\frac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n}\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)\right).$$

■

We next see how we can reconstruct the primitive F of f through the process of Fundamental Theorem of Calculus when f is improper Riemann integrable over an interval. First we show that the convergence $\lim_{n\to\infty} Q_n(f)(x)$ is uniform for all $x \in [a,b]$, where $Q_n(f)(x)$ denotes the function when $Q_n(f)$ is applied on $[a,x]$ for each $x \in (a,b]$. In other words, we make use of the Riemann sum, $Q_n(f)$, to reconstruct the primitive F of f. More precisely, we prove the following:

Theorem 9 *If f is Riemann integrable over $[c,b]$ for each $c \in (a,b]$ and $f(a) = 0$. Given a uniformly regular matrix a_{nk} over $[a,b]$, if we write $F(x) = \lim_{n\to\infty} Q_n(f)(x)$, where $Q_n(f)(x)$ denotes the function when $Q_n(f)$ is applied on $[a,x]$.Then (i) $F(x)$ is continuous over $[a,b]$, (ii) f is HK-integrable over $[a,b]$, (iii) $\int_a^x f = F(x)$ and $F'(x) = f(x)$ over $[a,b]$.*

Proof. First, we note that $Q_n(f)(a) = 0$. For $x \in (a,b]$, the function $Q_n(f)(x)$ is continuous, and we see $F(x) = \lim_{n\to\infty} Q_n(f)(x) = \sum_{n=1}^{\infty} Q_n(f)(x)$. Since $Q_n(f)(x)$ is continuous for each $x \in (a,b]$,and the convergence of the infinite series is uniform, F is continuous over $[x,b]$ for each $x \in (a,b]$. It remains to show that F is continuous at $x = a$, $\int_a^x f = F(x)$ and $F'(x) = f(x)$.

(i) We claim that F is continuous at $x=a$ from the right. Let $x \to a^+$,

$$\lim_{n\to\infty}\left(\lim_{x\to a^+} Q_n(f)(x)\right) = \lim_{x\to a^+}\left(\lim_{n\to\infty} Q_n(f)(x)\right) = \lim_{x\to a^+}\left(\int_a^x f\right) = F(a) = 0. \tag{11}$$

Therefore F is continuous at $x=a$ from the right.

(ii) We note that $F(x) = \lim_{n\to\infty} Q_n(f)(x) = \sum_{n=1}^{\infty} Q_n(f)(x) = \int_a^x f$.

(iii) We only need to prove that $F'(a) = f(a) = 0$:

$$F'(a) = \lim_{x\to a^+} \frac{F(x)-F(a)}{x-a} = \lim_{x\to a^+} \frac{\int_a^x f - 0}{x-a} = f(a) = 0. \tag{12}$$

■

We apply Theorem 8 on the following two examples, which is to say that improper Riemann integrals are HK-integrable and we can use the quadrature mentioned above to compute their respective integrals.

Example 10 *We define $f(x) = 1/\sqrt{x}$ if if $x \neq 0$, and $f(0) = 0$. We choose $a_{nk} = \frac{2k}{n(n+1)}$, then $|f(u_{n1})a_{n1}| = \sqrt{\frac{2}{n(n+1)}} \to 0$. Thus f is integrable over $[0,1]$ and it is proved in [8] that $\int_0^1 f = \lim_{n\to\infty} Q_n^2(f) = 2$.*

Example 11 *We define $f(x) = \dfrac{e^x}{\sqrt{x+1}}$ if if $x \neq 0$, and $f(0) = 0$. We choose $a_{nk} = \frac{2k}{n(n+1)}$, then*

$$|f(u_{n1})a_{n1}| = \frac{2e^{\frac{2}{n(n+1)}}}{\sqrt{\frac{(2+n^2+n)(n(n+1))}{n(n+1)}}} \to 0. \tag{13}$$

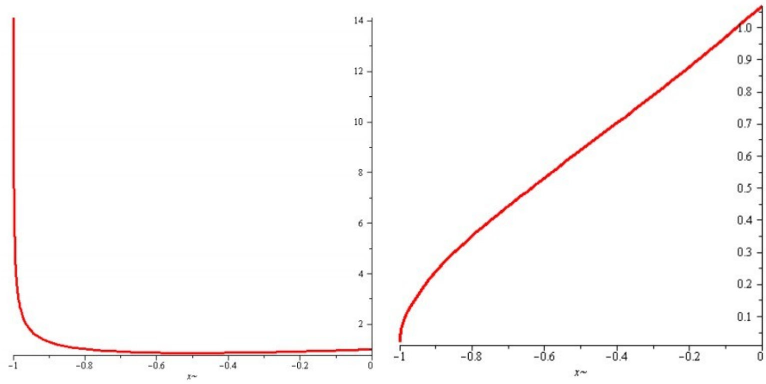
Therefore, it follows from Theorem 8 that f is improper Riemann integrable but f is HK-integrable over $[0,1]$, and it can be shown that $\int_{-1}^0 f = \lim_{n\to\infty} Q_n^2(f) \approx 1.076$. We further note that if we define $F(x) = \lim_{n\to\infty} Q_n(f)(x)$, then we are able to use a computational tool to simulate the graph of primitive F of f as follows as we did in Section 1.1. We plot $y = f(x)$ and $y = Q_{100}(f)(x)$ over the interval $[-1,0]$ in Figures 3(a) and 3(b), respectively.

2.2 Highly Oscillating Functions

In this section, we discuss those highly oscillating functions with one singularity near one end of the interval. The following example shows that dividing the entire interval $[a,b]$ unevenly by using a_{nk} is not sufficient for computing a highly oscillating, non-absolute integrals.

Example 12 *We define $f(x) = \frac{1}{x}\sin(\frac{1}{x})$ if $x \neq 0$, and $f(0) = 0$. We shall show that f is HK-integrable though not Lebesgue integrable over $[0,1]$ when we prove its two dimensional extension in the next section. We demonstrate how we approximate the integral $\int_0^1 f$. We first note the followings:*
(1) We can't apply the uneven partition and quadrature over the interval $[0,1]$ in one step. Instead, we construct a sequence $\{x_n\}$ converges to 0.
(2) In other words, we select

$$x_i = 5^{-(i-1)} \tag{14}$$



(a) (b)

Figure 3: The graphs of f and its primitive F. (a): The graph of f; (b): The graph of the primitive F of f.

for $i = 1, 2, \ldots$. *We approximate the integral of* f *in each* $I_i = [x_{i+1}, x_i]$ *and denote the integral of* f *over* I_i *by* A_i *when applying the closed type quadrature* $Q_n^2(f) = \sum_{k=1}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)$, *where* $u_{n0} = x_{i+1}$,*and* $u_{nk} = x_{i+1} + \sum_{i=1}^{k} a_{ni}$.
(3) For each $x \in [x_{i+1}, x_i]$, *we write* $F_i(x) = \lim_{n\to\infty} Q_n^i(f)(x)$, *where* $Q_n^i(f)(x)$ *denotes when* $Q_n(f)$ *is applied on* $[x_{i+1}, x]$.*Then*

$$\int_0^1 f \approx \sum_{i=0}^{r} F_i(x) = \sum_{i=1}^{r} A_i, \tag{15}$$

for some r. *If we use the matrix* $a_{nk} = \dfrac{2k}{n(n+1)}$ *and the closed type quadrature in each* $I_i = [x_{i+1}, x_i]$ *for* $i = 1, 2, \ldots 6$,*and with the help of Matlab and the closed type* $Q_n^2(f)$, *we have shown in [8] that* $\int_0^1 f$ *is approximately equal to* $6.247327401459105e - 001$.

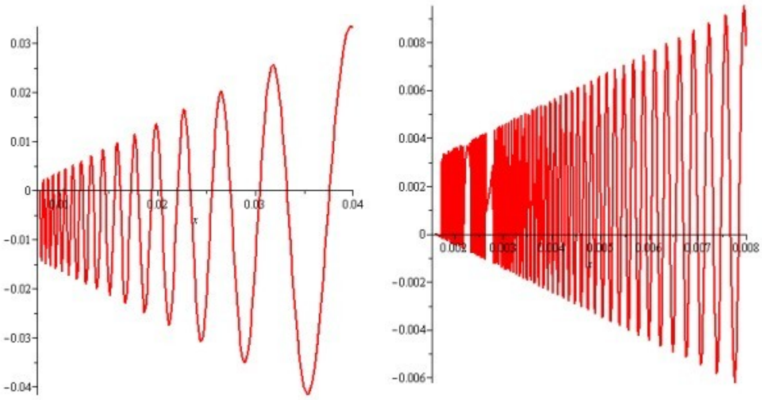
Remark: We demonstrate the convergence of F (for the function in above Example) being uniform in the interval of $[5^{-3}, 5^{-2}]$ graphically in `http://mathandtech.org/Yang/Henstock_integral/henstock2010_Feb1.html`, but the convergence is not uniform in $[0, 1]$.In other words, F is a countable union of uniformly continuous function. We can use a computable tool to demonstrate that the graphically that

$$y = Q_n^2(f)(x) \text{ is getting closer to } y = F_2(x) \text{ in } [5^{-3}, 5^{-2}], \text{ and} \tag{16}$$
$$y = Q_n^3(f)(x) \text{ is getting closer to } y = F_3(x) \text{ in } [5^{-4}, 5^{-3}], \tag{17}$$

although it is evidently that the second graphical representation will consume much more computation time. We demonstrate the graphs of $y = F_2(x)$ and $y = F_3(x)$ are shown respectively in Figures 4(a) and 4(b).

In view of Example 12 , we have the following definition.

Definition 13 *If a function* F *is said to be generalized uniformly continuous (UCG) on a set* X *if* X *is a countable union of* X_i, *or* $X = \bigcup_{i=1}^{\infty} X_i$, *such that* F *is uniformly continuous on each* X_i.



(a) (b)

Figure 4: The graphs of $y = F_2(x)$ and $y = F_3(x)$. (a): The graph of $F_2(x)$; (b): The graph of $F_3(x)$.

Theorem 14 *Let $\{x_r\} \to a^+$, and $A_r = \lim_{n\to\infty} Q_n(f)$ in $[x_{r+1}, x_r]$, where $x_0 = b$. If $\sum_{r=0}^{\infty} A_r$ converges, then f is HK-integrable over $[a, b]$ and*

$$\int_a^b f = \sum_{r=0}^{\infty} A_r. \tag{18}$$

Furthermore, for each $x \in [x_{r+1}, x_r]$, if we write $F_r(x) = \lim_{n\to\infty} Q_n^r(f)(x)$, where $Q_n^r(f)(x)$ denotes when $Q_n(f)$ is applied on $[x_r, x]$.Then (i) $F_r(x)$ is uniformly continuous (UC) on $[x_{r+1}, x_r]$, (ii) $F_r(x_r) = A_r$,(iii) $\int_a^b f = \sum_{r=0}^{\infty} F_r(x_r) = \sum_{r=0}^{\infty} A_r$. In other words, if we write $F(x) = \sum_{r=0}^{\infty} F_r(x)$, then F is UCG on $[a, b]$.

Proof. The proof of the first part of this Theorem is done in [8].

As $A_r = \lim_{n\to\infty} Q_n(f)$ in $[x_{r+1}, x_r]$, we have that f is HK-integrable over $[x_{r+1}, x_r]$. So for each $x \in [x_{r+1}, x_r]$, $\lim_{n\to\infty} Q_n^r(f)(x)$ exists and $\int_x^{x_{r+1}} f = \lim_{n\to\infty} Q_n^r(f)(x) = F_r(x)$.

So the primitive $F_r(x)$ of f is continuous in $[x_{r+1}, x_r]$, and $F_r(x)$ is UC there from the Theorem 5.48 in [7].

In addition, $\int_{x_{r+1}}^{x_r} f = \lim_{n\to\infty} Q_n^r(f)(x_r) = F_r(x_r) = A_r$, and $\int_a^b f = \sum_{r=0}^{\infty} F_r(x) = \sum_{r=0}^{\infty} A_r$. ∎

Theorem 15 *Assume the conditions of the preceding Theorem are met, and we write $F(x) = \sum_{r=0}^{\infty} F_r(x)$, then $F'(x) = f(x)$ for all x in $[a, b]$ and F is generalized uniformly continuous on $[a, b]$.*

Proof. For each $x \in (a, b]$, there exists a N and $x \in [x_{N+1}, x_N]$. Then $F(x)$ can be rewritten as:

$$F(x) = \begin{cases} F_0(x) & x \in [x_1, x_0] \\ F_1(x) + A_0 & x \in [x_2, x_1] \\ \ldots\ldots & \\ F_N(x) + \sum_{r=0}^{N-1} A_r & x \in [x_{N+1}, x_N] \end{cases}.$$

In $[x_{r+1}, x_r], r = 0, 1, .., N$, as every $F_r(x)$, is continuous and is the primitive of f, so $(F_r(x))' = f(x)$ and $F_r(x)$ is UC. So for each $x \in (a, b]$, $F'(x) = f(x)$.

$F_r(x)$ is UC in $[x_{r+1}, x_r]$, let $r \to \infty$ then we have $F'(a) = f(a)$. So $F'(x) = f(x)$ for all x in $[a, b]$ and F is generalized uniformly continuous on $[a, b]$. ■

We conclude the discussion of FTC by reminding readers that theoretically, we have a more general theorem regarding the HK-integrable functions below. In other words, we can allow more singularities in an interval and the theorem can be extended into higher dimensions, which is so called the Divergence Theorem, see [5].

Theorem 16 *If $F' = f$ on $[a, b]$, then f is HK-integrable over $[a, b]$ and*

$$\int_a^b f = F(b) - F(a). \tag{19}$$

Proof. This follows from the definition of HK integration, see ([5], Theorem 6.1.2, page 103). ■

3 Conclusion

In this paper, we described how we can replace the 'even partitions' with 'an uneven partition' in Riemann sum definition to reach a HK-definition. This allows us to handle some improper Riemann integrals. We also discussed the properties for these special types of primitive functions, which are generalized uniformly continuous functions, which are natural extensions from the primitive functions of Riemann integral functions. The approaches here are intuitive without the knowledge of Lebesgue measure and integration theories, which we think are more accessible to undergraduate students. Since the primitive function for a highly oscillatory with singularity at one end point and non-absolute HK-integrable function is generalized uniformly continuous, our future works will involve the use of different uneven partitions in each subintervals.

References

[1] Davis and Rabinowitz (1983), *Method of Numerical Integration*, 2nd ed., Academic Press, New York, ISBN978-0122063602

[2] P. Y. Lee (1989), *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, ISBN 9971-50-892-3.

[3] P. Y. Lee & R. Výborný (2000), *Integral: An Easy Approach after Kurzweil and Henstock*, Cambridge University Press, Cambridge UK, ISBN 0 521 77968 5.

[4] R. K. Miller (1971), *On Ignoring the Singularity in Numerical Quadrature,* Mathematics of Computation, Volume 25, Number 115, July.

[5] W.F. Pfeffer (1993), *The Riemann Approach to Integration: Local Geometric Theory,* Cambridge University Press, ISBN 0-521-44035-1.

[6] P. Rabinowitz and I. H. Sloan (1984), *Product Integration in the Presence of Singularity*, Siam J. Numer. Anal. Vol 21, No. 1, February.

[7] B. S. Thomson, J. B. Bruckner and A. M. Bruckner (2000), *Elementary Real Analysis*, 1st edition, Prentice Hall, ISBN 0-13-019075-6.

[8] W.-C. Yang, P.-Y. Lee and X. Ding (2009), *Numerical Integration On Some Special Henstock-Kurzweil Integrals*, Electronic Journal of Mathematics and Technology (eJMT), ISSN 1933-2823, Issue 3, Vol. 3.

[9] W.-C. Yang and J. Lewin, *Exploring Mathemaics with Scientific Notebook*, page 91, Springer Verlag, 1998. ISBN 981-3083-88-3.