Construction of Quadratic Curves Using the Analysis

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Abstract: *This research aims to observe the process in which students participate in constructing three quadratic curves tended to be overlooked in the current climate of school instruction. In this process, our aim was to have the students learn how to construct the quadratic curves logically by using the Analysis, and recognize its definition and properties of several geometric figures that they had learned while proving justification of the constructing. We taught two 10th grade girls who hadn't learned quadratic curve, about the parabola, ellipse, and hyperbola. Then, we had those girls construct quadratic curves on a plane. In conclusion, the Analysis suggested effective processes for the students to find out how to construct quadratic curve. Finally, it was found that GPS facilitated the students to learn the construction more conveniently by offering the numerical and visual feedback*

1. Introduction

1.1 Purpose and Significance of the Research

Quadratic curves in Korea's current high school curriculum are instructed so that the geometric relationship between the focal points and curves is induced first by an equation, then by considering the properties of each figure, and the related problems that ask for the focal point or the distance is solved. However, in observing the processes that the students go through in solving problems that inquire about the definition or concept of quadratic curves, it is quite clear that the students do not have a full understanding of the relationship between the points that consists the quadratic curves. Therefore, in the instruction of quadratic curves, classroom activities that involve constructing the points of quadratic curves from the definitions given in each problem are strongly called for. However, it is quite difficult for teachers to properly utilize different kinds of manipulative material in order to embody each curve by their definitions. Many strategies have been developed but most of them were deemed hard to use, so instruction would not go beyond such statements as 'This is what we get from first dictating the definition and displaying it on a plan.' Such instructional method is also a violation of the purpose of analytic geometry that argues that mathematical concepts are better understood after students personally construct the quadratic curves and then confirm them algebraically.

In particular, construction activities of quadratic curves are activities that require finding the points that correspond to the definitions of each quadratic curve using the several properties of the geometric figures that were learned earlier. Thus these activities could become useful tools that will allow students to reaffirm or reexamine concepts that they learned earlier and go on to observe what kind of a relationship the given conditions and the point that matches the definition of the quadratic curves have. However simply constructing by a construction process cannot replace an actual construction activity. This is because the most important part of a construction activity is the students' self-discovery of the constructing process. Only by making such a discovery may students confirm several mathematical properties as a way of proving that their strategies are correct and finally understand the concept of quadratic curves.

However discovering the construction process alone is very difficult for most students. In fact, it is not an easy task for mathematicians to find a new theorem or concept from the massive volume of pre-existing knowledge. Mathematicians would use mathematical heuristic in all case of problem solving. Mathematical heuristic is a powerful method that includes useful discoveries or inventions in the problem solving process and rules, strategies and tactics. Analysis is a powerful mathematical heuristic skill that helps solve construction or proof problems. Analysis, as also called retroduction is the method used by repeated hypothesizing what conditions is required to acquire the wanted results under the premise that the object to be proven is true and the answer that is in pursuit is already known.

In construction, students would first draw their target, explore the relationships with each condition, and finally find the conditions that are needed for construction. The core part of discovering the construction process through Analysis is drawing as if constructed. If the figures are drawn as if constructed accurately their relationships with each condition can be more easily understood than otherwise. However in a pencil and paper environment, it is very difficult to draw an accurate figure and because it is impossible to move and manipulate a drawing once it's drawn there is the downfall that Analysis cannot be utilized unless one is extremely well trained. Recently, dynamic geometry programs GSP(Geometer's Sketchpad) and Cabri II was devised as methods to overcome the shortcomings of utilizing Analysis in a pencil and paper environment.

Software that provide dynamic geometry programs including GSP and Cabri II can serve as powerful tools to help students achieve constructions more liberally, which would have been otherwise difficult in a pencil and paper environment. Students can make their own visual models in a concrete way by simply manipulating the computer mouse in the dynamic geometry program. Furthermore, accurate construction is easily achieved and the program can also be used as a tool for analyzing and modifying the constructed figure and therefore students can easily engage themselves in active exploring and reasoning.

This research aims to observe the process in which students participate in constructing three quadratic curves tended to be overlooked in the current climate of school instruction, namely parabolas, ellipses and hyperbolas. It will look at how the students discover and justify the construction process on their own using Analysis and the dynamic geometry environment such as GSP. And it will be remarkable method for students to develop the mathematical discovery and mathematical justification on the dynamic geometry environment such as GSP.

2. Theoretical Background

2.1 The Analysis and Synthesis

Problem solving requires a process of discovering an interpretation to the problem. The methods, rules, and strategies of discovery and invention useful to the process of interpretation are called Heuristic.

One of the oldest mathematical Heuristic is Analysis. It was used by Pythagoras, accented of its importance by Plato and finally made into a theorem by the mathematician Pappus in the third century BC. While Analysis was used to find proof to prove theorems in geometry in Greece, its use was expanded by Descartes to also solve problems using equations. Resultantly Analysis has had a phenomenal influence on the advancement of modern mathematics and science (Nam, 2006). In Analysis, we first hypothesize that the answer has already been found or what is to be proven is true, and inquire what conditions need to be met in order to obtain the desirable result. Such process is continuously repeated and the hypothesis is finally reached at the end.

Contrarily in Synthesis, we start at the proposition that is already known to us or that is true and follow a logical and natural process to finally reach the answer or the proposition to be proven.

3. Method and Procedures

3.1 Method

This research has its objective in looking at how the participating students learn to logically construct quadratic curves reflecting on the properties of several geometric figures that they had learned about earlier and prove the validity of their construction to eventually arrive at a clear understanding of the concepts behind quadratic curves, using the explorative software GSP. As the research aims at observing and analyzing the construction process, a qualitative approach has been selected for the method of research.

3.2 Participants

Two tenth grade female students who had not yet learned about quadratic curves were selected as the research informants. These students were selected by recommendation by the students' homeroom teacher and mathematics teacher, the degree of which they enjoy math and through interviews with the students. The informant, student A maintains a math score within the top 5 percentile and also excels in all other subjects. The student B, the other informant, maintains a math score within the 23 percentile. Her grades in all other subjects are mostly good, but her math score is not necessarily impressive. B could concentrate on topics and subjects that she liked very well and she regarded mathematics as a subject that provided her with a sense of achievement.

3.3 Procedures

As it was difficult to conduct the research during regular school hours, this research was carried out after school in the students' classroom. The lesson consisted of one preliminary period and two full periods. One computer was given to the two students so that they could solve problems while collaboratively communicating with each other.

The first preliminary period was an initial introduction to using GSP and lasted for approximately 30 minutes. The first official lesson involved solving a worksheet on parabolas and ellipses and lasted approximately 100 minutes. During this lesson the teacher explained about the definition of analysis and the students had a chance to briefly practice analysis in a pencil and paper environment under the teacher's instruction. The second lesson involved solving a worksheet on hyperbolas and was conducted for approximately 40 minutes.

4. Analysis

The process that the students went through constructing the quadratic curves can be classified into analysis and synthesis. The process students went through analyzing and synthesizing ellipses will be discussed in the present paper.

4.1 The Construction of Ellipses using Analysis

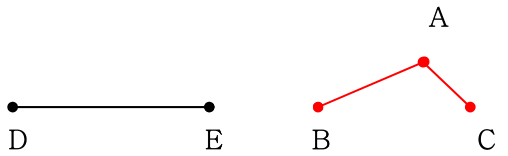
When two random points (In [Figure7] points B and C, which are the focal point of the ellipse) and a random lines (In [Figure7] line DE, its length is same with the sum of the distance from any point on the ellipse to the two focal points) are given, the construction of an ellipse refers to constructing a point where the sum of the distance from two focal points to that point and the length of the given line are the same. After constructing this point, the lesson was carried out so that the students would use the ‘Leave traces’ function of the GSP to confirm that the traces of this point would become an ellipse.

4.1.1 Constructing the hypothesized point

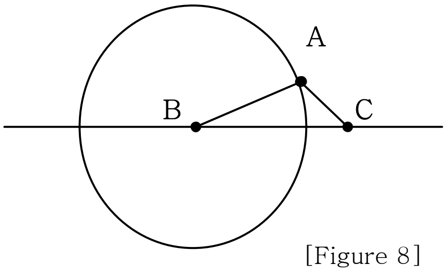
The students seemed to acknowledge that the activity of making the point hypothesized to be constructed on the monitor is quite useful. First, students made the line DE, its length is same with the sum of the distance from any point (A) on the ellipse to the two focal points, and then made the points B and C, which are the focal point of the ellipse and the point A. Next, they measured the distance of line DE, AB and AC by using the ‘Measure’ function of the GSP. Moving the point A, they found the place where the sum of the lengths line AB and line AC is equal to the length of the line DE [Figure 7].

[Excerpt 1] The beginning of analysis

1 Teacher: You understand the definition of an ellipse, right?
2 Student A: If we try drawing first like we did before….
3 Student B: Should we try drawing at least roughly first? The sum of the lengths should be fixed… (omitted)
4 Student A: **We need to find point A, right? Can’t we hypothesize that these (line AB and line AC) are the same?**
5 Student B: **But that’s a special case**. (Moments later)
6 Student B: **Couldn’t we just draw a circle here?**
7 Student A: That seems right
8 Student B: It does, right?
9 Student A: **The length here is different, how do we draw the circle? Just any circle?**



[Figure 7]



[Figure 8]

In <Excerpt 1> the students constructed the hypothesized point on their own (2-3). After doing so they greatly concentrated on drawing a circle, which was a very important activity in the construction of parabolas. Although the lengths were not regular they considered drawing a circle and therefore discussed the possibility of hypothesizing that the lengths were the same.(4-9) They went on to draw a circle with point B as the center and line AB as the radius.([Figure 8])

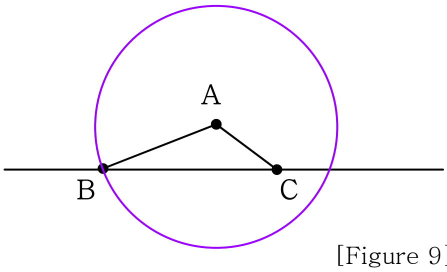
4.1.2 Finding properties from the hypothesized point

The students set that the sum of the lengths of line AB and line AC had to equal the length of the given line DE. It seemed that they came to such reasoning from the fact that in constructing parabolas the most decisive factor came from the definition given in the problem. The two students experimented with the monitor and their hands and found that lines AC and AB had to be straight lines. After that they drew a circle with point A in the center, as displayed in [Figure 9], they tried to find a way to move line AC.

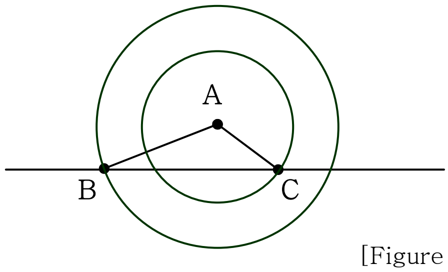
Excerpt 2. The sums of the lengths of the two lines are the same as the lengths of the given lines

1 Student A: **I think it will work if we change the two lines to one line**
2 Student B: Let’s try drawing it

3 Student A: Let’s first draw the straight line (line AB) that passes through this point and this point… The line (line AB) that connects this point and this point, the line (line AC) that connects this point and this point….
4 Student A: **Don’t you think we can move this line (line AC) over this straight line?** **We could designate a point and make the length the same… But**… (She points at point A as the end point of a line elongated from straight line AB and the point where the length equals line AC) **with this point the lines become equilateral…**
5 Student B: **Aha, then we can draw a circle**
6 Student A: If we draw a circle, what do we do… What about drawing two circles?
7 Student B: **What if we took this (point A) as the center and draw two circles** (each radius equals the lengths of lines AB and AC) **then we get the same radius (line AB) on the straight line and then the same radius as this (line AC)…**



[Figure 9]



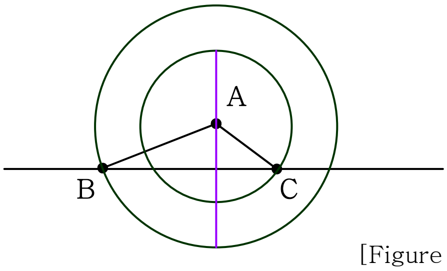
[Figure 10]

In <Excerpt 2>, the students, after observing the circle they had drawn earlier[Figure 8], started from the idea that in order for the sum of the lengths of the two lines to be fixed, the two lines needed to be able to move over to one straight line. Observing the figure drawn on the screen, the students attempted to move line AC towards the line that included line AB. (1-4) Then they drew a circle (Figure 9) with point A at its center and line AB as its radius and another circle [Figure 10] with point A as its center and line AC as its radius, based on the fact that a line of which length is the same as line AC, which was to be moved, (5-7).

4.1.3 Discovery of necessary condition

The students found the most important conditions for constructing an ellipse based on their idea to move both lines to one straight line. Of course the teacher did intervene by asking a few questions but in this process students were able to gradually find the necessary condition on their own.

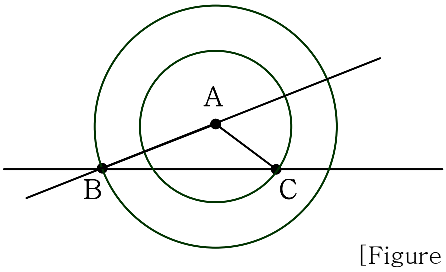
Excerpt 3. Moving the two lines over the straight line
1 Student B: (She points the screen)This is the radius, right? These are the same. And these are the same, so the two lengths can be moved over to this straight line
2 Student A: What do you mean?
3 Teacher: Why don’t you draw it for us?
4 Student B: **This is the radius so they are the same** (line AB and purple line) **and since this is also the radius these are the same** (line AC and short purple line, see up) **as well, right? So they can become a straight line**



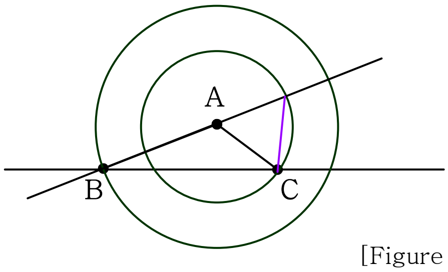
[Figure 11]

Excerpt 4. The length of a line of which the sum of the lengths of two lines are given

1 Teacher: **Is it necessary that we draw a straight line over here?**
2 Student A: **Ah, I guess we can move it over to this straight line (line AB).** Since the lengths of these two are the same.
3 Student B: Then we can mark the intersectional point of this circle (small circle) and the straight line.
4 Teacher: Which point do we have to find?
5 Student A: Point A.
6 Teacher: And how do we find that point?
7 Student B: **Since the lengths of this (line AC) and this** (the length between the center point and the intersection point between the small circle and the straight line) **are the same, we can draw a line and make a perpendicular bisectional line….**
8 Student A: That sounds right. Let's try it out.
9 Teacher: Then can you try it over from the beginning?
10 Student B: **But how do we find this point (the intersection between the small circle from [figure 13] and the straight line.**



[Figure 12]

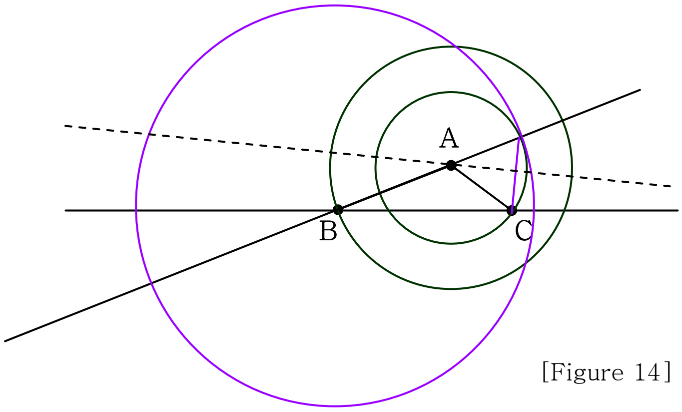


[Figure 13]

<Excerpt3> shows student B explaining that the two lines can be moved over to the one straight line. At first the students explained with her hands pointing to the monitor, but when student A didn't seem to understand, she went on to explain by drawing a line on the monitor (1-4). For a long time the students were not able to move on beyond the state of [Figure 11]. This was because, although they had to find a particular point where the sum of the distances of point B and point C (focal point) had to be found, the vertical line of [Figure 11] had no relation with point B and C. In other words, [Figure 11] did not help the students but rather interfered in their thinking process and did not provide them with any decisive hints.
At this time, as in <Excerpt 4> the teacher advised the students about the position of the straight line when moving the two lines. As a result of listening to such advice student A learns that the two lines can be moved to the line that is an elongation of line AB (1-3). Then the students spoke about how to construct point A [Figure 13] (7-8). Student B at this time posed a decisive question on how to find the intersection point between the small circle and the straight line (10).

Excerpt 5. Finding the point for the perpendicular bisectional line.
1 Teacher: So what did you do first?

2 Student B: We draw a straight line and then a circle. **But since the first circle has A at its center it won't work without A**
3 Teacher: What is the radius of the big circle?
4 Student A: The length of line AB
5 Teacher: What did you think was the most important at first?
6 Student A: That the sum of these two lengths are the same as this.
7 Teacher: Can you find how long that length is from the screen?
8 Student B: From here to here.(blue line)
9 Student A: **Then what if this length is taken as the radius?**



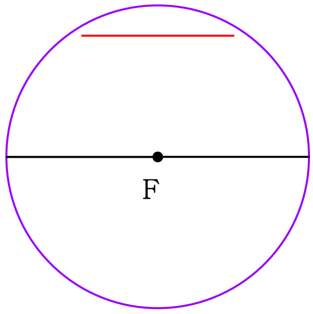
[Figure 14]

10 Student B: **Then should we take this length as the radius with B in the center?**
11 Teacher: What if you took the length of the radius as the length of the given line (line DE, in figure 7)?
12 Student A: **That works. Since they're the same**…
13 Student B: Then is this intersection point the point We're looking for?
14 Student A: Let's try it.

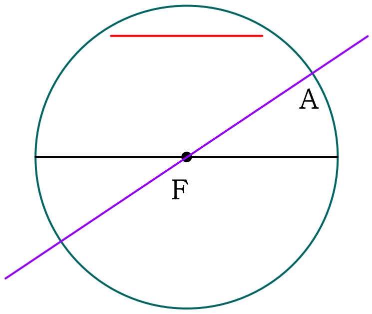
<Excerpt 5> shows the students conversing about how to find the point for a perpendicular bisectional line. However the students were not able to find any hints and were not able to move on for a while. At this time the teacher advised the students to look back on each of the steps they had gone through so far. Student B stressed the fact that if point A disappears then the point for the perpendicular bisectional line would also disappear. However they were still not able to find an answer (1-2), so the teacher asked the students what would be the most important fact to keep in mind in constructing the ellipse. When the students mentioned that the sum between the distances needed to be fixed the teacher suggested that they reconfirm the sum of the lengths on the monitor (3-8). In this questioning and answering process student A thought of using the big circle indicated in bold in [Figure 14] and learned how to construct a point for a perpendicular bisectional line (9-10). At the end the teacher reminded the students that the radius of the circle is the same as the length of the given lines. The teacher had the students wrap up their analysis process (11-14).

4.2 The Construction of Ellipses using Synthesis

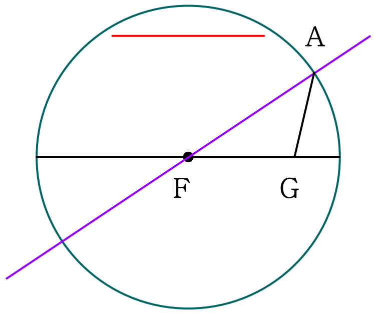
The students were able to construct an ellipse by making inferences in reverse of the direction of analysis, based on several facts they had learned through analysis.



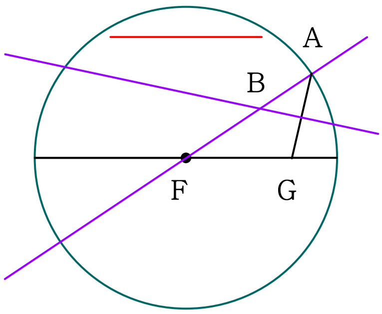
[Figure 15] Draw a circle with point F at its center (a focal point) and the length of the given line (red line) as its radius



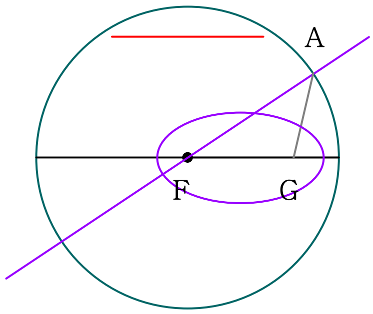
[Figure 16] Draw the straight line (line AF) that passes through the point A and point F



[Figure 17] Draw the line (line AG, point G is another focal point.).



[Figure 18] Draw the perpendicular bisectional line of line AG. And mark the point B that is the intersection between the line AF and the line AG.



[Figure 19] Tracing the point B would become an ellipse by using 'Leave traces' function of the GSP.

5. Findings and Analysis

5.1 Analysis provided the students with effective ideas in finding the construction process.

By drawing the point hypothesized to be constructed, the students were able to easily figure out the relationship between the figures and the given conditions. Such construction served as a decisive hint in finding the necessary conditions for construction. Analysis taught the students how to think in a logical and concrete way. Therefore including Analysis in instruction is imperative and moreover the instruction of Analysis through construction activities is very effective.

5.2 Learning construction became easier in a dynamic geometry environment as construction is made easy and accurate.

The dynamic geometry environment allowed students to construct easily and accurately and helped students to acknowledge the importance of the construction process, rather than simply regarding construction as an activity. Moreover, as students' assumptions or ideas could be easily displayed on the screen, the students were able to actively explore and make their own assumptions within such an environment.

5.3 Previously learned content and present content are connected through the construction activity.

Students will most likely use their knowledge on the many properties of geometric figures that they had learned in middle school in the construction process. The students themselves were surprised that they could actually complete the construction of the quadratic curves with their own prior knowledge and seemed to appreciate the value of what they had learned in earlier years. Therefore it was identified that the construction activity could serve as a bridge between earlier learning activities and present learning activities.

5.4 New content is acknowledged to be easy.

Although they had not yet learned about quadratic curves at school, the students did not acknowledge quadratic curves to be difficult. This was because they were able to construct quadratic curves by using of the knowledge that already had. Moreover, by doing the constructions on their own and using the function called 'Leave traces' they were able to construct curves, and upon such experience they ultimately got to know about the properties of the quadratic curves. Therefore the exposure to quadratic curves through the construction activity without complicated equations helped the students regard quadratic curves in a more easy way.

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