

# A System for Helping Concept-Building of 3D Linear Algebra by Connecting Graphic, Symbolic, and Verbal Representations

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**Abstract:** Many students in the college two of the authors work for have suffered lower performance in three-dimensional linear algebra for a long time. It is partially because they have learned linear algebra as a set of symbolic procedures isolated from graphical or verbal representations, and also from real world applications. Procedural knowledge without understanding the concept seems to evaporate just after the examinations.

To help students building conceptual knowledge of linear algebra, we have been constructing a learning system, which shows the dynamic relations of three representations: vector equations, 3D graphic objects represented by the equations, and verbal explanations of the features of equations and graphic objects. The system serves a series of activities, where students are able to touch or transform some parts of the graphic objects in virtual space, observe animated transformations of the graphic objects and simultaneous changes of symbolic expressions, and connect them to a list of verbal explanations. By experiencing the activities in the system, the students are expected to recognize the links between the three representations in 3D linear algebra.

The system is a combination of several software programs including web-browser/server, a database management system, a dynamic geometry software, and a computer algebra system. The structure and several activities of the system are presented in this talk.

## 1 Introduction

Linear algebra, especially three-dimensional (3D) vectors and matrices, is the most poorly performed field of pre-college level mathematics in annual INCT (Institute of National College of Technology, Japan) achievement tests, which has been given to all the third grade students (18 years old) of colleges of technology in Japan since 2007 [1]. We strongly suspected the cause of the poor performance is surface learning [2] spreading to the learning of many subjects in Japan because the students' answer sheets have more blank answers, fewer drawings, and fewer explanatory sentences especially in linear algebra. They seem to memorize as many formulas and solutions as possible to be ready for the examinations, but forget most of them after the term-end examinations. Since INCT achievement tests are given at least half a year after their learning of target fields, and the knowledge of 3D linear algebra is not used in engineering subjects of third grade, such isolated knowledge does not remain at the time of the achievement tests.

To improve the situation, we needed to design a lesson plan, where students learned the material more deeply and meaningfully. We proposed a new lesson plan [3] directed from actual applications to abstract mathematical ideas; from graphic objects and the explanation of their characteristics to symbolic expressions based on the MODEM theory [4], and have been conducting trial lessons according to the plan.

The system described in this paper is intended to be used in the new lesson plan to provide students with the opportunity to handle 3D graphic objects by themselves and to help them to identify the relationships between three representations: graphic objects, verbal explanations, and symbolic expressions.

It is a combination of several software programs including web-browser/server, a database management system, a dynamic geometry software, and a computer algebra system.

## 2 New Lesson Plan And The Preliminary Result

### 2.1 New Lesson Plan

Because the system is a part of the new lesson plan, we would like to describe it shortly first. The new lesson plan consists of seven steps described in Table 1 and takes 7.5 hours of lesson time plus additional outside-class activities. It introduces vectors in a real-world application, moves to model them with real objects in classroom and with graphic objects in 3D virtual space, and discusses the symbolic representations and their mechanisms at the later steps. It also blends traditional classroom activities with technology-based ones.

Table 1 New Lesson Plan of 3D Linear Algebra for Engineering Students

Step	Activity	Lesson time	
1	Actual Application	Students recognize the usefulness of linear algebra to solve engineering problems and are motivated to learn	0.75H * 2 + outside class
2	Real Objects in Classroom	Students handle simple real objects and are accustomed to the idea of parallel and perpendicular	0.75H
3	Graphic Objects in 3D Virtual Space	After observing animated constructions of 3D graphic objects, students use dynamic geometry software to construct 3D graphic objects by themselves using the idea of parallel and perpendicular	1.5 H+ outside class
4	Identifying and Producing the Relation of Different Representations	Students identify the relation of graphic objects, explanatory sentences and symbolic expressions, and producing it in small quizzes	0.75H + outside class
5	Deducing Formulas	The teacher explain the deducing process of formulas using the chalkboard	1.5 H
6	Pencil-and-paper Exercises	Students solve traditional paper-and-pencil problems using pencil-and-paper with more description of the processes	0.5H * 3 + outside class
7	Mutual Mini-lectures	Students explain their own solutions to the other students in a small group	outside class

On the first step: Actual application, we introduce 3D vectors in real-world applications because engineering students usually need actual applications to feel the reality in mathematics. They already have Newton's mechanics and electro-magnetism for algebra and calculus, and alternating electrical circuits for trigonometry and calculus, but they used to have no application for linear algebra.

An application, which is popular among our students, is deciding tactics in Robocup soccer games. The game is played with five automatically remote-controlled robots in each team, and a robot or a ball leaves a straight line as its trace. Because a competing robot may intercept the ball if the ball-path is too close to the robot in a game, we have to calculate the distance of the path of the ball to the opponent robots before the move. Optimizing virtual players in a battle-game is another application [1]. In the game “tournament”, a virtual player competes each other in different kind of battles at every stage of the tournament, for example, boxing or Sumo wrestling match, solving puzzles or chess match, or singing competitions or musical auditions. Every student in the class owns a player as his/her *avatar*, and designs its characteristics before attending the tournament. A 3D vector expresses the characteristics: physical strength, thinking ability, and musical skills.

On the second step: Real objects in classroom, students handle real objects in the classroom. We use a thin wooden stick for a straight line and a plastic plate for a plane. We also set a corner of each student’s desk as the origin of 3D space with two sides of the desktop as  $x$  and  $y$  axes, and a stick fixed to stand vertically from the corner as  $z$  axis. “Imagine to hold a wooden stick or a plastic plate at a certain position and angle over your desk using extra sticks and pieces of clay!” is the first activity.

In the activity, students are to recognize that they need two extra sticks to hold a stick in the space and three extra sticks to hold a plate. If we force them to hold the end of the extra sticks at the origin, they remind students of position vectors in the later lessons.

Some students may claim that they can hold a plane with only one extra stick if they fix the direction of the plane to the extra stick with clay. The claim is valuable because it will be lead to the idea of expressing a certain vector equation of a plane. We should encourage them to find as much facts as possible by themselves in the activity. The experience is to help the students to imagine the objects in 3D virtual space.

On the third step: Graphic objects in 3D virtual space, after the activities with real objects, students draw objects in 3D virtual space with a dynamic geometry software program. Position vectors are drawn always from the origin, show the coordinates of end points, and resemble wooden sticks fixed to the corner of their desks in classroom. If they animate a ball of Robocup soccer in virtual space, its trace is a straight line with a vector showing its velocity and time as the parameter.

On the fourth step: Identifying and Producing the relation of different representations, after some free activities in virtual space, students conduct guided experiments in virtual space and confirm the relations of graphic objects, verbal explanation of them and vector equations. The experiments use either a Web-system or a stand-alone software program. On the Web-system graphic objects are created with Cabri 3D [5] and directly manipulative, and some of the graphic objects are animated to show their features. The stand-alone program is Mathematica [6] notebook connected to a remote database program. The program shows simultaneous change of graphic objects and related symbolic expressions, and serve small quizzes adapted to each student.

The step five: Deducing formulas, is composed of traditional lectures, where we symbolically express vector equations of straight lines and planes in 3D space. We also confirm that there are several types of symbolic expressions to show a line or a plane, and they can be rewritten to each other. Each symbolic expression could be linked to a certain feature of the graphic object, and the link should be explained verbally. We often reuse the 3D graphics objects already shown to the students during the experiments in

virtual space. The students, who know a real application and 3D graphic image of the objects, accept vector equations easier than the former students learnt with traditional lesson plan.

On the sixth step: Pencil-and-paper exercises, we also set up pencil and paper exercises after the lectures because it is important to write their own thought on paper to think deeply and to make it structured. In the exercises, we encourage them to start from a drawing of target objects, add necessary vectors, and express them in symbolic expressions at the final step. We forbid them to use formulas written in their textbooks as the starting point of their solutions because it hardly explains necessary mathematical ideas by itself. If they need symbolic manipulations, they have to write every step of the rewriting process on their paper, too.

On the seventh step: Mutual mini-lectures, we organize a small group of students to teach each other in the mini-lectures, where every student takes the role of lecturer in tern to explain a topic of linear algebra. A teacher or a teaching assistant organizes the group and helps when they need.

## 2.2 Preliminary Result

In the term-examinations, our students improved their scores to the problems similar to problem: **Problem 1** Calculate the distance  $d$  between a plane in 3D space expressed by an equation  $ax + by + cz + D = 0$  and the origin, as shown in Figure 1.

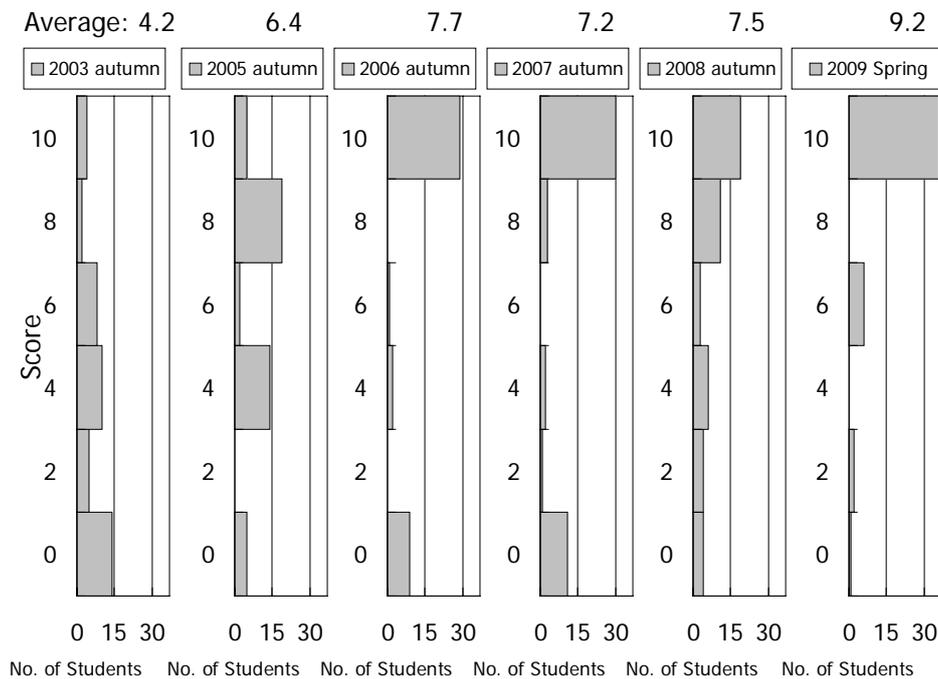


Figure 1 Score Distributions Term-end examinations

The data of 2003 and 2005 academic years are the reference, where only step 6: Pencil-and-paper exercises of the lesson plan were included. The students who learned linear algebra in 2005 academic year are the students who showed low performance at INCT achievement test in 2007.

In autumn semester of 2006, step 5: Deducing formulas was added in lessons, and the result was measured at INCT achievement tests in 2008 and 2009. Although they had higher scores at the term-end examinations at 2006 and 2007 academic years as shown in the middle graphs in Figure 1, their performances at INCT achievement tests stayed as low as the one of 2007. We thought at first that they had understood the mechanism of the vector equations because of the explanations, but they still seemed to depend only on the procedure of solving the given problems.

In autumn semester of 2008 and spring semester of 2009, step 2: Real objects in classroom was introduced in class to all the students, and step 3: Graphic objects in 3D virtual space was used to slow learners for their extra work outside the class using our system. About one third of the students in the class did the extra work with the system. Because the system is under development, we expect the activities in 2009 are more effective than the ones in 2008.

As the result, although the average score of the term-end examination of 2008 academic year stayed the same, fewer students scored zero points. In the term-end examination of spring semester in 2009, the average score increased more, the description in their answer sheets got richer, and the students of lower scores decreased. We expect the students to have a little deeper conceptual knowledge and show higher performance at INCT achievement test in 2010 and 2011.

Adding to that, step 1: Actual application using the game module described in section 3.3, and step 4: Identifying and producing the relation of different representations using the module described in section 3.2 are due to be included in 2009 autumn semester as pilot activities.

### 3 Activities Served by the System

#### 3.1 Module to Visualize the Construction of 3D Vector Equation



Figure 2 Web-pages with 3D graphic objects embedded to the pages

In the steps 1, 3 and 4 of the lesson plan, we use the first module of our system for the students' activities. In step 3: Graphic objects in 3D virtual space of the lesson plan, we use Web-pages which have 3D graphic objects embedded into them. Each Web-page shows several vectors to characterize the position and direction of a graphic object in

virtual 3D space. Most of the graphic objects are animated to show their features. And some of the vectors composing the graphic object can also be manipulated by dragging the end-points with a mouse operation for the students to examine them more in detail. For example, Figure 2 shows an animation of a spirally moving position vector, the end-point of which is restricted on a plane in 3D space. Because the spiral movement is limited on the surface of the plane, remaining points on its trace make a rough image of the plane. If the students rotate the graphic objects and change the viewpoint by mouse operation, they are able to recognize them as a plane from the screen shown on the right side picture of Figure 2.

### 3.2 Module Showing Dynamic Link of 3D Graphic Objects and Vector Equations

Figure 3 shows the second module of our system used in step 3: Graphic objects in 3D virtual space, and step 4: Identifying and producing the relation of different representations of the lesson plan. It connects graphic objects in 3D-space with vector equations. When the students change a parameter in an equation with one of the *sliders* on the screen, the vector equation change the related parameter, and the graphic object also changes the position or direction in the 3D-space simultaneously. In addition, it serves quizzes related to the contents for reproducing what they have learned until then. The module is an application of Mathematica and uses its new feature of linking graphic objects and symbolic expressions dynamically.

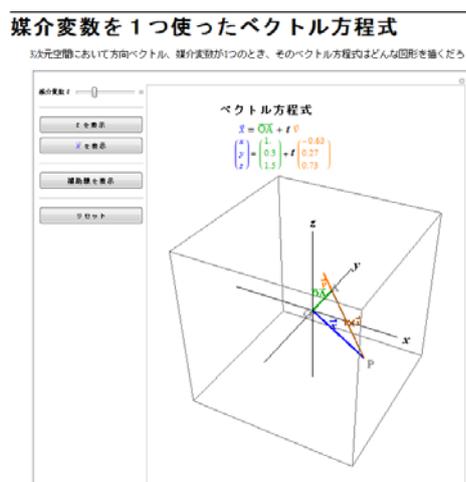


Figure 3 Guided experiments in 3D virtual space

Using the new feature of Mathematica, the module is able to serve identification and production activities for the students [3]. In the identification stage, it is especially important for the students to identify the concepts of parallel and perpendicular, and to express them verbally and also symbolically. In the production stage, the students refine their concept by relating and converting different expressions for the same object in various ways.

### 3.3 Game Module for Introducing 3D Vector Operations

We are also constructing a game module for the step 1: Actual application of the lesson plan because it is especially important for motivating engineering students to convince them that there are actual applications of 3D vectors and vector equations/operations

before their learning (Figure 4).

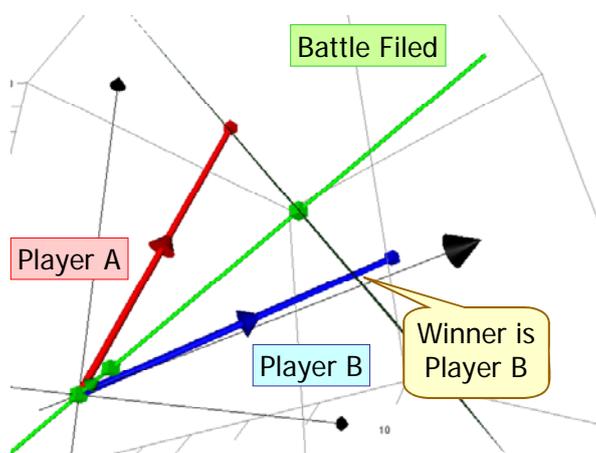


Figure 4 Game module which uses 3D virtual space

In the game module, each student has his/her own player as avatar whose characteristics can be manipulated by 3D vectors. The players compete each other in various kinds of battlefields in the game. When two players fight each other in a battlefield, the strength of each player is calculated by dot product of characteristic vector of each player and one of the battlefields, and the winner of the game is decided by comparing the strength of two competitors. This simple application shows students that a multidimensional vector is a convenient way to express the characteristics of an object, for example, the characteristics of a player in virtual world, which young students visit regularly.

#### 4. Structure of the System

We use two types of software modules in the system.

The first module is a part of a Web-browser/server system. A module constructed with Cabri 3D is embedded in a Web-page and works on the Web-page with the help of Cabri Plug-in software. Cabri 3D objects can be manipulated with mouse-operations of students and they give the students the feel of direct operation. They are preferable to be used in step 3: Graphic objects in 3D virtual space of the lesson plan.

Although the objects in the module are linked to each other and students can learn from the dynamic link, the module itself is a static component in the Web-page. It is embedded into the HTML code describing a Web-page statically and the content is always the same. In another words, it doesn't change according to the student or his/her need. Students are supposed to select the content from the menu on the homepage.

The second module is implemented as a Mathematica notebook and uses its new feature of linking graphic objects and symbolic expressions to each other. The changes of graphic objects and symbolic expressions are controlled by the operation tools, for example the *sliders*, and simultaneous. Although they don't give the students the feel of direct operation, it is enough to be used after the activities with Cabri 3D module.

The second module works as a front-end program of a client/server system that has communication link between front-end programs and a remote database server. The server stores all the learning contents and also the learning histories of registered

students, so the content of the module can be dynamically changed and optimized for each student. Using the database-link, it also serves quizzes to the students, which is also optimized to each student.

## 5. Discussions

The low performance of our students in linear algebra is the direct result of our teaching/learning strategy, where we handle mainly procedural knowledge using compact symbolic expressions. The knowledge may be precise but lacks connections to other mathematical knowledge and real life phenomena. It also tends to attract surface learning because it looks cost-effective if we only think it as the means to earn high enough scores at the term-end examinations.

In that sense, it is not appropriate to evaluate the effectiveness of new lesson plan in the scores of term-end examinations. We need to wait the result of INCT achievement tests. We also would like to discuss more qualitative evaluations of our students learning in 3D linear algebra in this chapter. There are at least two concepts that hinder many students from learning.

### 5.1 Concept of Parallel Vectors

There is a gap between our students' mental image of parallel vectors (left side picture of Figure 5) and the vector equation of parallel vectors (right side picture of Figure 5).

The students' image tells them the two vectors are separated to each other and the start-points of two vectors are *not* the same. On the other hand, the vector equation  $\vec{b} = k\vec{a}$  gives our students the impression that the starting points of two vectors are the same.

Although they are taught that usual vectors only have the direction and the magnitude but not fixed starting point, it doesn't seem to affect their default mental image of Figure 5b. Our practice tells that traditional lessons and exercises of symbolic manipulation are not effective to change their mental image, but the operations with graphic objects are.

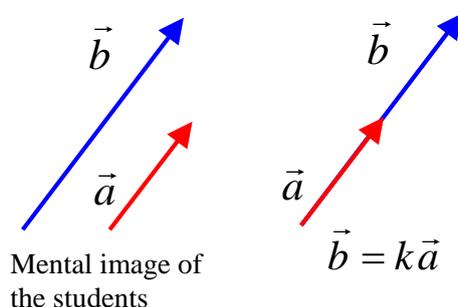


Figure 5 Students' Image of Parallel Vectors

### 5.2 Concept of a Position Vector

The difference of a position vector and a general vector is another point of confusion for our students at first. All the vector equations express how the position vector pointing an

arbitrary decided point on the graphic object behaves, but it is not easy for the students to imagine the position vector graphically. They seem to have the tendency to assume that both the start-point and end-point of the target vector stay in the graphic object, and are misguided by that graphical image.

Observing many graphic examples of position vectors would help them to change their hidden assumption.

## 6. Conclusion

We have been developing a system for students to deeply understand the concept of 3D vectors and use 3D vector equations/operations with confidence in engineering situations. It is meant to be used in several steps of our new lesson plan of introducing 3D linear algebra for engineering students to provide them with the opportunity to handle 3D graphic objects by themselves and to help them to identify the relationships between three representations: graphic objects, verbal explanations, and symbolic expressions.

In this paper, we described some of the features of the system, and preliminary result of using it as the score distributions of term-end examinations. We expect the students who used the system to show better performance in the INCT achievement tests in 2010 and 2011, too.

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