

Questions, Answers, and Mathematical Discourse

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Abstract

Technology has often been accused of just giving students answers without the students knowing why. In this talk, we examine problems in which this is a very good thing. Specifically, we discuss problems in which technology provides the answer very quickly. At that point, the initial conditions are changed- but the answer remains the same! As a result, the answer becomes less important than figuring out why the answer is not changing as conditions change. What will emerge from this pattern is that these problems require students to focus on proof and reasoning, and engage in mathematical discourse.

Introduction

In this paper, we discuss how technology can be used to enhance mathematical discourse. Specifically, we examine problems in which technology reveals something of interest to students of mathematics, engaging their curiosity with a surprising result. Students then engage in mathematical discourse in their search for an explanation.

The Golden Ratio

With students in groups of two, have each student pick a random positive integer. Enter these integers into their HP39gs or other graphing calculator. Now add the two integers to get a third integer. The HP 39gs has a COPY command that simplifies this operation. Have the students add the second and third integer to get a fourth. Figures 1 and 2 illustrate the process. Students should repeat the process a dozen or so times. Once they have completed a dozen iterations, have them divide their last two integers in any order. Imagine their surprise when you tell them the first four digits after the decimal place are 6, 1, 8, and 0! Not only does it not matter what integers they use to start, the order of the numbers in the division is irrelevant as well.

The whole process will take students less than 3 minutes, but it gives the teacher a wealth of new directions in which to take the class. For example, the results from Figures 1 and 2 that are shown in Figure 3 reveal that the division done one way returns 0.6180... while reversing the order returns 1.6180.... One of these is one less than the other.



Figure 1



Figure 2

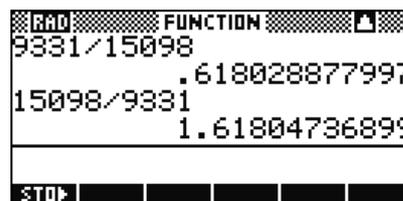


Figure 3

The teacher can ask the students to write and solve an equation that says “the reciprocal of a number is one less than the number”. The resulting equation resolves to a quadratic which gives the familiar result for the Golden

Ratio: $\frac{1+\sqrt{5}}{2}$ or approximately 1.61803.

So how are these two results related?

Why does it not matter which two integers one uses to start?

The only satisfying answers to these questions require mathematical discourse! We used technology to quickly get to a surprising result, but then we search for a mathematical explanation.

One way to approach the problem utilizes the Fibonacci sequence. It is relatively easy to define and explore this sequence using technology.

In Figure 4, we have stored the value 1 in both A and B. We now define the Fibonacci sequence in terms of A and B. Figure 5 shows the definition on the HP 39gs.

It is known that the quotient of consecutive members of the Fibonacci sequence converges on the Golden Ratio. In Figure 6, we define U2(N) as the sequence of such quotients.

Figure 7 shows the two sequences numerically. The second sequence can be seen to converge to 4 decimal places by the 12th iteration.

We now store -38 in A and 255 in B (Figure 8). Returning to the Numeric view of the Sequence applet, we see Figure 9.



Figure 4

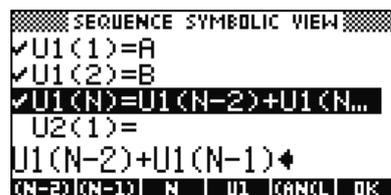


Figure 5

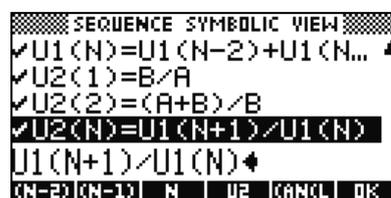


Figure 6

N	U1	U2
10	55	1.618182
11	89	1.617978
12	144	1.618056
13	233	1.618026
14	377	1.618037
15	610	1.618033

Figure 7

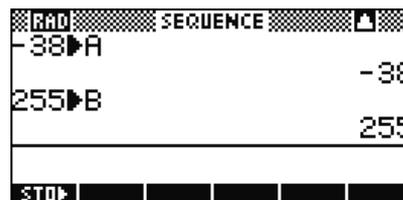


Figure 8

N	U1	U2
10	7872	1.617505
11	12733	1.618236
12	20605	1.617957
13	33338	1.618063
14	53943	1.618023
15	87281	1.618038

Figure 9

Linear Systems

I first saw this problem in a lesson study done by Dan Kennedy at a recent NCTM conference in the United States.

Suppose the linear system

$$Ax + By = C$$

$$Dx + Ey = F$$

is such that A , B , and C are terms in an arithmetic sequence and D , E , F are terms in another arithmetic sequence. Figure 10 shows one such system entered in the HP 39gs Linear Solver application. Note the solution $(-1, 2)$. Figure 11 shows another such system - with the same solution!

Imagine asking each student in your classroom to create and solve such a system only to find out that the solution to each and every one of their systems is $(-1, 2)$. They are sure to be surprised and intrigued.

Here again, technology is used to quickly confront the student with a surprising and intriguing result. The only satisfying explanation will be a mathematical one, generating mathematical discourse.

Here, for example, you could rewrite $Ax + By = C$ as $Ax + (A + B)y = A + 2B$ to reflect the arithmetic sequence of the parameters. The right side of this equation contains the term $2B$. The only way for the left side to correspond is if $y=2$. But then the left side will have a term of $2A$. The only way to reduce it back to $1A$ is if $x=-1$.

Now students can be asked to rewrite the second equation in a similar fashion and solve for both x and y using substitution or a similar method.

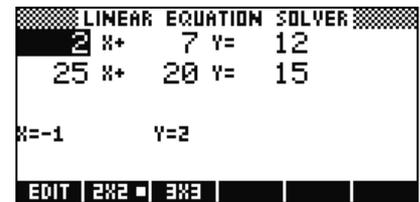


Figure 10

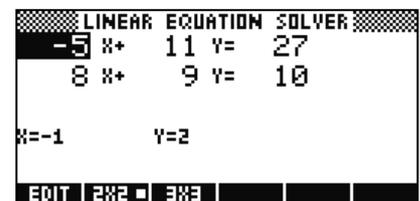


Figure 11

Linear Iterations

Have students pick any random number, then multiply it by 0.2 and subtract 3 from the result. Figure 12 shows the result if the initial random number chosen is 5. Now repeat this process iteratively. On most graphing calculators, this can be accomplished simply by pressing the ENTER key repeatedly. Figures 13 and 14 show the result quickly converges on -3.75. The surprise for the students is that the sequence of iterations converges on -3.75 regardless of the initial number chosen.

In fact, if a number x is chosen at random and is used as the start of an iterative process of the form $A*x+B$, then either it converges or its inverse converges. Of course, there are restrictions on A and B . A cannot be -1, 0, or 1 and B cannot be 0.

Consider the iterative process shown in Figure 15. Clearly, this process will not converge on a single value. Figure 16 shows the inverse process, which converges quickly to -1.

If used as the initial input for the original process, -1 will produce $-1*3+2=-1$. Students are intrigued by this behavior and want to know what is happening mathematically.

Let's return to our original example using $0.2*x-3$. If this iterative process converges on a value x , it simply means that the input value and the output value are equal, that is, $0.2*x-3=x$. From this statement, it can be easily seen that $x=-3.75$.

But this algebraic view does not by itself explain why there is convergence either in the process or its inverse!

For a graphical view of the iteration process, we turn to the Function and Sequence applets. Figure 17 shows the graphs of $y=0.2*x-3$ and $y=x$. From the initial value of $x=5$, we drop a vertical segment to the graph of the line to see $y=-2$.

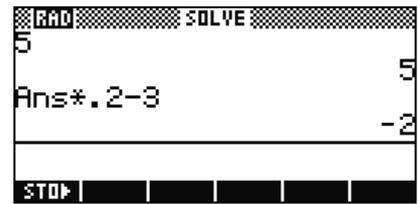


Figure 12

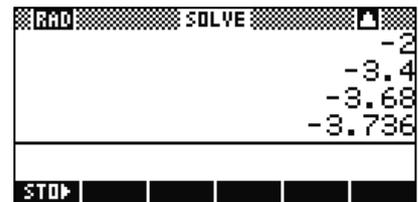


Figure 13

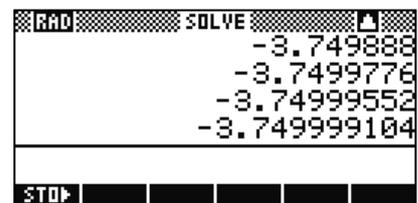


Figure 14

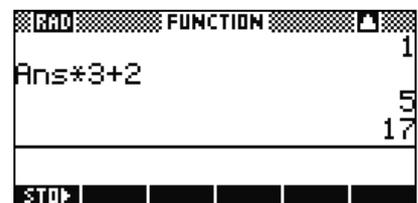


Figure 15

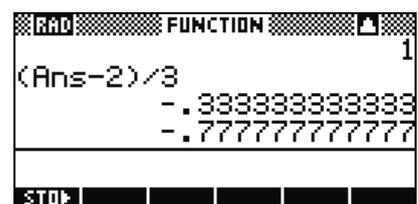


Figure 16

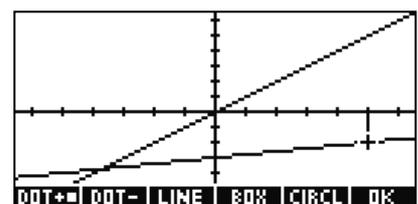


Figure 17

This represents our first iteration graphically. Now to proceed to our next iteration, we want to use the current y -value as our next x -value. Figure 18 shows a horizontal segment drawn from the graph of our iterative function to the line $y=x$. At this point, the x -value is now -2 as well, so we are ready to repeat the process. As you can see in Figure 19, the iterations are approaching a fixed point and that fixed point is at the intersection of the two graphs. This brings us around full circle to our equation $0.2*x-3=x$, but with a graphical interpretation.

Why did we not get convergence in our other example? Figure 20 shows the graphs of $y=3*x-2$ and $y=x$. This time, the first iteration is moving away from the intersection, not towards it. Using the inverse allows us to “go backward” toward the intersection.

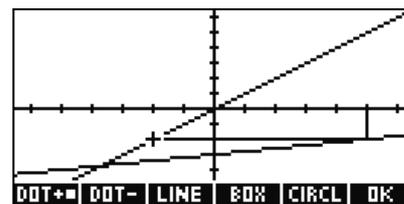


Figure 18

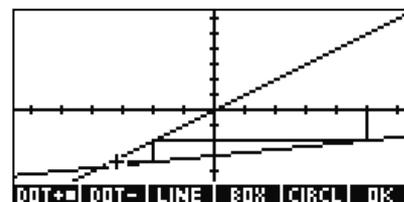


Figure 19

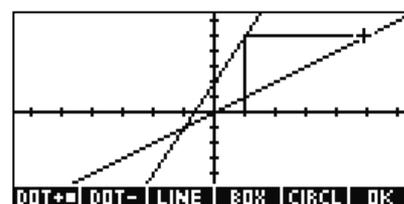


Figure 20

Conclusion

In each of the examples above, we have tried to show how technology can quickly get students to an answer that is surprising and activates their curiosity. Once their curiosity is aroused, the only thing that will satisfy it is good mathematical discourse; that is, an argument based in proof and reasoning. It is hoped that activities such as these can increase the level and frequency of mathematical discourse in the secondary classroom, increasing student engagement and participation.