Volumes of Solids of Revolution via Summation Methods

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Abstract: In this paper, we will show how to calculate volumes of certain solids of revolution without using direct integration. The traditional method of such volume computation uses definite integrals as given by Disk Method or Shell Method in a calculus course. However, instead of direct integration, we will calculate these volumes as a limit of a summation. Even though somewhat longer then the traditional method in general, this method emphasizes the fundamental idea behind a definite integral, i.e. the definite integral as the limit of a sum. We will also use the computer algebra system *Mathematica* to facilitate and verify our calculations.

1. Introduction

In this section, we will briefly review the Disk Method and the Shell Method of finding the volume of a solid of revolution (see [3]).

Consider a nonnegative continuous function $y = f(x)$ defined on a closed interval $[a,b]$ where a and b are real numbers with $a < b$. Let $\Re$ be the region bounded by the graphs of $y = f(x)$, $y = 0, x = a$, and $x = b$. Let V be the volume of the solid obtained by rotating the region $\Re$ around the x-axis. See the following figure:

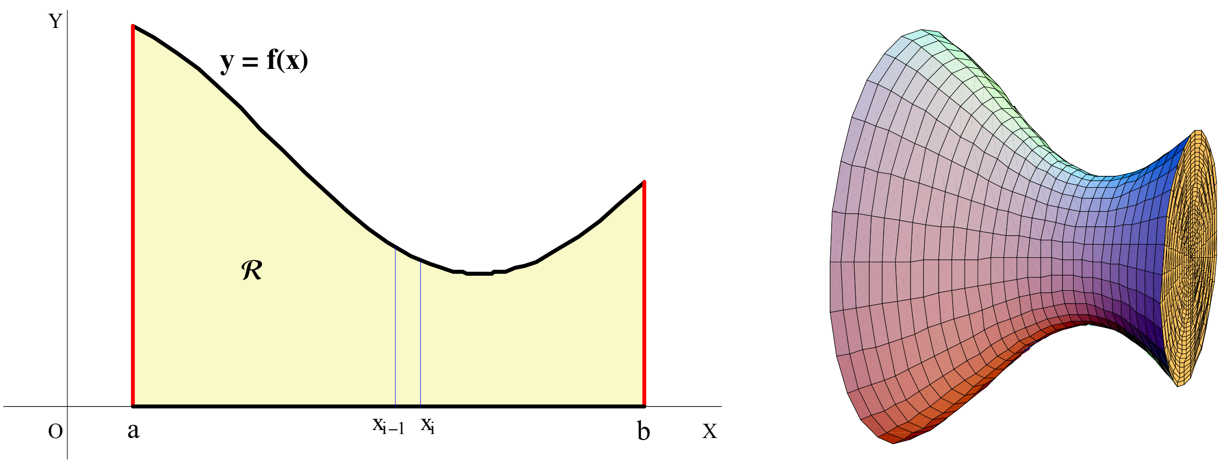

Figure 1.1 The region $\Re$ under the graph of $y = f(x)$, and the solid obtained by revolving this region around the x-axis

According to the Disk Method, the volume *V* is given by the following definite integral (see [3]):

$$V = \pi\int_a^b [f(x)]^2\,dx \qquad (1.1)$$

The above formula (1.1) is based on the volume of a cylinder, which is given by $\pi(radius)^2(height)$. For example, suppose we divide the interval $[a,b]$ into *n* equal pieces, where *n* is a natural number, using the partition $a = x_0 < x_1 < ... < x_{1-1} < x_1 < ... < x_n = b$. Let Δx be the length of any subinterval $[x_{i-1}, x_i]$, $i = 1, 2,...., n$. Then we have the following formulas:

$$\Delta x \;=\; (b-a)/n \qquad (1.2)$$

$$x_i \;=\; a \;+\; i(b-a)/n \qquad (1.3)$$

We can cut the solid into slices perpendicular to the *x*-axis, at the numbers x_i where $i = 1,2,...,n$. Each slice is approximately a thin cylinder with height Δx and radius $f(x_i)$, so its volume ΔV_i is approximately given by $\pi[f(x_i)]^2\Delta x$. Therefore, an approximation for the volume *V* of the solid is obtained by adding all the smaller volumes ΔV_i, as given below:

$$V \;\approx\; \sum_{i=1}^{n} \pi[f(x_i)]^2\,\Delta x \qquad (1.4)$$

The actual volume *V* of the solid is given by taking the limit of the above summation as $n \to \infty$:

$$V \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2\,\Delta x \qquad (1.5)$$

In the next section, we will illustrate how to compute certain volumes using the above formula (1.5), without directly evaluating the integral given by (1.1).

Now to illustrate the Shell Method, let *W* be the volume of the solid obtained by rotating the above described region $\Re$ around the *y*-axis. According to the Shell Method formula *W* is given by the following integral (see [3]):

$$W = 2\pi\int_a^b x\,f(x)\,dx \qquad (1.6)$$

The above integral is based on the volume of a shell, i.e. the space between two cylinders, which is given by $2\pi(radius\ of\ the\ shell)(height\ of\ the\ shell)(thickness\ of\ the\ shell)$ (see [3]). Corresponding to equation (1.5), the summation version of the integral (1.6) is given by the following equation:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\, x_i\, f(x_i)\, \Delta x \qquad (1.7)$$

In the next sections we will illustrate how to use formulas (1.5) and (1.7) to calculate volumes corresponding to several types of functions $f(x)$, instead of using direct integration.

2. The Volume of a Solid Generated by Revolving the Region Under a Square Root Function Around the *x*-Axis

Consider the function $f(x) = k\sqrt{x}$ over the closed interval $[0, b]$, where k and b are real positive constants. Let $\Re$ be the region bounded by the graphs of $y = f(x), y = 0, x = 0$, and $x = b$. In this section, we will compute the volume V of the solid obtained by rotating the region $\Re$ around the *x*-axis, using equation (1.5). See the following figure:

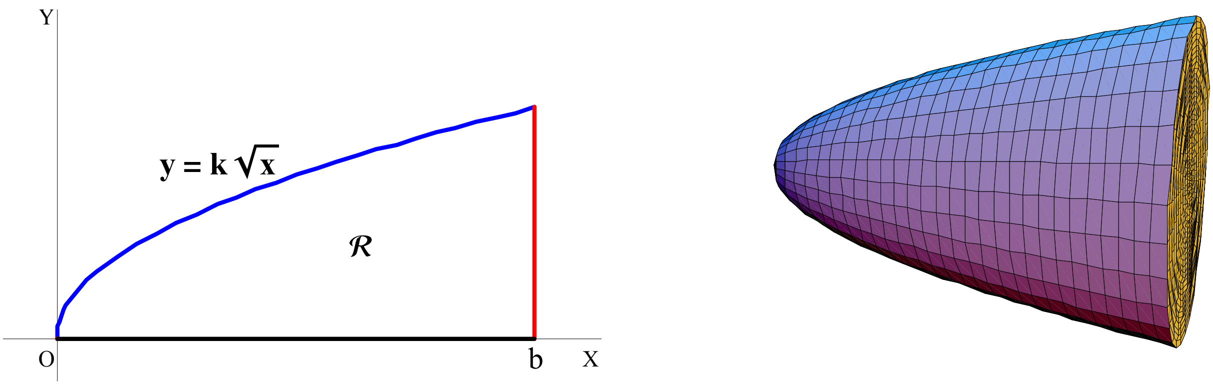

Figure 2.1 The region $\Re$ under the graph of $f(x) = k\sqrt{x}$, and the solid obtained by revolving this region around the *x*-axis

By employing the same notation as given in section 1, we can see that $\Delta x = b/n$, and $x_i = a + i(b-a)/n = ib/n$, for $i = 1, 2, \ldots, n$. Then using equation (1.5), one can obtain the following:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[k\sqrt{\frac{ib}{n}} \right]^2 \frac{b}{n} = \lim_{n\to\infty} \left\{ \frac{\pi k^2 b^2}{n^2} \left(\sum_{i=1}^{n} i \right) \right\} \qquad (2.1)$$

The summation involved in equation (2.1) is a well-known quantity in mathematics, and is given by the following (see [1] and [3]):

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (2.2)$$

By using (2.2) in equation (2.1), we are able to calculate the required volume V:

$$V = \lim_{n\to\infty}\left\{\frac{\pi k^2 b^2}{2}.\frac{(n+1)}{n}\right\} = \frac{\pi k^2 b^2}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{\pi k^2 b^2}{2}$$

One can of course, verify the above result by direct integration: Using the Disk Method formula give by equation (1.1), $V = \pi\int_0^b (k\sqrt{x})^2\,dx = \pi k^2\,[x^2/2]_0^b = \pi k^2 b^2/2$, which agrees with the result just obtained.

In the next section, we will consider a slightly more challenging problem arising from a quadratic function. Obviously, as the function $f(x)$ becomes more complicated, the corresponding summation and limit calculations become tedious. However, in the course of the paper, the reader will be surprised to observe that one can still use the summation method to find the volumes corresponding to a wider class of functions, including polynomials, exponential functions, logarithm functions, and also sine and cosine functions.

3. The Volume of a Solid Generated by Revolving the Region Under a Quadratic Function Around the *x*-Axis

Consider the function $f(x) = k\,x^2$ over the closed interval $[0, b]$, where *k* and *b* are real positive constants. Let $\Re$ be the region bounded by the graphs of $y = f(x), y = 0, x = 0,$ and $x = b$. In this section, we will compute the volume *V* of the solid obtained by rotating the region $\Re$ around the *x*-axis, using equation (1.5). Using the same notation as in section 1, we can express the volume *V* as the limit of a summation:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[k\left(\frac{ib}{n}\right)^2\right]^2\frac{b}{n} = \lim_{n\to\infty}\left\{\frac{\pi k^2 b^5}{n^5}\left(\sum_{i=1}^{n} i^4\right)\right\} \tag{3.1}$$

The summation $\sum_{i=1}^{n} i^4$ arising in equation (3.1) refers to the sum of the fourth powers of the first *n* positive integers, for which an expression can be found in most standard mathematical handbooks (see [1]). Alternatively, one can use a computer algebra system (CAS) such as *Mathematica* to evaluate this summation (see [2] and [5]). The *Mathematica* built in function "**Sum**" can calculate reasonable finite or infinite sums. Specifically, the input line **Sum [i ^ 4, {i, 1, n}]** calculates the required sum. Each *Mathematica* command can be executed by pressing "**Shift-Enter**" at the end of the command line. Thus, we can obtain the following result:

$$\sum_{i=1}^{n} i^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \tag{3.2}$$

Using equation (3.2) in (3.1), we are in a position to finish the computation for V:

$$V = \lim_{n\to\infty}\left\{\frac{\pi k^2 b^5}{30n^4}(n+1)(2n+1)(3n^2+3n-1)\right\} = \frac{\pi k^2 b^5}{30}\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(3+\frac{3}{n}-\frac{1}{n^2}\right)\right\} = \frac{\pi k^2 b^5}{5}$$

One can of course, verify the above result directly using the Disk Method formula (1.1). The method outlined in this section can be used to find the volumes of solids corresponding to any polynomial function $f(x)$, not just a quadratic function. However, as the degree of the polynomial gets higher, the hand calculations become more tedious, and therefore, one can use a CAS for advantage. The following *Mathematica* program automates the procedure for finding the volume by summation method (see [2] and [5]):

Program 3.1

```
Clear[f,a,b,deltax,v]
f[x_]:=  3x^3 + 2x^2 + x + 2
a =1;
b =3;
deltax = (b-a)/n;
x [i_]:= a + i(b-a)/n;
v[i_]:= Pi * f[x[i]]^2*deltax
vapprox = Simplify[Sum[v[i], {i, 1, n}] ];
v = Limit[ vapprox, n->Infinity]
```

In this program, the user can enter his or her own inputs for f[x_], a and b, and the program calculates the volume of the solid generated when the corresponding region is rotated about the *x*-axis. In the above, as an example, we have chosen $f(x) = 3x^3 + 2x^2 + x + 2$, $a = 1$, and $b = 3$. The program can be executed by pressing "**Shift-Enter**" at anywhere in the command lines. As the output, we obtain the volume as $36208\pi/7$. One can also verify this answer directly using the "**Integrate**" command of *Mathematica*, and the Disk Method formula (1.1), as shown below:

Input: Integrate[Pi(3x^3+2x^2+x+2)^2,{x,1,3}]

Press "**Shift-Enter**" at the end of the command line to obtain the output as $36208\pi/7$, confirming the previous answer.

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4. The Volume of a Solid Generated by Revolving the Region Under an Exponential Function Around the *x*-Axis

In this section, we would like to challenge ourselves by considering an exponential function for $f(x)$. Consider the function $f(x) = k e^{mx}$ over the closed interval $[0, b]$, where *k*, *m*, and *b* are real constants with *k*, *b* positive, and *m* nonzero. Let $\Re$ be the region bounded by the graphs of $y = f(x), y = 0, x = 0,$ and $x = b$. We would like to compute the volume *V* of the solid obtained by rotating the region $\Re$ around the x-axis, using the summation method. The equation (1.5) again implies the following:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(k\, e^{\frac{mbi}{n}} \right)^2 \frac{b}{n} = \lim_{n\to\infty} \left\{ \frac{\pi k^2 b}{n} \left(\sum_{i=1}^{n} e^{\frac{2mbi}{n}} \right) \right\} \tag{4.1}$$

Observe that $\sum_{i=1}^{n} e^{\frac{2mbi}{n}}$ is a geometric series with the initial term, and the common ratio each equal to $e^{2mb/n}$. Using the fact that the sum of the first n terms of a geometric series with initial term a and the common ratio r is given by $a(1-r^n)/(1-r)$ where $r \neq 1$, we obtain the following (see [3]):

$$\sum_{i=1}^{n} e^{2mbi/n} = \frac{e^{2mb/n}\left(1-e^{2mb}\right)}{\left(1-e^{2mb/n}\right)} \tag{4.2}$$

Substituting equation (4.2) into (4.1), we are able to compute the volume V, by making use of L'Hopital's Rule (see [3]):

$$\begin{aligned} V &= \lim_{n\to\infty} \left\{ \pi k^2 b(1-e^{2mb}) \frac{e^{2mb/n}}{n(1-e^{2mb/n})} \right\} = \pi k^2 b(1-e^{2mb}) \lim_{n\to\infty} \left\{ \frac{1}{n(1-e^{2mb/n})} \right\} \\ &= \pi k^2 b(1-e^{2mb}) \lim_{n\to\infty} \left\{ \frac{1/n}{(1-e^{2mb/n})} \right\} = \pi k^2 b(1-e^{2mb}) \lim_{n\to\infty} \left\{ \frac{-1/n^2}{-e^{2mb/n}(-2mb/n^2)} \right\} \\ &= \pi k^2 b(1-e^{2mb}) \left[\frac{-1}{2mb} \right] = \frac{\pi k^2 (e^{2mb}-1)}{2m} \end{aligned} \tag{4.3}$$

One can check the accuracy of the above calculation, using Program 3.1. By choosing, $f(x) = k\,e^{mx}$, $a = 0$, $b = b$, the resulting output is the same as volume V obtained above. Another way to verify the result directly is, by using the Disk Method formula (1.5), and the *Mathematica* "**Integrate**" command, as shown below:

Input: Integrate[Pi* (k*Exp[m*x])^2, {x,0,b}]

Press "**Shift-Enter**" anywhere in the command line to obtain the same answer above, verifying our calculations. ■

5. The Volume of a Solid Generated by Revolving the Region Under a Logarithm Function Around the *x*-Axis

Consider the function $f(x) = k \ln x$ over the closed interval $[1, b]$, where k, b are real constants with k positive, and $b > 1$. Let $\Re$ be the region bounded by the graphs of $y = f(x)$, $y = 0$, $x = 1$, and $x = b$. We would like to compute the volume V of the solid obtained by rotating the region $\Re$ around the x-axis, using the summation method. Using the same notation as in section 1, we see that $\Delta x = (b-1)/n$, and $x_i = 1 + i(b-1)/n$. Once the equation (1.5) is used to compute the volume V, the difficulty is immediately clear, since we are now faced with calculating $\sum_{i=1}^{n}[\ln(1+i(b-1)/n]^2$. It is very difficult to simplify this summation even with the help of a CAS, so we will approach the problem in an indirect way.

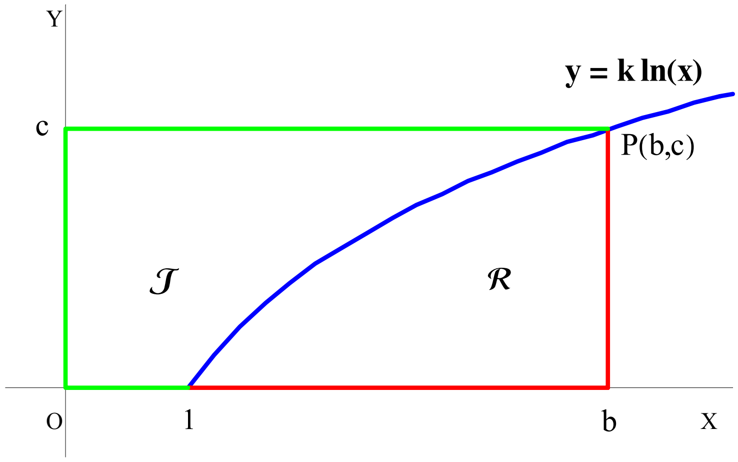

Figure 5.1 The regions $\Re$ and $\Im$ associated with the graph of $f(x) = k\ln x$

As given in the above figure, let $P(b,c)$ be the point on the graph of $f(x) = k\ln x$ corresponding to $x = b$. This means that $c = k\ln b$, or equivalently $b = e^{c/k}$, We now consider the region $\Im$ bounded by the graphs of $y = f(x)$, $y = c$, $y = 0$, and $x = 0$. Let V_1 be the volume of the solid generated when the region $\Im$ is rotated around the x-axis, and let V_2 be the volume of the cylinder generated when the region bounded by the graphs of $y = c$, $y = 0$, $x = 0$, and $x = b$ is rotated around the x-axis. Clearly, $V_2 = \pi c^2 b = \pi k^2 b(\ln b)^2$, and the required volume V is just equal to $V_2 - V_1$. See the following figure:

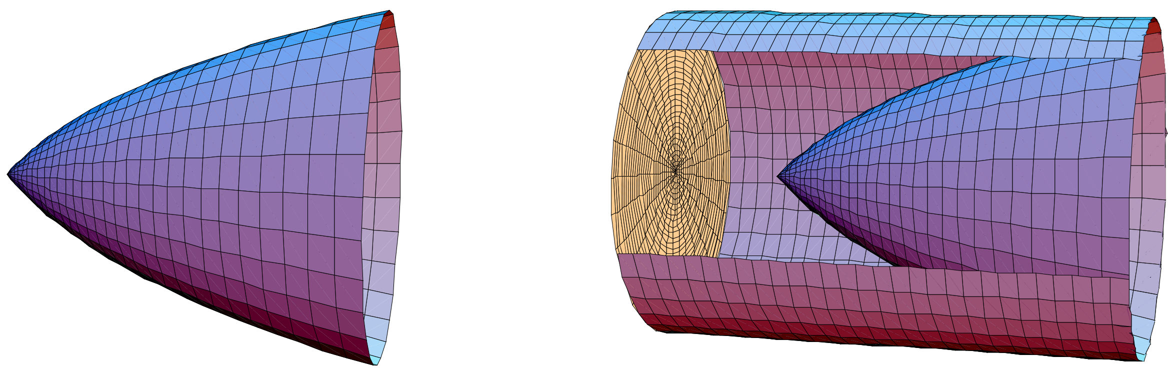

Figure 5.2 The solid obtained by revolving region $\Re$ around the x-axis, with volume V (on left), and the solid obtained by revolving the region $\Im$ around the x-axis, with volume V_1 (on right)

In order to find the volume V_1, we will use the summation version of the Shell Method formula given by equation (1.7), with *x* and *y* variables interchanged. Divide the interval $[0, c]$ of the *y*-axis into *n* equal pieces using the partition $0 = y_0 < y_1 < ... < y_{1-1} < y_1 < ... < y_n = c$, where *n* is a natural number. Let Δy be the length of any subinterval $[y_{i-1}, y_i]$, $i = 1, 2, ..., n$. It follows that $\Delta y = c/n$ and $y_i = ci/n$. When using the Shell Method as described in section 1, note that the radius of each shell is equal to y_i, while the height is equal to $e^{y_i/k}$. Thus, the volume element of a typical shell is approximately equal to $2\pi\, y_i\, e^{y_i/k} \Delta y$. By adding all these volumes, and taking the limits as $n \to \infty$, we can calculate V_1 as follows:

$$V_1 = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\, y_i\, e^{\frac{y_i}{k}}\ \Delta y = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \left(\frac{ci}{n}\right) e^{\frac{ci}{kn}} \left(\frac{c}{n}\right) = \lim_{n\to\infty} \left\{ \frac{2\pi c^2}{n^2} \left(\sum_{i=1}^{n} i\, e^{\frac{ci}{kn}} \right) \right\} \quad (5.1)$$

A series such as $\sum_{i=1}^{n} i\, e^{ci/(kn)}$ in above formula (5.1) is called an arithmetico-geoemtric series, because it is a hybrid of an arithmetic series and a geometric series, and can be computed by hand without much difficulty. However, we opt to use the "**Sum**" command of *Mathematica* to compute its sum, where the result is given below:

$$\sum_{i=1}^{n} i\, e^{\frac{ci}{kn}} = e^{\frac{c}{kn}} \left[\frac{1 - e^{c/k}}{(1 - e^{c/(kn)})^2} - \frac{n\, e^{c/k}}{(1 - e^{c/(kn)})} \right] \quad (5.2)$$

By substituting equation (5.2) into (5.1), we can proceed with the calculation for V_1. In the following, we used the L'Hopital's Rule to obtain $\lim_{n\to\infty} 1/[n(1 - e^{c/(kn)})] = -k/c$, similar to the calculation of equation (4.3), and the fact that $c = k \ln b$.

$$V_1 = 2\pi c^2 \lim_{n\to\infty} \left\{ e^{c/(kn)} \left[\frac{1 - e^{c/k}}{n^2 (1 - e^{c/(kn)})^2} - \frac{e^{c/k}}{n(1 - e^{c/(kn)})} \right] \right\} = 2\pi c^2 \left[(1 - e^{c/k}) \left(\frac{-k}{c} \right)^2 - e^{c/k} \left(\frac{-k}{c} \right) \right] \quad (5.3)$$

$$= 2\pi \left[k^2 (1 - b) + k^2 b \ln b \right] = 2\pi k^2 (1 - b + b \ln b)$$

We are finally in a position to calculate the required volume *V* as the difference of the volumes V_2 and V_1, as remarked earlier.

$$V = V_2 - V_1 = \pi k^2 b (\ln b)^2 - 2\pi k^2 (1 - b + b \ln b) = \pi k^2 [b (\ln b)^2 - 2b \ln b + 2(b - 1)]$$

As done in the previous sections, one can of course use the "**Integrate**" command of *Mathematica* to verify the accuracy of the above answer.

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6. The Volume of a Solid Generated by Revolving the Region Under a Sine or Cosine Function Around the *x*-Axis

As a final illustration, let us consider some trigonometric functions. Consider the function $f(x) = k\,Sinx$ over the closed interval $[0,\ \pi]$, where *k* is a positive real constant. Let $\Re$ be the region bounded by the graphs of $y = f(x),\ y = 0,\ x = 0,$ and $x = \pi$. Let us compute the volume *V* of the solid obtained by rotating the region $\Re$ around the *x*-axis, using the summation method. Using the same notation as in section 1, we find that $\Delta x = \pi / n$ and $x_i = \pi\, i / n$ for $i = 1,\ 2, \ldots, n$ where *n* is a natural number. Then equation (1.5) yields the following expression for the volume *V*:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[k\,Sin\left(\frac{\pi\, i}{n}\right)\right]^2 \frac{\pi}{n} = \lim_{n\to\infty} \left\{ \frac{\pi^2 k^2}{n} \left(\sum_{i=1}^{n} Sin^2\left(\frac{\pi\, i}{n}\right)\right)\right\}$$
$$= \lim_{n\to\infty} \left\{ \frac{\pi^2 k^2}{2n} \left(\sum_{i=1}^{n} \left[1 - Cos\left(\frac{2\pi\, i}{n}\right)\right]\right)\right\} = \lim_{n\to\infty} \left\{ \frac{\pi^2 k^2}{2n} \left(n - \sum_{i=1}^{n} Cos\left(\frac{2\pi\, i}{n}\right)\right)\right\} \tag{6.1}$$

Note that $\sum_{i=1}^{n} Cos(2\pi\, i / n)$ refers to a sum of the cosines of a sequence of angles in arithmetic progression, which can be computed by the following well-known formula, where α and β are real numbers with $Sin(\beta/2) \neq 0$ (see [1] and [4]):

$$\sum_{i=1}^{n} Cos\left[\alpha + (i-1)\beta\right] = \frac{Cos[\alpha + (n-1)\beta/2]\ Sin(n\beta/2)}{Sin(\beta/2)} \tag{6.2}$$

Using the above equation with $\alpha = \beta = 2\pi / n$, we can easily see that the required sum $\sum_{i=1}^{n} Cos(2\pi\, i / n)$ is equal to zero. Thus, the equation (6.1) yields the desired volume *V*:

$$V = \lim_{n\to\infty} \left(\frac{\pi^2 k^2}{2n} n \right) = \frac{\pi^2 k^2}{2} \tag{6.3}$$

The “**Integrate**” command of *Mathematica* readily verifies the above result for the volume *V*.

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Even though not included here, one can use the same method to find the volume of the solid of revolution corresponding to a cosine function.

Conclusion

In this paper, we showed how to calculate the volumes of certain solids of revolution using a summation method. The traditional method directly calculates these volumes via certain definite integrals based on either Disk Method or Shell Method, and is generally faster than the proposed method. However, the advantage of our method is that it emphasizes one of the most fundamental aspects of definite integral, i.e. the definite integral as the limit of a summation. Another important aspect of our method is that it gives students an excellent opportunity to deal with two other important aspects of calculus, namely summations and limits. Also our method creates a new appreciation for the definite integral and the Fundamental Theorem of Calculus (FTC), as FTC provides an effective short-cut for the summation method. The paper also emphasizes the usage of a CAS, without sacrificing the hand calculations altogether. The student is encouraged to try the summation method described in this paper to calculate volumes of solids of revolutions arising from other types of functions.

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