Elementary PDEs with CAS technology

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Partial differential equations (PDEs) are arguably one of the most powerful tools for mathematical modeling of diverse phenomena that include such diverse applications as traditional mechanical and electro-magnetic settings, bio-medical and financial applications, and even social dynamics. Ever since the days when technology meant pencil and paper, the study of PDEs and their solutions has been a major driving force for the development of new mathematical theories that underlie analytical and computational tools for solving and simulating solutions of PDEs. For elementary PDEs such as the wave and heat equation on very simple, regular domains have traditionally been solved using methods related to Fourier analysis. These start with elementary separation of variables techniques, identification of bases of eigenfunctions and representations of solutions by infinite series. For less regular domains, descendants of the Riemann mapping theorem and harmonic analysis are an example of analytic tools. Nonetheless, analytic techniques are not only arduous to use, but often also do not provide the desired hands-on information about solutions. Thus, the numeric simulation of solutions of diverse PDEs has in many areas become the predominant approach, and it has spawned the new discipline of modern scientific computing.

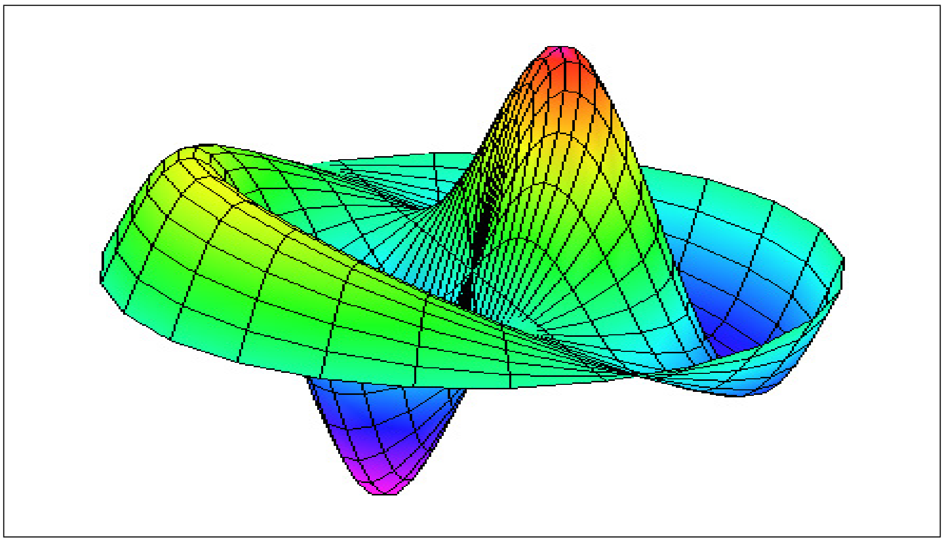

Figure 1: A not-rotationally symmetric, exact solution of a vibrating drum

While giving due respect to numeric simulation techniques, this presentation focuses on the opportunities provided by modern computing technology to enhance the analytic treatment of PDEs. We

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demonstrate that computer algebra systems (CAS) are predestined to vastly enhance the applicability of analytic Fourier techniques. Even if not yielding a complete solution right away, we argue that a dedicated use of *computer algebra first* is in many settings more powerful and effective than simple brute force numeric simulations.

The first example illustrates this approach at the hand of classical problems such as vibrating membranes. We focus, in particular, on non-rotationally symmetric solutions of vibrating circular disks. The exact solutions in terms of Bessel functions traditionally were beyond typical first courses in PDEs. Yet the availability of computer algebra systems such as MAPLE allows the user to focus on the underlying principles such as orthonormal sets of eigenfunctions, without having to worry about the specific rules for doing algebraic and calculus with them. While vastly extending the reach of these techniques beyond problems whose solutions can be written in terms of simple trigonometric and hyperbolic functions, this also elevates the learning environment from focus on calculus to doing analysis. In a nutshell, if CAS are available, then anyone who really *understands* Fourier analysis, can handle any problems where trigonometric functions are replaced by any of the many other elementary functions. We will also provide dramatic and beautiful animations of such solutions, which are the icing on the cake.

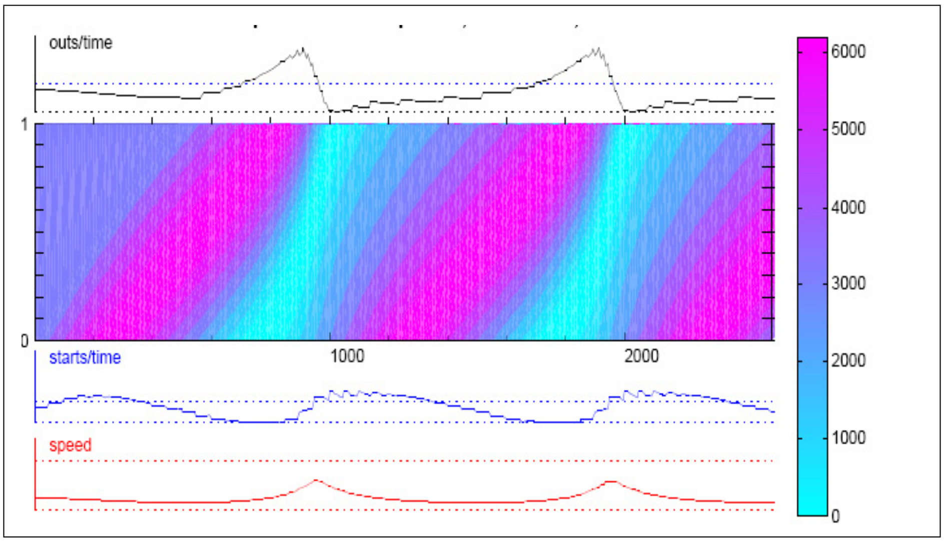

Figure 2: Simulation of a hyperbolic conservation law modeling a supply chain

As a second example we shall consider systems of hyperbolic conservation laws that arise e.g. as mathematical models for traffic flows, and more recently have been utilized to model supply chains in the semiconductor manufacturing industry. While frequently brute-force numeric simulations are employed to study such systems, we will demonstrate the utility of first using CAS to simplify the problems. Of particular interest are *shocks* that commonly can develop in such settings, and meaningful approximations of weak solutions of such PDEs. Again, a characteristic feature of utilizing CAS is the much enlarged set of *elementary functions* that are available for calculus. In this case, LambertW functions appear everywhere – and with CAS at our fingertips they are as almost easy to work with as classical elementary functions such as trigonometric and exponential. Consequently, the use of numerics can be delayed as more exact solutions are readily available.