Exploring and Visualizing Hilbert Geometry in a Triangle

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ABSTRACT

In this paper we explore Hilbert geometry in a triangle, using Maple to illustrate some concepts such as Hilbert distant, projective and affine coordinates, unitary circle, etc. and to introduce new "trigonometric functions" for this geometry.

1. Introduction.

There has recently been growing interest in Hilbert Geometry [1] and many research papers were published, see [2 - 11] to cite just a few of them.
We begin by recalling the Hilbert geometry. Let H be a nonempty bounded open convex set in $\mathbb{R}^n$, $n \geq 2$. The Hilbert distance "$distH$" on H was introduced by D. Hilbert as follows. For any $P \in H$, let $distH(P,P) = 0$. For distinct points P and Q in H, assume the line passing through P, Q intersects the boundary $\partial\bar{H}$ at two points X, Y such that the order of these four points on the line is Y, P, Q, X, see figure 1.
Denote the cross-ratio of these points by:

$$(PQXY) = \frac{\overline{PX}}{\overline{XQ}} : \frac{\overline{PY}}{\overline{YQ}}$$

Where the bar on letters means Euclidean distant on $\mathbb{R}^n$. Then the Hilbert distance is defined by:

$$distH(P,Q) = \left|\log(PQXY)\right|$$

This is a well defined distance under which the points at boundary are "at infinite".
The metric space $(H, distH)$ is called Hilbert geometry. When H is the unit open ball $\left\{(x_1, x_2, \ldots x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 < 1\right\}$, $(H, distH)$ is the Klein model for the hyperbolic geometry.

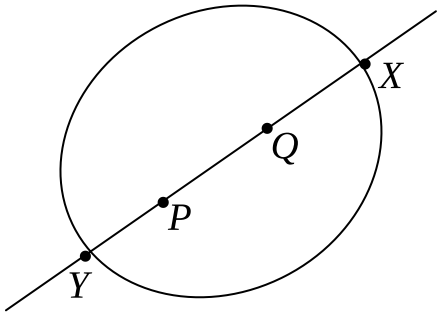


Figure 1

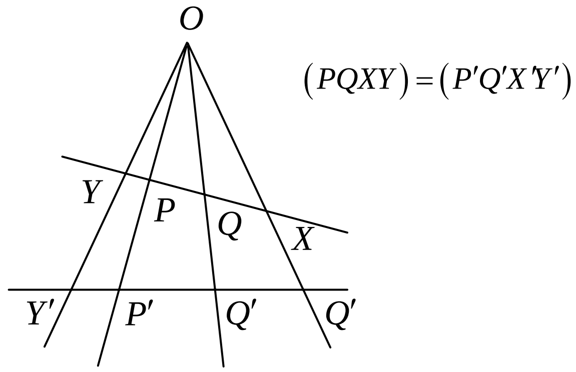


Figure 2

2. Distance and coordinates.

To explore Hilbert geometry in a triangle it will be convenient to recall alternative definition for the Hilbert metric. Since the cross-ratio is invariant under any projective mapping T, $(H, distH)$ and $(T(H), distT(H))$ are isometric as Hilbert geometry (figure 2). We will use this property of the cross-ratio to introduce coordinates for points in $(H, distH)$ when H is a triangle.

We know that given two points A and B in a segment, then a point Q divides this segment in a ratio k if $\frac{\overline{AQ}}{\overline{QB}}=k$. Solving for Q we have $Q=a\cdot A+b\cdot B$ with $a+b=1$. Where $a=\frac{1}{1+k}$ and $b=\frac{k}{1+k}$. These relations are independent of the chosen origin of coordinates. In the same way, given the vertices A, B and C of a triangle H, a point P inside of it can be written as:

$$P=a\cdot A+b\cdot B+c\cdot C\,,\ a+b+c=1,\ a\geq 0, b\geq 0, c\geq 0$$

We will call (a,b,c) the ***projective coordinates*** of a point P. Using these coordinates we can see, for example that points with $(0,b,c)$ correspond to the side $\overline{BC}$ etc.

Let be P_C the projection of a point $P=(a,b,c)$ onto the side $\overline{AB}$ then:

$$P_C=\frac{a}{1-c}A+\frac{b}{1-c}B$$

The point P_C divides the segment $\overline{AB}$ in a ratio $k=\frac{b}{a}$ (figure 3).

PROPOSITION 1: If $P=(a_1,b_1,c_1)$ and $Q=(a_2,b_2,c_2)$ are two points inside the triangle H and the straight line that joins them intersect the sides $\overline{AC}$ and $\overline{BC}$ then:

$$distH(P,Q)=\left|\log\left(\frac{b_2}{a_2}\cdot\frac{a_1}{b_1}\right)\right|. \qquad (2.1)$$

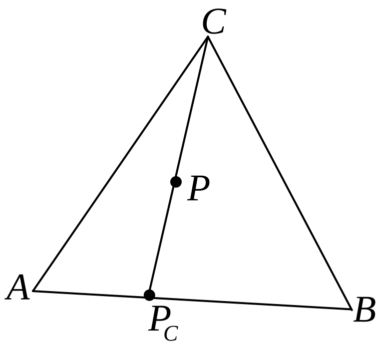


Figure 3

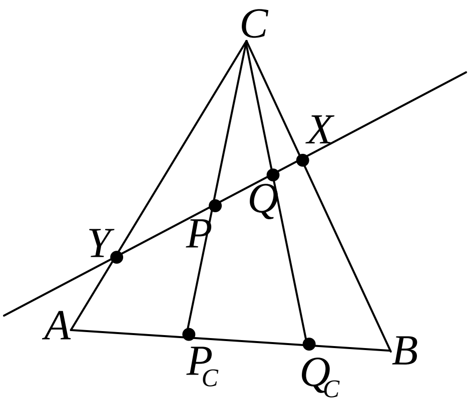


Figure 4

Proof:
By projection from C we will have (figure 4):

$$distH\left(P,Q\right)=\left|\log\left(PQXY\right)\right|=\left|\log\left(P_CQ_CBA\right)\right|=\left|\frac{\overline{P_CB}}{\overline{BQ_C}}:\frac{\overline{P_CA}}{\overline{AQ_C}}\right|$$

But $\dfrac{\overline{P_CB}}{\overline{P_CA}}=\dfrac{1}{k}=\dfrac{a_1}{b_1}$ and $\dfrac{\overline{AQ_C}}{\overline{BQ_C}}=l=\dfrac{b_2}{a_2}$ so, $distH\left(P,Q\right)=\left|\log\left(\dfrac{b_2}{a_2}\cdot\dfrac{a_1}{b_1}\right)\right|$ ■

In the same way, if the straight line that joins points $P=\left(a_1,b_1,c_1\right)$ and $Q=\left(a_2,b_2,c_2\right)$ intersects the sides $\overline{AB}$ and $\overline{BC}$ then:

$$distH\left(P,Q\right)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)\right| \qquad (2.2)$$

If the straight line joining these points intersects the sides $\overline{AB}$ and $\overline{AC}$ then:

$$distH\left(P,Q\right)=\left|\log\left(\frac{b_1}{c_1}\cdot\frac{c_2}{b_2}\right)\right| \qquad (2.3)$$

To explore the metric space $\left(H,distH\right)$ we construct the Maple procedure "disT" (see appendix A), which consider all these cases.

PROPOSITION 2: If $P=\left(a_1,b_1,c_1\right)$ and $Q=\left(a_2,b_2,c_2\right)$ are two points inside the triangle H There exists a point $Z=\left(a_3,b_3,c_3\right)\in H$ such that P, Q and Z are not collinear and:

$$distH\left(P,Q\right)=distH\left(P,Z\right)+distH\left(Z,Q\right)$$

Using the procedures "disT" (appendix A) and "ttn" (appendix B) we colored the inside of the triangle H according to the value of the function:

$$distH\left(P,Z\right)+distH\left(Z,Q\right)-distH\left(P,Q\right)=\Phi\left(Z\right) \qquad (2.4)$$

The results can be seen in the figure 5. Inside the quadrilateral represented in the figure, we have $\Phi\left(Z\right)=0$.

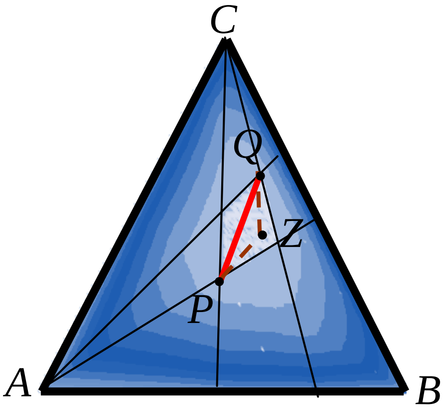


Figure 5

Proof:
Taking Z inside the quadrilateral as it is shown in the figure we have:

$$distH(P,Z)+distH(Z,Q)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_3}{a_3}\right)\right|+\left|\log\left(\frac{a_3}{c_3}\cdot\frac{c_2}{a_2}\right)\right|=$$

$$=\log\left(\frac{a_1}{c_1}\cdot\frac{c_3}{a_3}\right)+\log\left(\frac{a_3}{c_3}\cdot\frac{c_2}{a_2}\right)=\log\left(\frac{a_1}{c_1}\right)+\cancel{\log\left(\frac{c_3}{a_3}\right)}+\cancel{\log\left(\frac{a_3}{c_3}\right)}+\log\left(\frac{c_2}{a_2}\right)=$$

$$\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)=\left|\log\left(\frac{a_1}{c_1}\cdot\frac{c_2}{a_2}\right)\right|=distH(P,Q)$$

If Z is outside of the quadrilateral then the distance must be calculated using different formulas because the straight line joining the points P,Z and Q,Z intersects different pairs of sides of the triangle ■

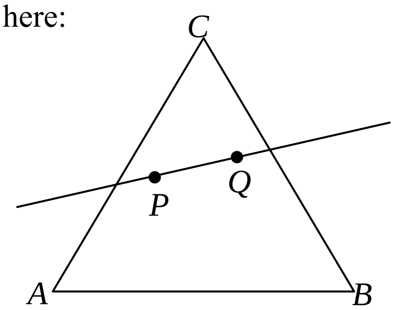
Together with the projective coordinates we will consider the ***affine coordinates*** of a point P. If $P=(a,b,c)$ with $a+b+c=1$ then affine coordinates for P are:

$$P=[x,y] \quad \text{with} \quad \begin{cases} x = a/c \\ y = b/c \end{cases} \qquad (2.5)$$

We suppose P is strictly at the interior of the triangle, where a, b, c are positives.
If we know affine coordinates $[x,y]$ then we can obtain the projective coordinates by the following formulas:

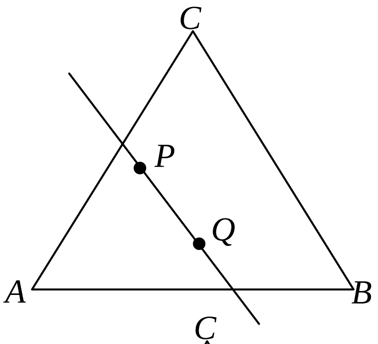
$$\begin{cases} a = x/(1+x+y) \\ b = y/(1+x+y) \\ c = 1/(1+x+y) \end{cases} \qquad (2.6)$$

These affine coordinates has the advantage to be two (and not three as projective coordinates) in a space of dimension two.
The formula for distance “distH” in these coordinates however varies and we write it down here:

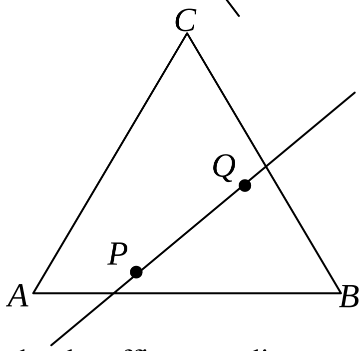


$$P=[x_1,y_1] \quad Q=[x_2,y_2]$$

$$distH(P,Q)=\left|\log\left(\frac{y_2}{x_2}\frac{x_1}{y_1}\right)\right| \qquad (2.7)$$



$$distH\left(P,Q\right)=\left|\log\left(\frac{y_1}{y_2}\right)\right| \qquad (2.8)$$



$$distH\left(P,Q\right)=\left|\log\left(\frac{x_1}{x_2}\right)\right| \qquad (2.9)$$

For example, the affine coordinates of the points in figure 8 for $r=1$ are:
$O=$ $[1,1]$; $P_1=$ $[1,e]$; P_2 = $\left[e^{-1},1\right]$; P_3 = $\left[e^{-1},e^{-1}\right]$; P_4 = $\left[1,e^{-1}\right]$; $P_5=\left[e,1\right]$;
P_6 = $[e,e]$.

3. Circles and Disks.

For our Maple explorations of Hilbert geometry we will use an equilateral triangle with vertices at points $A=\left(0,0\right)$, $B=\left(1,0\right)$ and $C=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Using the procedure "disT" (appendix A) together with the procedure "tt2"(appendix C) we explore the shapes of disks centered at the points $O=\frac{1}{3}\cdot A+\frac{1}{3}\cdot B+\frac{1}{3}\cdot C=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (see figure 6) and $O'=\frac{3}{7}\cdot A+\frac{1}{8}\cdot B+\frac{25}{56}\cdot C=\left(\frac{3}{7},\frac{1}{8},\frac{25}{56}\right)$ (see figure 7) with different radius. Note that the "circles" are really hexagons.

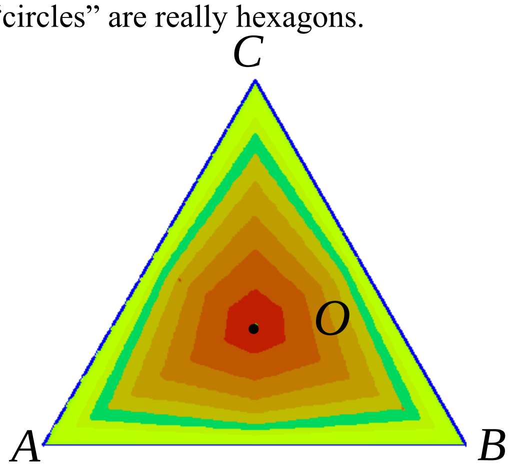


Figure 6

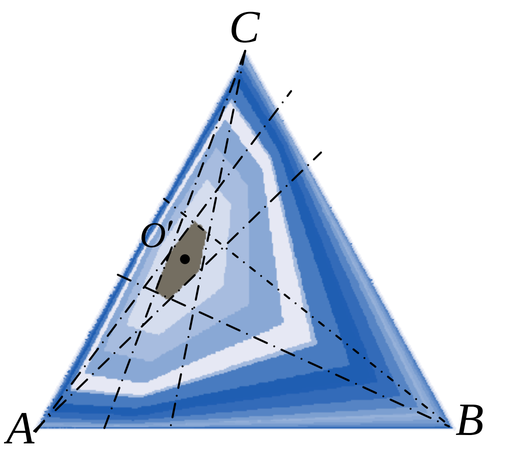
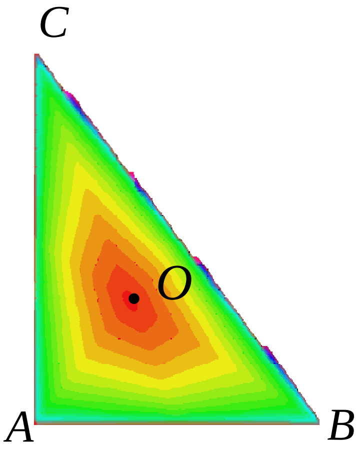
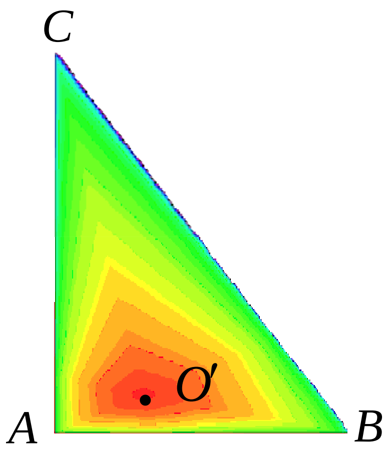


Figure 7

This property is independent of the used triangle. If we use another triangle for example, with vertices at points $A=(0,0)$, $B=(1,0)$ and $C=(0,1)$ we obtain again "circles-hexagons" as in the following figures:





We can parameterize "circles" (boundary of the hexagons) centered at O and radius r using the following formulas (see figure 8):

$$\begin{aligned}
\overline{P_1P_2} &= \left(t, e^r\cdot t, 1-t-e^r\cdot t\right); \quad \overline{P_2P_3} = \left(t, 1-t-e^r\cdot t, e^r\cdot t\right)\\
\overline{P_3P_4} &= \left(1-t-e^r\cdot t, t, e^r\cdot t\right); \quad \overline{P_4P_5} = \left(e^r\cdot t, t, 1-t-e^r\cdot t\right)\\
\overline{P_5P_6} &= \left(e^r\cdot t, 1-t-e^r\cdot t, t\right); \quad \overline{P_6P_1} = \left(1-t-e^r\cdot t, e^r\cdot t, t\right)
\end{aligned} \tag{3.1}$$

with $\dfrac{1}{2\cdot e^r+1} \le t \le \dfrac{1}{e^r+2}$

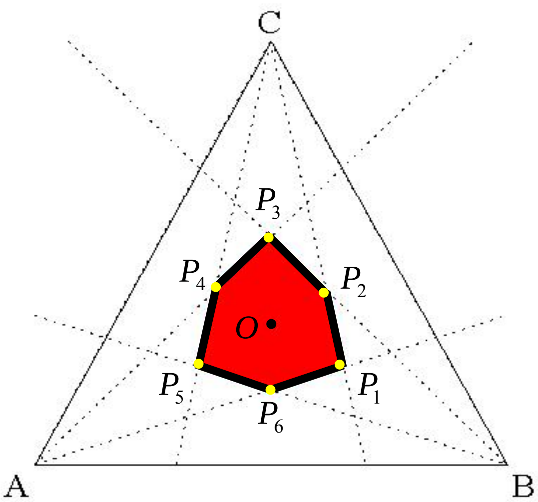


Figure 8

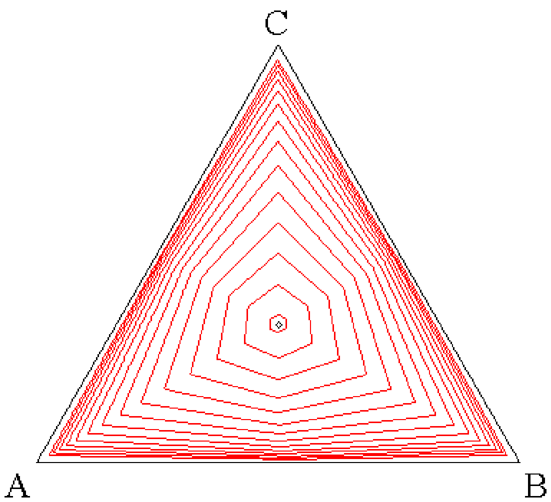


Figure 9

Using these formulas with Maple commands given in the appendix D we obtain circles with center O and radius r as it is shown in figure 9.

4. Trigonometric Functions.

Consider now the following definition of an angle for the Hilbert geometry in a triangle. Let's take two semi-straight lines that intersect each other in a vertex P. By an isometry we can take P to the central point O; the semi-straight lines intersect the unitary circle in two points defining an arch in the circle. The length of this arch will be the measure of the angle.

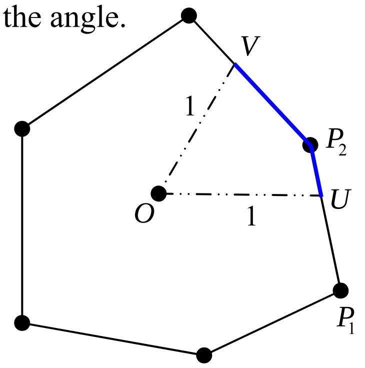


Figure 10

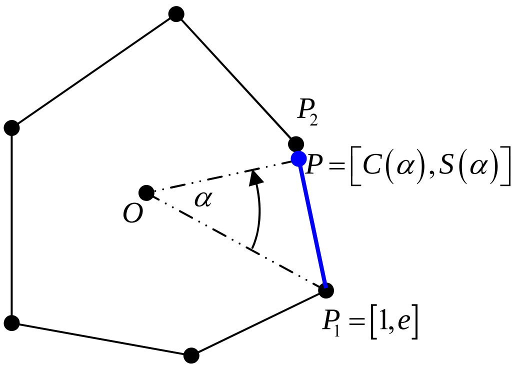


Figure 11

$$\alpha = \angle UOV = distH(U, P_2) + distH(P_2, V) \qquad (4.1)$$

It is easy to show that $\sum_{i=1}^{5} distH(P_i, P_{i+1}) + distH(P_6, P_1) = 6$ so the complete angle measures 6.

As in the classic case, we now consider a semi-straight line OP_1 that starts to move in a counter clockwise direction. We define $[C(\alpha), S(\alpha)]$ as the affine coordinates of the point P if $\angle P_1OP = \alpha$, (figure 11).

Using the procedure "Arch" (appendix E) we can determine the angle α for any point P on the unitary circle given its affine coordinates. This procedure uses the formulas (3.1), automatically determines to which segment of the unitary circle P belongs and applies the correct formula for the distance of P from a point P_1. With this procedure we can construct the graphics of functions $C(\alpha)$ and $S(\alpha)$ as it is shown in figures 12 and 13.

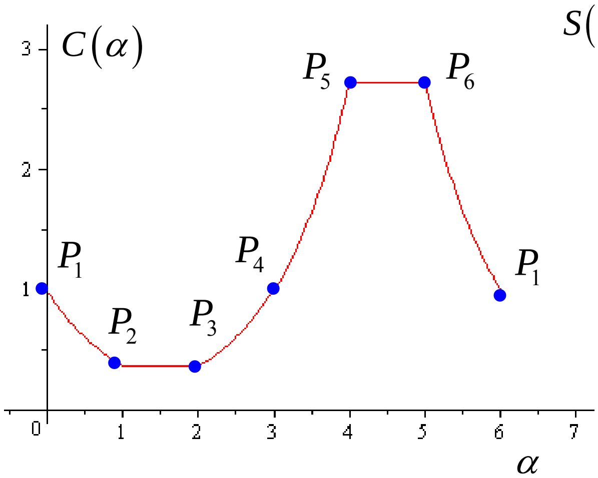


Figure 12

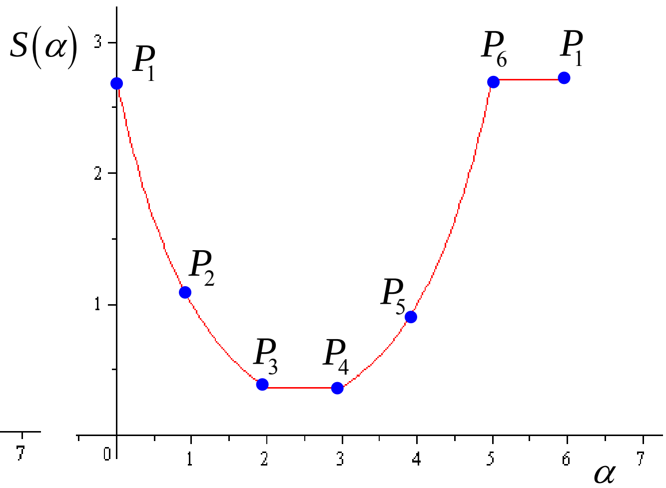


Figure 13

Moreover, these functions can be given in analytic form as it is shown in the following proposition.

PROPOSITION 3: The trigonometric functions take the following values:

$$\begin{cases} C(\alpha)=e^{-\alpha} \\ S(\alpha)=e^{1-\alpha} \quad \text{if} \quad 0\le\alpha\le 1 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{-1} \\ S(\alpha)=e^{1-\alpha} \quad \text{if} \quad 1\le\alpha\le 2 \end{cases}$$

$$\begin{cases} C(\alpha)=e^{\alpha-3} \\ S(\alpha)=e^{-1} \quad \text{if} \quad 2\le\alpha\le 3 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{\alpha-3} \\ S(\alpha)=e^{\alpha-4} \quad \text{if} \quad 3\le\alpha\le 4 \end{cases}$$

$$\begin{cases} C(\alpha)=e \\ S(\alpha)=e^{\alpha-4} \quad \text{if} \quad 4\le\alpha\le 5 \end{cases} ; \quad \begin{cases} C(\alpha)=e^{6-\alpha} \\ S(\alpha)=e \quad \text{if} \quad 5\le\alpha\le 6 \end{cases}$$

These formulas can be extended by periodicity for $\alpha \le 0$ or also for $\alpha \ge 6$, (see figures 14 and 15).

Proof:

We will verify for instance formula (1). By using different versions of distance (2.7 – 2.9) can be proved in the same way the rest of the formulas

Using formula 2.7 we have:

$$distH(O,P) = distH\left([1,1],\,[e^{-\alpha},e^{1-\alpha}]\right)$$

$$= \left|\log\left(\frac{1}{1}\cdot\frac{e^{1-\alpha}}{e^{-\alpha}}\right)\right| = \log(e) = 1$$

Using formula (2.9)

$$distH(P,P_1) = distH\left([1,\,e],\,[e^{-\alpha},\,e^{1-\alpha}]\right)$$

$$= \left|\log\left(\frac{e^{-\alpha}}{1}\right)\right| = \log(e^{\alpha}) = \alpha\,. \blacksquare$$

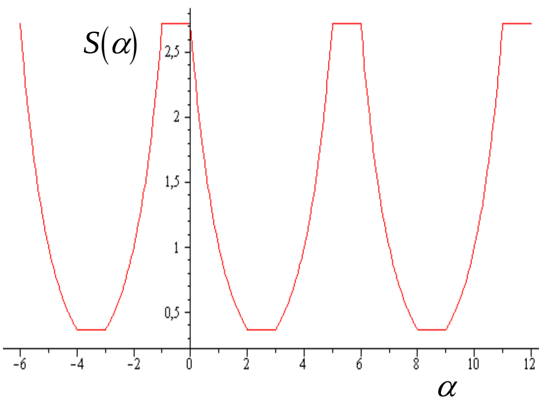


Figure 14

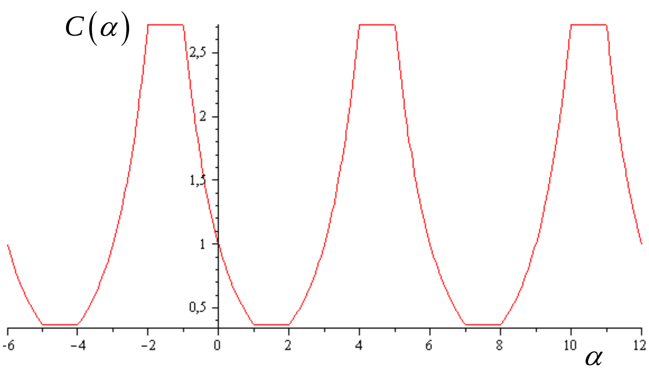


Figure 15

Using $S(\alpha)$ and $C(\alpha)$ we can define new functions:

$$T(\alpha)=\frac{S(\alpha)}{C(\alpha)}\ ; \qquad Ct(\alpha)=\frac{C(\alpha)}{S(\alpha)}.$$

The graphics of these functions are shown in figures 16 and 17 respectively.

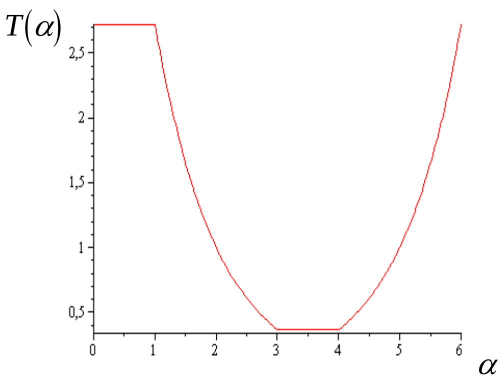


Figure 16

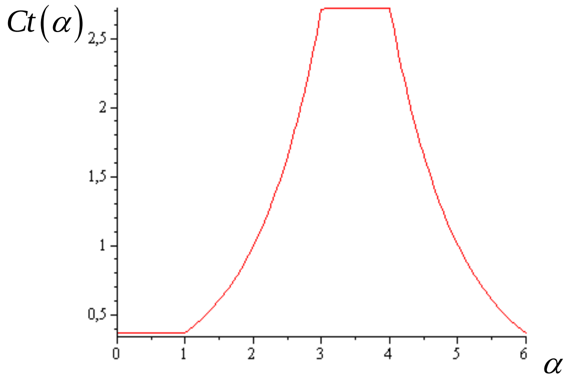


Figure 17

With the proposition 3 and the graphics of the functions $S(\alpha)$, $C(\alpha)$, $T(\alpha)$ and $Ct(\alpha)$ we can proof the following proposition:

PROPOSITION 4: The functions $S(\alpha), C(\alpha)$, $T(\alpha)$ and $Ct(\alpha)$ satisfy the following relations:

$$C(\alpha-1)=S(\alpha);\ C(\alpha+1)=Ct(\alpha)$$
$$S(\alpha-1)=T(\alpha);\ Ct(\alpha-2)=T(\alpha+1)$$

5. Alternative definitions for trigonometric functions.

With the view to gaining more familiar properties of trigonometric functions for Hilbert geometry in a triangle, we could adopt the following alternative definitions:

$$s(\alpha)=\ln(S(\alpha)); \qquad c(\alpha)=\ln(C(\alpha))$$

$$t(\alpha)=\frac{s\left(\alpha-\frac{1}{2}\right)}{c(\alpha)}=\frac{\ln\left(S\left(\alpha-\frac{1}{2}\right)\right)}{\ln(C(\alpha))};\ ct(\alpha)=\frac{c(\alpha)}{s\left(\alpha-\frac{1}{2}\right)}=\frac{\ln(C(\alpha))}{\ln\left(S\left(\alpha-\frac{1}{2}\right)\right)}$$

The graphics for these functions are shown in the figures 18 - 21:

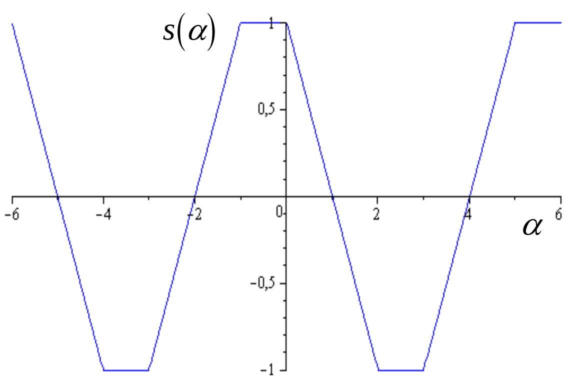


Figure 18

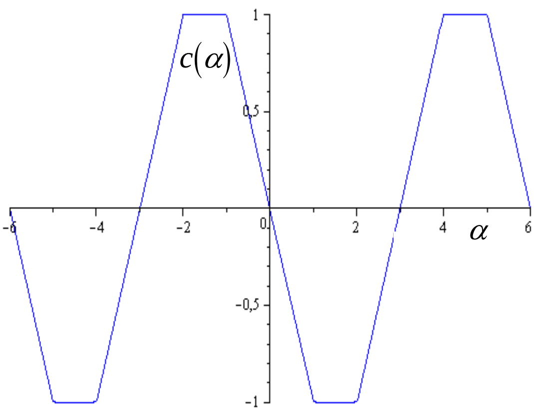


Figure 19

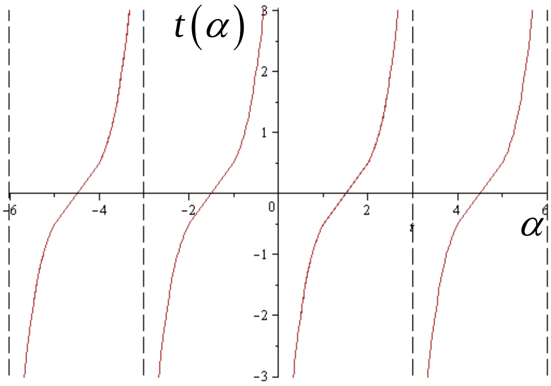


Figure 20

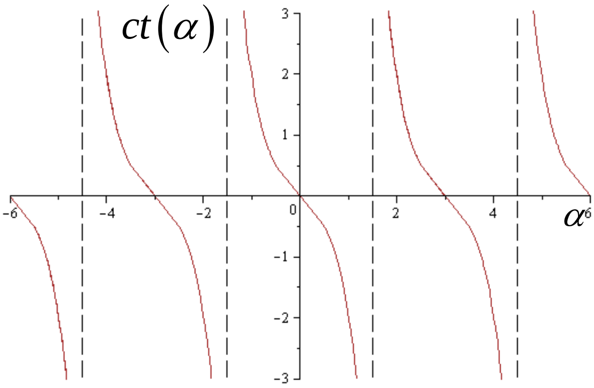


Figure 21

PROPOSITION 5: The functions $s(\alpha), c(\alpha)$, $t(\alpha)$ and $ct(\alpha)$ satisfy the following relations: $s(\alpha)=s(\alpha+6)$; $c(\alpha)=c(\alpha+6)$; $c(\alpha-1)=s(\alpha)$; $c(-\alpha)=-c(\alpha)$; $s\left(\alpha-\frac{1}{2}\right)=s\left(\frac{1}{2}-\alpha\right)$; $t(\alpha)=t(\alpha+3)$; $ct(\alpha)=ct(\alpha+3)$.

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***The appendixes can be downloaded from the Web site:**
www.casioacademicochile.com/hilbert/appendix-hilbert-geom.pdf