Content Knowledge, Creativity and Technology

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Abstract

It is always debatable how much math content knowledge we expect from a future math teacher before he or she starts teaching. Many educators do not agree that we solely rely on examinations to test one's mathematical knowledge, which gives reasons for stuffing lots of contents into a math curriculum. The real question is how a person can build life long interests in mathematics and be creative when it comes to solving their daily problems. Could innovative use of technological tools provide some possible solutions? In short, we can come to the following consensus for mathematics curriculum in general:

1. We shall see more innovative use of technological tools to be integrated in a mathematics curriculum to enhance one's mathematics knowledge horizon.

2. An important task is motivating a learner to appreciate mathematics as early as possible, so a learner can be motivated in discovering more mathematics.

1 Introduction

We have often heard remarks such as "You didn't do well on a math test so you are not smart" or "You are not smart enough to do mathematics". These remarks often turn away students' interests in mathematics very quickly. It is also reflected in the traditional way of assessing students' learning, where teachers 'control' most of the learning content and environments, and students simply 'receive' or 'obey' what they are being taught.

On the other hand, technological tools advance rapidly, we 'empower' learners to expand their learning horizons and means of acquiring necessary information. Learners can research appropriate content from the internet anytime and anywhere, where they not only learn traditional content that one expects from a course, but could also gain added knowledge while exploring the topics with technological tools. Therefore, it is time for us to consider developing hybrid course content, where we include *static content*-fixed topics which are agreed to be covered but also include *dynamic content*-evolving content when new discoveries are added when technological tools advance.

It is not how much content we can squeeze into a mathematics curriculum but how much students can absorb and apply. While there is no specific formula on how much content is needed for a future math teacher in each country, it is definitely affirmative that the more content knowledge a teacher possesses, the better it is for our education system. The 2D-3D visualization tools have been the main driving force for assisting learners in comprehending abstract and complex concepts. While basic algebraic manipulation skills are essential to student success in mathematics, it is also clear that students discover more mathematics only when they comprehend how a theorem or formula is derived through exploration not through hours of drill doing repetitive exercises. Teachers, parents and communities should encourage students to explore and discover more mathematics with innovative (timely and proper) use of technological tools.

In Section 2, we describe the content knowledge for a future math teacher from various countries. In Section 3, we describe the importance of creativity and give examples to support our views.

2 Content Knowledge

Here are some problems in the US educational system

1. One article, "Educational Crisis" (see [1]), correctly points out many fundamental problems of math teachers education in the US.

2. Another article "U.S. Middle School Math Teachers Are Ill-prepared Among International Counterparts" (see [2]). William Schmidt, Michigan State University, distinguished professor says "Our future teachers are getting weak training mathematically and are just not prepared to teach the demanding mathematics curriculum we need for middle schools if we hope to compete internationally in the future".

3. In 1999 a study showed that 49% of seventh-grade math teachers did not have the equivalent of a minor in mathematics (see [3]).

4. It is a fact that many college bound students has to take remedial math courses. (see [4]).

5. In the US, we can't demand future middle school math teachers to finish even the sequence of Calculus courses. The rationale is if they do not need to teach Calculus, why would they need to finish Calculus courses?

6. Set standards for licensing teachers that are high enough to recruit from the top third of college graduates-See 10 steps to World-Class Schools (see [7]).

In view of these problems, many educators in the US wonder how we can expect future math teachers to do well in teaching if they do not have enough content knowledge. So how much math should future middle (or high) school teachers be required to learn? It is beyond the scope of this paper debating on the answer to this question but let's take a look at the situation in South Korea. The high level math courses demanded by Seoul National University (provided by Kwon Oh Nam) for a future middle school math teacher include Abstract Algebra, Real Analysis, Topology and etc. The immediate reaction from math educators in the US is that it is impossible to demand the same requirements in the US. We further note that one major difference between high school students in the US and other countries (let's take South

Korea for example) is that '*A Korean high school student does not have the luxury of selecting mathematics courses as electives; mathematics is something he or she must master to succeed, regardless of his or her future goals. Simply put, no mathematics means no college education in Korea*', (see [5]). It is beyond the scope of this paper to explore why we see this big difference between the system in the US and South Korea but one key reason for this happening in the US is that many students in the US choose professions other than teaching because the salary in the teaching field is not as competitive when compared to other professions.

There are many reasons why we see more and a higher level of content being required in the curriculum in many countries in Asian Pacific regions. Many countries do require some kind of entrance examinations for universities or even high schools. The high expectation from parents, teachers and communities ensure that students will spend longer hours in doing homework. Subsequently, it is not surprising that the International Mathematical Olympiad (IMO) Bu-Shi Ban has become so popular in China. (For a brief history of IMO in China, (see [6]). Remember many students in Asia 'obey' and 'receive' what are being taught. They tend to have fewer questions in a classroom and teacher can cover more content at the end. Consequently, students have no choice but to accept and to finish taking whatever subjects that are required before they graduate. Many students will simply memorize formulae in order to pass a test without understanding how a formula is derived.

One immediate criticism from this is that students are less creative and innovative when it comes to solving real-life applications. In addition, too much emphasis on examinations will definitely turn away more potential students who might be interested in mathematics in the future. Students should be given opportunities to integrate what they have learned in class with real-life applications. Creative and innovative thinking skills become crucial to some mathematics curricula.

3 Creativity does not come from drill but from exploration

It is evident that content knowledge and creativity are important to a mathematics curriculum. It is mentioned that basic skills and basic knowledge are two basics in [16]. We need to constantly examine and update which skills and knowledge are needed for the learner. Indeed, we see some mathematics reform towards more technology based in many Asian Pacific regions.

- In 2000, Ministry of Education in Taiwan: Mathematics should be accessible to 80% of students. Complicated algebraic manipulations can be replaced by using calculators and/or computers.

- In 2001, 'the standard of the National Curriculum is set preliminarily' in China: The "fill the duck style" (rote learning style) should be replaced by solving more real life problems. The following types of questions have disappeared from the exams when scientific calculators were allowed in Shanghai area: Compare $6^{0.7}, 0.7^6$, and $\log_{07} 6$; find $\log(20) + \log_{100} 25$. They have also proposed diverse standards of measuring students' success instead of basing success on testing alone.

- Ministry of Education in Singapore as of September, 2009: A-level candidates can use approved graphic calculators (non-CAS enabled). These are solely for A-level Mathematics exams and not other subjects (such as Physics). The graphics calculators are not approved in O-level exams. (Thanks to Ang Keng Cheng for this information.)

- The various Australian states have their own rules regarding calculator use in the senior secondary years. Several states such as Western Australia, Victoria, Tasmania, and Queensland are using or moving towards CAS use in schools, and in external examinations, which are critical for university entrance. Timetables differ from place to place. Syllabuses and assessments are different among states. There is no national examination or syllabus. (Thanks to Barry Kissane for this information).
- In India, 7000 high schools are experimenting with MathLab projects.
- Malaysian MOE also actively provided graphics calculators to school teachers many years ago. Technological tools have been incorporated into math curriculum. The goal of achieving 'one laptop for one math or science teacher' was realized.
- France in 2004: MOE added technology competency as part of the oral examination (in math graduate programs) for future math teachers.

In view of what we described we see technological tools playing important roles in major examinations in many countries already. Consequently, teachers have to redesign their content accordingly and hope more classroom explorations (with technological tools) will keep more students interested in mathematics. In the next two subsections, we describe some classroom explorations which might help students comprehend difficult concepts or even discover some mathematics.

3.1 Memorizing a formula will only last for a test

I conjecture that if we introduced more geometric and graphic representations before algebraic manipulations, many students may have not lost their interests in mathematics so early. We hope evolving technological tools can assist pre-service teachers in learning adequate content in mathematics before they start teaching. We should be reminded that there are new areas that await each learner to discover thanks to new advanced technological tools.

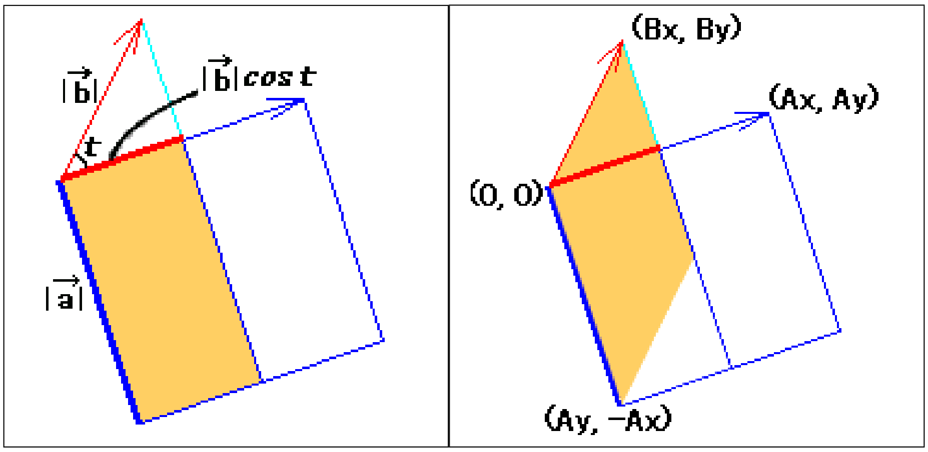
Confucius said 'Give a man a fish and you feed him for a day; teach a man to fish and you feed him for a lifetime'. I would say: Memorize a formula and you pass one exam; comprehend a formula and you discover more mathematics in a lifetime.

I recalled when I was in high school, I was told the distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}, \tag{1}$$

I, like many other classmates, hardly cared how this formula was derived because there is no time to worry about the derivation during a 50 minute test.

There are two important concepts in vector calculus, one is the dot product, $\overrightarrow{a} \cdot \overrightarrow{b}$, and the other is the cross product, $\overrightarrow{a} \times \overrightarrow{b}$,where $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors in R^n. However, most students would only remember the definition of $\overrightarrow{a} \cdot \overrightarrow{b}$ as $\sum a_i b_i$ or $|a|\ |b| \cos\theta$; or interpret this wrongly as the area of the parallelogram determined by $\overrightarrow{a}$ and $\overrightarrow{b}$ when $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors in R^2. Some students disagree that $\overrightarrow{a}\cdot\overrightarrow{b}$ represents the area of the parallelogram determined by $\overrightarrow{a}$ and $\overrightarrow{b}$, not because they understand completely the geometric interpretations of $\overrightarrow{a}\cdot\overrightarrow{b}$ and $\overrightarrow{a}\times\overrightarrow{b}$ respectively, but because they memorize the formula that the area of the parallelogram determined by $\overrightarrow{a}$ and $\overrightarrow{b}$ should be $\left\|\overrightarrow{a}\times\overrightarrow{b}\right\|$ but not $\overrightarrow{a}\cdot\overrightarrow{b}$. Thus we will explore $\overrightarrow{a}\cdot\overrightarrow{b}$ and $\overrightarrow{a}\times\overrightarrow{b}$ in more detail. For two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in 2D, $\overrightarrow{a}\cdot\overrightarrow{b}$ represents the area of the parallelogram determined by $\overrightarrow{a}$ and $\overrightarrow{b}_{\perp}$,where $\overrightarrow{b}_{\perp}$ is the vector that is perpendicular to $\overrightarrow{b}$, and with the same magnitude as $\overrightarrow{b}$. We summarize this by observing the Figures 1(a) and 1(b), and encourage readers to explore this further by using the Java applet developed by IES of Japan (see [8]).



Figures 1(a) and 1(b) Dot product

For two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in 2D, it follows from the definition of cross product $\overrightarrow{a}\times\overrightarrow{b}=(\|a\|\,\|b\|\sin\theta)\,\overrightarrow{n}$, where $\overrightarrow{n}$ is a unit vector perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$ and whose direction is given by the right-hand rule, that $\left\|\overrightarrow{a}\times\overrightarrow{b}\right\|$ *is equal to the area of the parallelogram determined by* $\overrightarrow{a}$ *and* $\overrightarrow{b}$. The java applet in [9] allows us to visualize this fact from a geometric point view.

Combining the ideas we explored on $\overrightarrow{a}\cdot\overrightarrow{b}$ and $\overrightarrow{a}\times\overrightarrow{b}$, it is not hard to understand why the volume of the parallelepiped determined by three vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ in 3D can be written as

$$\left|\overrightarrow{a}\cdot\left(\overrightarrow{b}\times\overrightarrow{c}\right)\right| \tag{2}$$

$$= \|a\|\left\|\overrightarrow{b}\times\overrightarrow{c}\right\|\left|\cos\theta\right|, \tag{3}$$

where θ is the angle between the normal vector $\overrightarrow{n}$ of the plane determined by $\overrightarrow{b}$ and $\overrightarrow{c}$ and the vector $\overrightarrow{a}$. This can be further explored by using a Java applet (see [10]). Furthermore, the quantity $\|a\|\,(|\cos\theta|)$ *represents the height of the parallelepiped and it can also be used to find the distance from a point to a plane.* Furthermore, we discover that the volume of the parallelepiped determined by three vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$, is actually equal to the area of a parallelogram determined by $\overrightarrow{a}$ and $\left(\overrightarrow{b}\times\overrightarrow{c}\right)_{\perp}$, which is a vector that is perpendicular to $\overrightarrow{b}\times\overrightarrow{c}$ and with the same magnitude as $\overrightarrow{b} \times \overrightarrow{c}$; we note that vector $\left(\overrightarrow{b} \times \overrightarrow{c}\right)_{\perp}$ can be any vector lying on the plane determined by $\overrightarrow{b}$ and $\overrightarrow{c}$. We 'discovered' an additional observation, which is not mentioned in a regular textbook. Thus, it is not hard to conjecture that more interesting mathematics concepts can be explored and discovered thanks to advancing technological tools.

We demonstrate below that the proof of a traditional theorem can sometimes be done from a geometric construction, which makes the proof more interesting and intuitive.

Example 1 *About the proofs for the Mean Value or Cauchy Mean Value Theorems. These two theorems are very often used in applied mathematics. However, when it comes to the proof of either one of these theorems, it is not surprising that not many people can recall the proof.*

For example, the Mean Value Theorem can be stated below:

Suppose the function $f : [a, b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point x_0 in (a, b) at which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

To prove this theorem, in many traditional text books, one introduces the function h defined at each number x by the following equation:

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a). \tag{4}$$

Then we use the fact that h satisfies the conditions for Rolle's theorem to deduce that there is a point c in (a, b) such that $h'(c) = 0$, and the Mean Value Theorem follows. However, we can inspire students to see how the function h is constructed from a geometric point of view. Suppose the blue curve (the darker curve) is given by $f(x) = \cos(x)$ and satisfies the conditions of the Mean Value theorem over the interval covering $(a, b) = (-\frac{\pi}{2}, 0.725)$ shown in Figure 2 below. We connect the line segment AB, where $A = (-\frac{\pi}{2}.0)$ and $B = (0.725, f(0.725))$ lying on $y = f(x)$ and ask the following question:

If we rotate line segment AB (while AB is attached to the graph of the function) *so that AB becomes a horizontal line segment, how would the graph of the original function appear?* We refer readers to the paper in [11], and explore that the curve in green or the lighter color is exactly the function $h(x)$ we are looking for (by observing distance EF = distance GD) and we apply the Rolle's Theorem on $h(x)$.

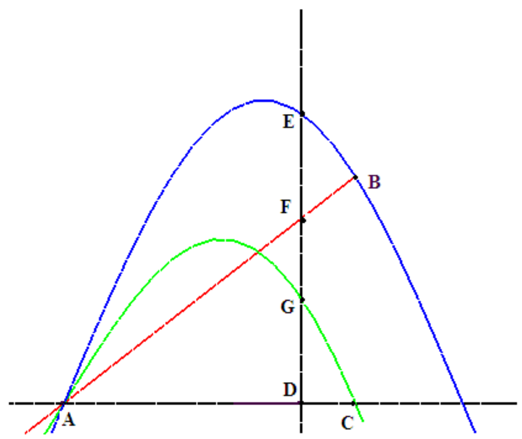


Figure 2. The graphs of two functions and a chord

We can prove the Cauchy Mean Value Theorem (CMV) in a similar manner. The statement of CMV can be seen below:

Suppose the function $f : [a, b] \to R$ and $g : [a, b] \to R$ are continuous and that their restrictions to (a, b) are differentiable. Moreover, assume that $g'(t) \neq 0$ for all t in (a, b). Then there is a point t in (a, b) at which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t)}{g'(t).}$$

We (again) see little geometric motivation of why we have two functions f and g and how the conclusion is obtained. In [11], we summarize how we can prove the CMV geometrically by extending the idea we described above when proving the MVT.

1. Assume functions f and g satisfy the condition of the Cauchy Mean Value Theorem, the Theorem holds, can be interpreted as any number t for which the parametric curve P defined by the equation
$$P(t) = [g(t), f(t)]$$
for $a \leq t \leq b$ has slope equal to the slope of the secant that runs from the point $(g(a), f(a))$ to the point $(g(b), f(b))$.

2. Equivalently, if we apply the Mean Value Theorem to the graph of a polar equation $r = h(t)$, by writing the polar equation in a parametric form
$$[x(t), y(t)] = [h(t)\cos(t), h(t)\sin(t)] = [g(t), f(t)], \tag{5}$$
we obtain the conclusion of Cauchy Mean Value Theorem (see [11] for details).

Through this example, we see that graphical and geometrical animations do make mathematics more accessible and interesting to more learners. Computer Algebra System (CAS) provides the thrust to the analytical proofs and makes mathematics challenging to some learners too.

3.2 Content evolve when technological tools advance

We further use the following two examples to demonstrate that traditional content may be integrated with more real-life applications. More importantly, learners will discover more mathematics as technological tools advance. This fits well with why the National Science Foundation of the USA is pushing the Science, Technology, Engineering and Mathematics (STEM) Program (see [12]). We need to integrate mathematics teaching with other applied disciplines. We demonstrate this by using the following example. Students learn the concept of finding the inverse for a function, $f(x)$, in Pre-Calculus is to find a function $g(x)$ so that the graphs of $y = f(x)$ and $y = g(x)$ are symmetric to $y = x$.

Example 2 *We extend the ideas of finding the inverse image of a parametric curve with respect to $y = mx+b$ to the idea of finding the reflection of a light source B on a curve C_1 with respect to a moving point P on C_2, which we call B'. The* ***locus of the point*** *B' is called the orthotomic curve and it is linked to the concept of a* ***caustic curve.*** *The concepts can be extended to the corresponding concepts in 3-D. The complete description on this problem can be found in [13].*

This problem can be explored with knowledge from complex concepts in the area of Optics in Physics with ease. We refer to Figure 3(a) below, where $C_1 = [x_1(t), y_1(t)] = [2\cos t - \cos 2t, 2\sin t - \sin 2t]$, $t \in [0, 2\pi]$ (shown as cardioid), and $C_2 = [s, f(s)] = [s, -2 + \sin(s)]$ (shown as a sinusoidal curve below). We pick a light source B on C_1, and set $C_3 = [p(t), q(t)]$ to be the reflection of C_1 with respect to the tangent line to C_2 at a point P. For a fixed point B on C_1 (light source at a point on C_1), the reflection of B on a curve C_1 with respect to a moving point P on C_2, which we call B', the **locus of the point** B' is called the *orthotomic curve.* Then we have the following observations:

1. The orthotomic curve of C_2 relative to B is shown in Figure 3(b) which can be experimented with by using a dynamic geometry software such as Geometry Expression (See [19) and verified by using a CAS such as Maple (See [20]).

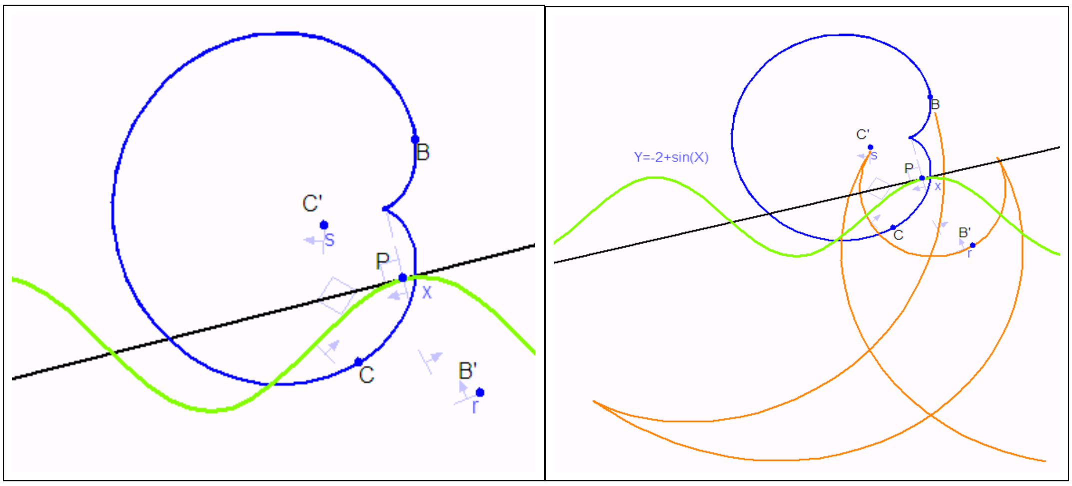


Figure 3(a) Original Curve and Figure 3(b) Original and its Orthotomic Curve

2. Picking another light source C on C_1 we obtain another orthotomic curve of C_2 relative to C (shown in black or darker curve in Figure 3). We immediately discover the following observation by the 'dragging' mode with Geometry Expression. (see Figure 3(c)): *As B approaches C, the orange orthotomic curve (or the lighter curve) approaches the black orthotomic curve (the darker curve).*

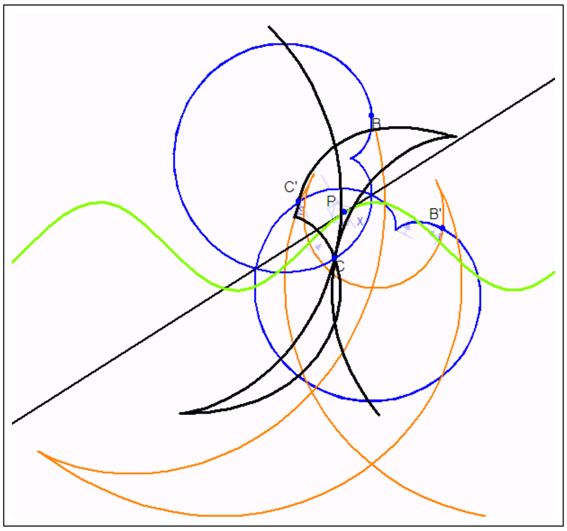


Figure 3(c) Two Orthotomic Curves

3. The sharp corner (cusp) of the black orthotomic occurs at the inflection point of C_2.

In the past, all of the observations above would have been difficult to be realized without a proper technological tool.

In Optics, we have often heard the term ‘caustic curve’. This can be viewed as the envelope of rays reflected by a curve. We first note that the *evolute of a curve* C is the set of all its centers of curvature; it is equivalent to the envelope of all the normals to C. It can be shown that *the caustic generated by rays reflected by a curve* C *from a light source* O *(caustic of* C *relative to* O*) is equivalent to finding the evolute of the orthotomic of* C *relative to* O*. Or equivalently, the caustic curve is the centers of curvature of the family of orthotomic normals of a given curve relative to* O. We shall use a java applet (see [15]) to explore the relationship between the orthotomic and caustic curves, which are difficult concepts otherwise.

- Step 1: Create the original curve so that it roughly resembles an ellipse (see Figure 4):

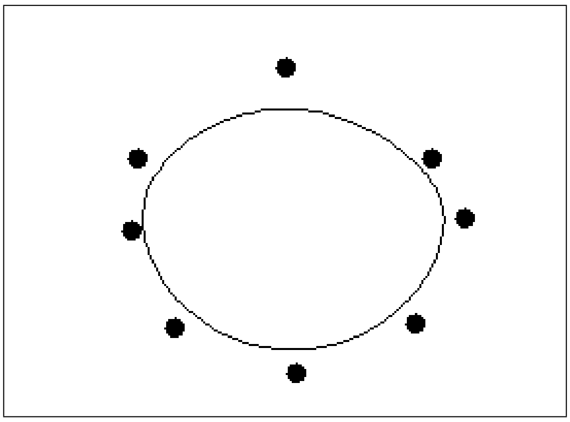


Figure 4. Creating a curve

- Step 2: Next, select the curves and click on 'Orthotomic Curve', the picture should look like Figure 5. Note the red center dot is the light source.

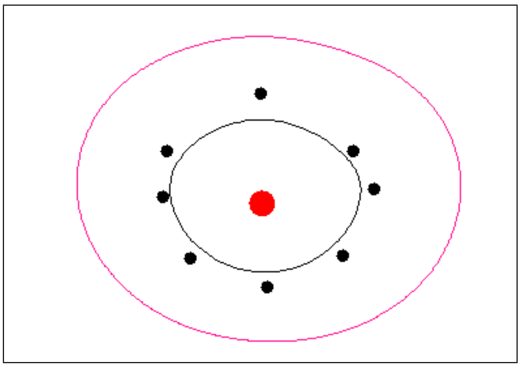


Figure 5. Original curve and its orthotomic curve

- Step 3: Now choose the Family of Orthotomic Normals, the picture should look like Figure 6:

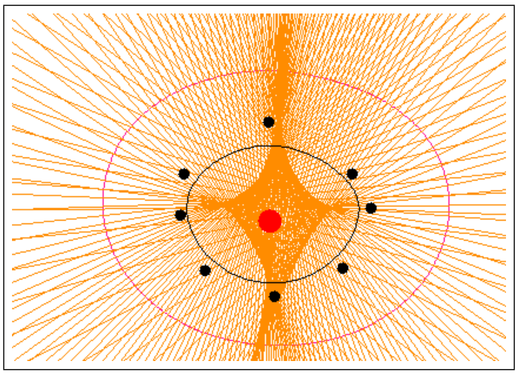


Figure 6. Original curve, its orthotomic and the evolute of the orthotomic

- Step 4: Click on the Caustics and note the following graph, we see that the caustic curve is the set of *centers of curvature of the family of orthotomic normals of a given curve relative to O* (the red or center dot).

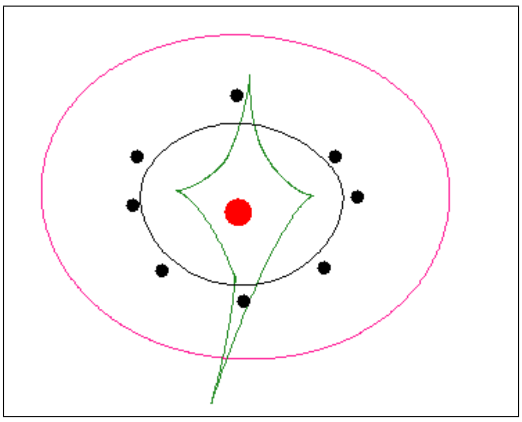


Figure 7. Original curve, its orthotomic and the caustic curve

These geometric observations coincide with algebraic approaches and can be verified when a computer algebra system such as Maple is used. For example, we consider the ellipse $[\frac{7}{5}\cos s, \frac{6}{5}\sin s]$, where $s \in [0, 2\pi]$, the orthotomic curve of the ellipse relative to the origin, O, can be shown (see [13]) to be

$$\left[\begin{array}{c} -\sin(2\arctan(\dfrac{6\cos s}{7\sin s}))(-\frac{6}{5}\sin s - \dfrac{6\,(\cos(s)^2)}{5\sin s}) \\ -\cos\left(2\arctan\left(\dfrac{6\cos s}{7\sin s}\right)\right)\left(\frac{-6}{5}\sin s - \frac{6}{5}\dfrac{(\cos s)^2}{\sin s}\right) + \frac{6}{5}\sin s + \dfrac{6\,(\cos s)^2}{5\sin s}. \end{array}\right] \tag{6}$$

We plot the original ellipse (in green or inner ellipse), its orthotomic curve (in blue or outer ellipse), and its caustic curve, can be shown to be the curve shown in red (or in the center of the inner ellipse), together in Figure 8.

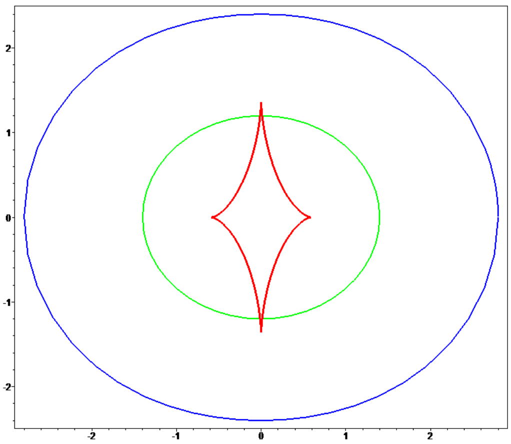


Figure 8. An ellipse, orthotomic, and caustic curve

We notice that Figure 7 is indeed similar to Figure 8 and yet Figure 7 was obtained by exploring the concepts from a geometric point of view and Figure 8 is obtained after some thorough theoretical and algebraic verifications. Obviously, geometric approaches provide critical intuition and motivations to learners and algebraic approaches challenge those students who want to do more. This example also shows that many complex concepts in applied disciplines can be explored from a mathematical point of view when learners are empowered with proper knowledge both in content and also in skills of manipulating latest technological tools.

Next we explore one example of finding the global minimum value of *total squared distances* among non-intersecting curves in the plane and surfaces in the space. The geometric interpretations will help students appreciate the use of the Lagrange multipliers method and concepts learned from Linear Algebra.

Example 3 *We are given four convex surfaces in the space, represented by the orange, yellow, blue and purple surfaces (shown in Figure 9) which we will call S_1, S_2 S_3 and S_4 respectively. We want to find points A, B, C and D on S_1 (orange or the one at the left lower corner), S_2 (yellow or the one at the right lower corner), S_3 (blue or the one at the upper right) and S_4*

(the purple or the one at the upper left) respectively so that the total distances $AB + AC + AD$ achieves its minimum.

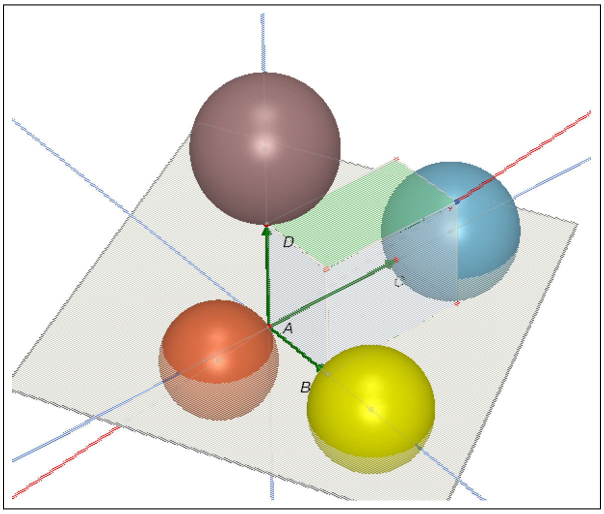
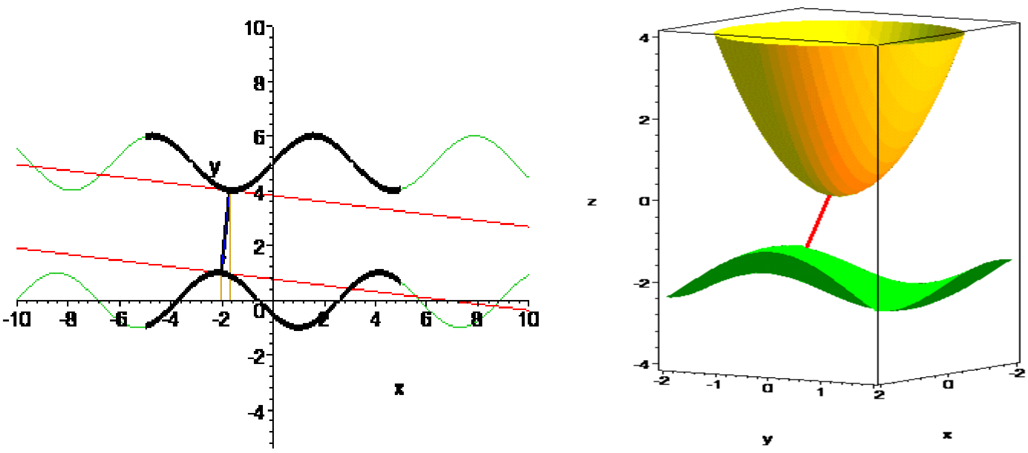


Figure 9. The shortest total squared distances among four convex surfaces (thanks to J.J. Dahan)

The details of this example can be found in [14]. This problem was started when we ask the following preliminary questions:

- If we are given two non-intersecting curves or surfaces in their respective domains, what is the shortest distance between these two curves and surfaces? With the exploration using a dynamic geometry software, it is not difficult to see the minimum distance occurs when the line segment connecting two points at two respective curves or surfaces is perpendicular to the each of the tangent lines or tangent planes at respective points. We demonstrate this by noting the following two Figures 10(a) and 10(b).



Figures 10 (a) and (b) Minimum distance between two curves and two surfaces.

- We extend the idea to finding the total shortest squared distances from one curve (say sinusoidal curve below) to two other curves-one is parabolic and the other one is a circle-by observing the following Figure 11. Does this diagram say anything about how we should position the normal vector at the point on the sinusoidal curve in relation to two other vectors? This will be clear later.

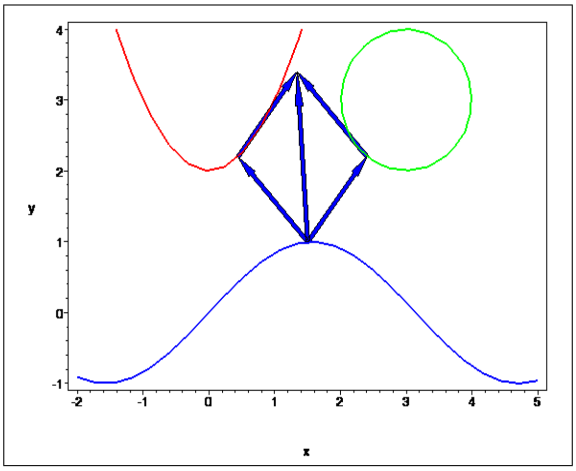


Figure 11. Shortest total squared distance from one curve to two other curves.

We note that the key observations for finding the shortest total squared distances $AB + AC + AD$, described in Figure 9, from a geometric point of view can be summarized as follows:

1. The vector AB should be parallel to the normal vector of the surface S_2 at B. This is equivalent to
$$AB = \lambda_2 (\nabla S_2 \text{ at } B) \text{ for some } \lambda_2. \tag{7}$$

2. The vector AC should be parallel to the normal vector of the surface S_3 at C. This is equivalent to
$$AC = \lambda_3 (\nabla S_3 \text{ at } C) \text{ for some } \lambda_3. \tag{8}$$

3. The vector AD should be parallel to the normal vector of the surface S_4 at D.This is equivalent to
$$AD = \lambda_4 (\nabla S_2 \text{ at } D) \text{ for some } \lambda_4. \tag{9}$$

To achieve the minimum distance for $AB + AC + AD$, we should also place point A so that the normal vector of S_1 at A is in the same direction of $AB + AC + AD$. This is equivalent to say we can find λ_1 so that

$$\lambda_1 (\nabla S_1 \text{ at } A) = \lambda_2 (\nabla S_2 \text{ at } B) + \lambda_3 (\nabla S_3 \text{ at } C) + \lambda_4 (\nabla S_2 \text{ at } D). \tag{10}$$

The following Theorem sums up what we discussed above, the proof can be found in [14].

Theorem 4 *If the total squared distances function*

$$f(\mathbf{x}_1, \mathbf{x}_2, , , , \mathbf{x}_p) = |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_1 - \mathbf{x}_3|^2 + ... |\mathbf{x}_1 - \mathbf{x}_p|^2 \text{ has a global value,} \tag{11}$$

where $\mathbf{x}_i = (x_1^i, x_2^i, ..., x_n^i)$, $i = 1, 2, ...p$,*subject to* p *constraints*

$$g_1(\mathbf{x}_1) = c_1, g_2(\mathbf{x}_2) = c_2, ..., \text{ and } g_p(\mathbf{x}_p) = c_p, \tag{12}$$

at $\mathbf{x}_0 = (x_1^*, x_2^*, , , , x_p^*)$ *in its closed and bounded domain, then we can find coefficients,* $\lambda_j, j = 1, 2, ...p$, *so that*

$$\lambda_1 \nabla g_1(\mathbf{x}_0) = \sum_{j=2}^{p} \lambda_j \nabla g_j(\mathbf{x}_0). \tag{13}$$

This example further shows that there are plenty of interesting problems awaiting us to explore if students are properly introduced to the latest technological tools and empowered with proper content knowledge. It shows the importance for students to be able to integrate concepts they learned in Calculus and Linear Algebra. Therefore, when we consider mathematics reform, we need to consider hyperlinking the content properly. This example further demonstrates that geometrical interpretations of a problem will provide crucial intuition and motivation for students grasping key concepts when solving a problem. The intuition will further assist students setting up conjectures and strategies of how to manipulate a CAS timely and properly to verify their conjectures.

4 Conclusion

In the advent of advancing and evolving technological tools, we are empowered to experiment with problems in analytical geometry, and even differential geometry (see [17], and [18]) which were difficult to do otherwise. As we have seen from the discussion above, mathematics curricula differ from country to country, we can only adopt global views to solve our local problems. Decision makers need to adopt the global views by knowing what technological tools are available and make the best possible decision in their respective countries. Many will agree it is necessary for future middle school math teachers to finish (at least) Calculus courses in the US. It is, however, difficult to implement this recommendation (for the time being). Nevertheless, it is possible that technological tools can assist us by introducing Calculus concepts intuitively from a graphical and a geometric point of view rather than from an algebraic point of view which is more abstract. While it is definitely applaudable that math curricula demand many in-depth content knowledge covered in many Asian Pacific regions, we also need to be reminded about the followings:

1. Do students simply memorize the proof of a theorem?

2. Are students allowed to do project oriented problems? It is not how much a teacher can cover within a semester, it is how much students are able to apply.

3. We are interested in students who can apply their math skills throughout their lives not just to pass an exam.

4. Teaching to the test will only turn more students away from liking mathematics.

5. Mathematics should be taught in cross-disciplinary manner. It should be a subject that can be applied in the real-world.

Providing adequate content knowledge and inspiring creative thinking skills are pivotal in a mathematics curriculum. Technology definitely can not solve all our problems but will assist us in achieving the balance between content knowledge and creativity. Implementing technological tools into teaching and learning is not a trivial task and it will be an on-going pedagogical issue for many years to come. Many students may have lost confidence or interests before entering universities because of the deficiency in algebraic manipulation skills. It is therefore imperative, in my view, to build a curriculum where teachers need to know when and how to introduce a subject with lots of intuition and motivations so mathematics is more accessible and interesting to more students at younger ages.

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