From $f(x)$ to xf: Using technology to promote advanced modern mathematical thinking

*Matthias Kawski**
kawski@asu.edu
Department of Mathematics and Statistics
Arizona State University
Tempe, AZ 85287-1804
USA

Abstract

This article discusses the virtual absence of modern mathematics through not only secondary, but also most curricula at the beginning postsecondary level. This is in stark contrast with the sciences whose curricula more or less emphatically embrace discoveries made in the last few decades. We propose to utilize modern computing technology to create the intellectual need for developing modern mathematical viewpoints. We discuss several examples, primarily at the level of second year of collegiate mathematics where judicious use of modern computing tools can change complacency into excitement about modern mathematical thinking. Mathematical topics include vectors, functions, actions, and transforms.

1 Introduction

We all deplore the silence or trivial views of mathematics when at a party or in the subway a conversion stumbles into mathematics. Whereas today's science curricula throughout include some coverage of discoveries of recent decades, say from DNA in biology to black holes in astronomy, modern curricula in mathematics are almost devoid of similar coverage of major breakthroughs in mathematics in the last half century. There are some exceptions such as a modest inclusion of chaotic dynamical systems in first courses in ordinary differential equations (ODE) or explorations of cryptology. However, even in the case of the recent emphasis on a *dynamical systems* point of view in the first course in ODEs, such changes are modest in the big scheme. It is no surprise that mathematics is struggling to attract and retain large numbers of the brightest and most inquisitive students, when most of its curricula through even the first years of postsecondary education have virtually not changed over many decades, and

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almost completely fail to include most modern aspects of the mathematical revolutions of the last century. Just to name a few, as any such list necessarily is far from complete: Modern life would be unthinkable without the Fast Fourier transform and its younger cousins. Similarly, modern information technology and networks could not work without modern number theory. Since the 1930s [8, 18, 19] analysis and geometry routinely change focus from the space itself to the algebra of smooth or continuous functions on the space, the theory of partial differential equations (PDEs) has been completely rewritten, see e.g. [17], by taking advantage of the theory of distributions. Stochastic models and methods pervade almost all areas. Computational methods and simulation techniques have completely changed mathematical applications. But it appears that a mathematical topic must be at least a hundred, better many hundred years old in order to be mentioned at the secondary and early postsecondary level. It does not help to blame research mathematicians who take too little interest in helping shape curricula at the earlier stages, or politicians, parents, and mathematics teachers and instructors who simply do not have the background to comprehend these revolutions. However, there is a need for bolder actions, for improved communication, and learning: The teachers need to be knowledgable and comfortable with modern mathematics in order to be able to instill excitement about mathematics in the students.
We claim that computing technology can make a huge difference, be an enabler. Unfortunately, all too often computing technology is mainly used to revisit the contents of old curricula, to repeat with the machine what usually was and is done by hand. Much too rarely is computing technology used to bring new exciting mathematics into the classroom, mathematics that connects with the forefront of current mathematical discoveries and developments.

This author has taught courses at a large public university in the United States from the algebra and calculus levels to a diverse set of advanced undergraduate and graduate courses. Admittedly, he has found it extremely hard to find ways to include modern ways of thinking into traditional syllabi. However, two observations apply routinely:

- There are many, many very bright students who have not been interested in mathematics because all they ever have seen about it was extremely dull, *more of the same*. But once they *saw the light*, quite a few became excited and many changed majors *to* mathematics.
- Judicious use of modern computing technology can help not only overcome hurdles resulting from missing prerequisites, but also establish the intellectual need which is the essential fertile ground from which to explore modern mathematics, and develop excitement about the discipline.

Further reiterating the first point is the common experience in the first theorem proving course in analysis (*advanced calculus*) that the brightest students come from other disciplines, who long ago chose *more intellectually stimulating* majors. In turn, many of the declared mathematics majors struggle. This should be no surprise as many decided to major in mathematics because they were good at, and even liked, the kind of mindless symbol manipulation which is all that they have seen over many years in mathematics. We cannot blame either group – it is our own fault that we portray our discipline this way. Do we really want to perpetuate this way? What kind of future mathematics teachers shall we educate?

Traditional mathematics curricula are distinguished by an extreme vertical structure, with prerequisites of prerequisites sitting on top of each other. Many mathematicians take the easy way out, claim that this is intrinsic and impossible to change – but look at biochemistry, biomedicine, physics, astronomy, . . . they managed to bring exciting discoveries into their curricula, and their breakthroughs routinely appear in the evening news on television, and on the title pages of popular news papers. We agree that there is a need for *adequate mathematical maturity* for each level of understanding of every mathematical concept. But each concept can be addressed at many levels – there is no need to do everything the first time! Think of the levels at which students first encounter DNA, black holes, or anti-angiogenesis drugs?

The main point of this article is the view that judicious utilization of modern computing technology can much accelerate this maturing process, establish the *intellectual need* [10] upon which progress is built, and stimulate the excitement about the disciplines among the students. Indeed, according to prevalent theories compare e.g. [6, 7, 24], the learning of mathematical topics follows common tracks from actions through processes to encapsulating these into objects. Our experience, and thesis, is that the judicious use of technology can much accelerate this development, getting the learner ready for the next stage.

With this big picture in mind, this article presents a few small selected examples of using computing technology to prepare for what traditionally are considered more advanced topics and ways of thinking. Suitable technologies include spreadsheets, numerical packages such as MATLAB, and computer algebra systems such as MAPLE and MATHEMATICA. Targeted at the audience of the *atcm* (Asian Technology Conference on Mathematics), especially faculty and teachers familiar with second year collegiate mathematics courses, mathematically we focus on ever more advanced ways of using the *function* concept, including the notion of an *action*, eventually ending with a discussion of why many professional mathematicians write xf rather than $f(x)$. This printed article does not include computer code or animations. Instead we refer the interested reader to the author's websitẽciteMKwww for implementations and demos in both MAPLE and MATLAB.

2 Linear algebra: vectors and functions

2.1 Introduction

The initial focus of the first course in linear algebra (LA) at the college level usually is on solving systems of linear equations. To analyze the structure of the solution set, the notions of vectors, linear (sub)spaces, bases, etc. are developed. For vectors, the common starting point are intuitive notions of vectors as arrows or directed line segments, and the mantra of introductory physics courses that *vectors are quantities with magnitude and direction.* The eventual goal are factorizations of linear transformations in more or less abstract vector spaces. Computing technology is widely used in this course, compare e.g. model projects developed by ATLAST [2]. The well-respected Linear Algebra Curriculum Study Group (LACSG) [3] recommends that this first LA course focus on matrices. Yet there is a clear demand for some abstraction in order to satisfy the needs of the many students who concurrently take a first course in ordinary differential equations (ODE). The latter makes significant use of e.g. the linear independence of a fundamental set of solution to a linear systems of ODEs – and this needs to be connected to the corresponding notion of linearly independent set of vectors in LA. There exists an extensive literature on the cognitive challenges and difficulties encountered by students in LA. We point the interested reader to the recent thesis [22] which includes a comprehensive analysis with a large number of references.

2.2 Functions as vectors

As a technical side note: For every fixed basis, a vector is a function from that basis to the underlying field. But here our interest is mainly in the other direction: Functions taking values in a field, or more generally, in a linear space, are naturally vectors themselves.

Our focus here is the critical task of aiding the student to go from vectors in the plane and 3-space to function spaces (an eventually, abstract vector spaces). We claim that with modern technology this step is much easier than with paper and pencil, chalk and blackboard only. Just like in calculus, there is a clear need for multiple representations, especially multiple graphical representations, of objects in linear algebra. Compare e.g. the emphasis in [23] on understanding both the row picture and the column picture for a system of linear equations. But to pave the way for connecting vectors and functions, let us start with one provocative way

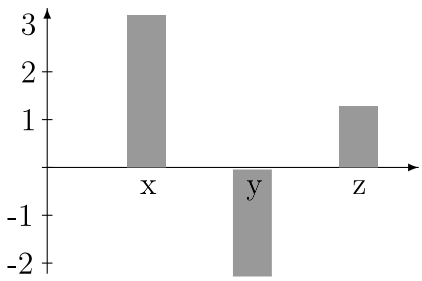

Figure 1: A "graph" of the vector $3\vec{\imath} - 2\vec{\jmath} + \vec{k}$

to graph the vector $3\vec{\imath} - 2\vec{\jmath} + \vec{k}$ as a column diagram, compare figure 1. (Incidentally, EXCEL is our preferred tool for instantaneously switching between different graphical representation of the same data – often suggesting very different points of view.) Clearly this is not the way this course usually thinks of vectors, and not the way to be cultivated at the heart of the LA course.

Yet this is a perfectly legitimate view which opens many doors, and which is much more important than it may initially seem: On one hand, it immediately connects with the outside world – students find vectors in the column charts in every newspaper, not only in the section on stock-indexes with their long columns of numbers, and hence find vector thinking useful in everyday life. Moreover, the componentwise (pointwise) algebraic operations for linear combinations of vectors (functions) are natural. But more importantly, in this picture, just like in the numerical view of column vectors, it is easy to go beyond three dimensions. This view is already helpful when extending least square approximation problems from the point on a given plane closest to a given point to the usual least squares fits of sampled data – again see [23] for a collection of masterfully crafted examples which straddle the different points of view.

Before such least square fitting problems, interpolation problems are a routine source of applications of systems of linear equations. For example the problem of finding a quadratic function $p(x) = c_1 + c_2 x + c_3 x^2$ that interpolates the points $(1,2)$, $(2,9)$, and $(3,22)$ is translated into the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 22 \end{pmatrix}. \tag{1}$$

In the author's experience it has proven to be very helpful to continually work with examples whose columns are easily recognized as the values of familiar functions at usual inputs. A slightly more elaborate example obtains the coefficients in the trigonometric identity $\sin^5 x = \frac{5}{8}\sin x - \frac{5}{16}\sin 3x + \frac{1}{16}\sin 5x$, formulated here as a least squares problem.

$$\text{minimize } \left\| \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4}(\sqrt{6}-\sqrt{2}) & \frac{1}{2}\sqrt{2} & \frac{1}{4}(\sqrt{6}+\sqrt{2}) \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{3} & 0 & -\frac{1}{2}\sqrt{3} \\ \frac{1}{4}(\sqrt{6}+\sqrt{2}) & -\frac{1}{2}\sqrt{2} & \frac{1}{4}(\sqrt{6}-\sqrt{2}) \\ 1 & -1 & 1 \\ \frac{1}{4}(\sqrt{6}+\sqrt{2}) & -\frac{1}{2}\sqrt{2} & \frac{1}{4}(\sqrt{6}-\sqrt{2}) \\ \frac{1}{2}\sqrt{3} & 0 & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4}(\sqrt{6}-\sqrt{2}) & \frac{1}{2}\sqrt{2} & \frac{1}{4}(\sqrt{6}+\sqrt{2}) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{64}(11\sqrt{6}-19\sqrt{2}) \\ \frac{1}{32} \\ \frac{1}{8}\sqrt{2} \\ \frac{9}{32}\sqrt{3} \\ \frac{1}{64}(11\sqrt{6}+19\sqrt{2}) \\ 1 \\ \frac{1}{64}(11\sqrt{6}+19\sqrt{2}) \\ \frac{9}{32}\sqrt{3} \\ \frac{1}{8}\sqrt{2} \\ \frac{1}{32} \\ \frac{1}{64}(11\sqrt{6}-19\sqrt{2}) \\ 0 \end{pmatrix} \right\|^2. \tag{2}$$

In this example, the columns of the coefficient matrix and the right hand side are recognized as the familiar values of the functions $\sin x$, $\sin 3x$, $\sin 5x$ and $\sin^5 x$ at integer multiples of $\frac{\pi}{12}$. In a typical class, we would use MATLAB or similar software to easily generate much larger matrices. A key trick is to make the columns so long that they no longer fit on one screen, but require some scrolling. In the author's experience this has much helped students to think of functions (of a real variable) as in some sense just very long column vectors. Without technology, this is much harder to experience: just because the teacher says that it is so, does not at all mean that the students deeply internalize this identification.

An intriguing aside is that sooner or later the class will address the common notational quandary: Could one actually write f_x and $v(i)$ for what commonly is denoted by $f(x)$ and v_i? Of course, the answer is "yes" – and deeply understanding the freedom to choose really helps in subsequent classes and opens many doors. Notation matters, and often determines the way one thinks about a problem. Our students found it easy to think of functions as corresponding to infinitely long column vectors – due to the uncountability of the reals, this seems fraud with danger, but it really does not cause any trouble – while for practical purposes, like in a table of function values generated for plotting its graph, one routinely uses only finitely many values.

2.3 Inner products and integrals

Using traditional course materials without routine use of technology, many students found it hard to connect the usual dot-product

$$\langle (u_1, u_2, u_3)^T, (v_1, v_2, v_3)^T \rangle = \sum_{i=1}^{3} u_i v_i \tag{3}$$

and more general inner products on $\mathbb{R}^3$ with the usual inner products of matrices $\langle A, B \rangle = \text{trace}(A^T B)$ and functions $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$. But students who are used to thinking of functions as very long column vectors found the latter most natural, by recognizing that

$$\langle (f_1, \ldots, f_n)^T, (g_1, \ldots, g_n)^T \rangle = \sum_{i=1}^{n} f_i g_i \,\Delta x \approx \int_{x_0}^{x_n} f(x)g(x)dx \tag{4}$$

is basically a Riemann sum approximating the integral, where we chose to write f_i for $f(x_i) = f(x_0 + i\Delta x)$. The scaling by $\Delta x = \frac{1}{n}(b - a)$ is easily seen as a means to make the inner product essentially independent of the number of points used, of the length of the column vector representation. With this point of view, orthogonality of polynomials and of functions in general also looses much of the usual mystique.

2.4 Changes of bases and transforms

Changes of bases are a core syllabus item in LA, most important in the task of diagonalizing matrices using bases of eigenvectors. However, this task is rarely connected with the topic of transforms that students encounter almost at the same time in their ODE course, most commonly Laplace transforms, but also Taylor expansions and Fourier transforms. Yet, with technology, by identifying (really: approximating) functions with (by) long column vectors, it is only a little step from the preceding example involving a trigonometric identity to understanding Fourier coefficients just as the coordinates of a given function with respect to a different basis, e.g. a set of functions of the form $\sin kx$ or $\cos kx$ for $k \in \mathbb{Z}$.

This also opens the door to more tantalizing subjects: In the case of orthogonal sets of basis vectors / functions, the coordinates of any given functions are found by simple inner products – compare the above example with the Fourier coefficients $b_k = \frac{2}{\pi} \int f(t) \sin kt \, dt = \langle \sin kt, f(t) \rangle$. From this point of view, the standard basis of vectors e_i in the limit becomes a set of delta distributions. Suggestively written $f = (f_1, \ldots f_N) = \sum_{i=1}^{N} f_i e_i$ and $f_i = \langle e_i, f \rangle = e_i^T f$ become $f(x) = \int_a^b \delta(x - t) f(t)\, dt$.

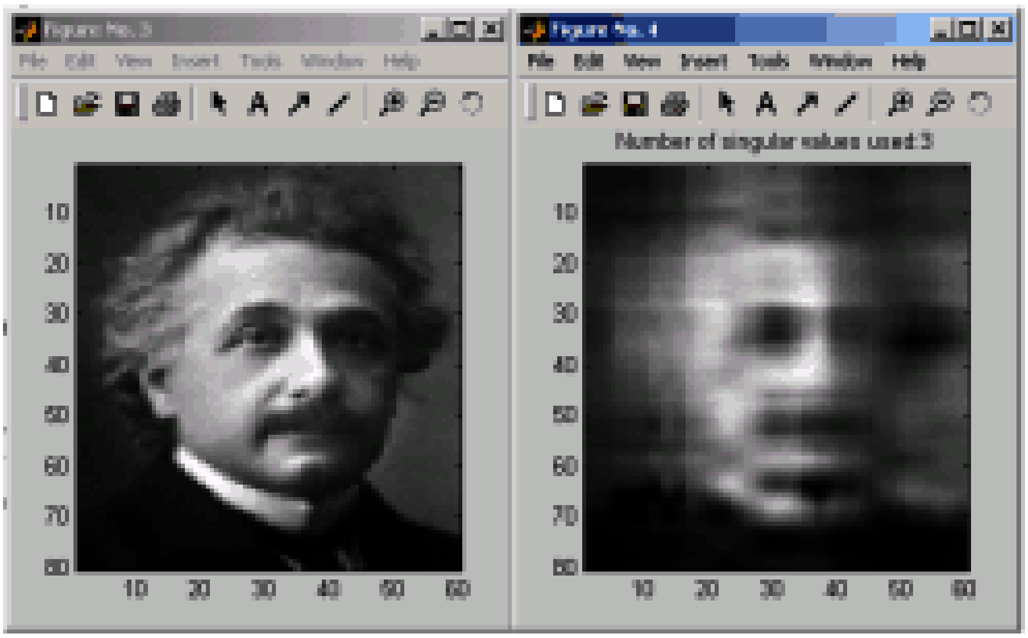

Figure 2: Image compression in linear algebra, here using SVD

2.5 Projection and data compression

We finally note that in the author's experience, many students actually find it not only much more exciting, but in fact also easier in several settings to work with functions of two variables. Particularly successful have been explorations and projects involving least squares approximations and compression of two-dimensional images. This has been much more successful than similar projects involving audio signals, even with the current iPod generation of students. Starting with bitmap grey-scale images from the WWW, we import these into MATLAB and represent them as mid-size matrices, typically of size 60 times 80, or similar, and recognize these as vectors in a 4800 dimensional vector space (or 4800 sampled values of functions of two variables). The objective is to reduce the size of these data (4800 numbers) without much detoriating the images. Routine solutions use Fourier approximations that *filter out* some high frequency content, that is projections onto subspaces spanned by discretized products of the form $\cos(2m\pi/60) \cdot \sin(2n\pi/80)$ etc.

Even more tangible are subspaces spanned by suitably chosen piecewise constant functions, e.g using the two dimensional Haar wavelets which are simple linear combinations of characteristic functions of suitable scaled and translated rectangles. Again, it is possible to choose orthogonal such sets, and it makes a nice project to implement *fast* versions of the Haar wavelet transform. Sample MATLAB implementations and demos of various such approximations are available online from the author's wwwsite [15].

In summary, we note that there are many ways in which modern technology can help the students to almost seamlessly expander/his horizons beyond finite dimensional vector spaces and matrices, and develop intuition for how the core LA material generalizes to much more tantalizing settings. But likely most important is that rather than having to be intimidating, technology allows the student to go far beyond the routine drudgery of working with cooked-up small matrices, and rather catch a glimpse of advanced modern ways of mathematical thinking, explore new horizons an set the sights for still unexplored territories, come up with novel applications.

3 Solutions of partial differential equations

The first course on partial differential equations (PDEs) typically studies elementary properties and solution methods for constant coefficient linear PDEs of elliptic, parabolic and hyperbolic type. In this setting numerical, graphical, and symbolic computation packages all can make a huge difference by enabling the student and making her/him comfortable to ask truly inquisitive questions that were unthinkable in the traditional setting of manual symbol shuffling.

3.1 Solutions that are not differentiable?

A typical example/exercise found in most textbooks involves he d'Alembert solution of the second order one-dimensional wave equation $u_{xx}-u_{tt}=0$ on the semi-infinite strip $[0,\pi]\times[0,\infty)$, with e.g. zero boundary conditions $u(0,t)=u(\pi,0)=0$ for all $t>0$ and initial shape defined by $u(x,0)=\frac{\pi}{2}-|x-\frac{\pi}{2}|$. It is routine to write down the d'Alembert solution as the average (superposition) of a left and a right traveling wave

$$u(x,t)=\frac{1}{2}\left(\frac{\pi}{2}-|x-t-\frac{\pi}{2}|+\frac{\pi}{2}-|x+t-\frac{\pi}{2}|\right). \tag{5}$$

Fine, done. The author cannot recall any student ever questioning how this can be the solution of a second order PDE as this function is not even once differentiable!

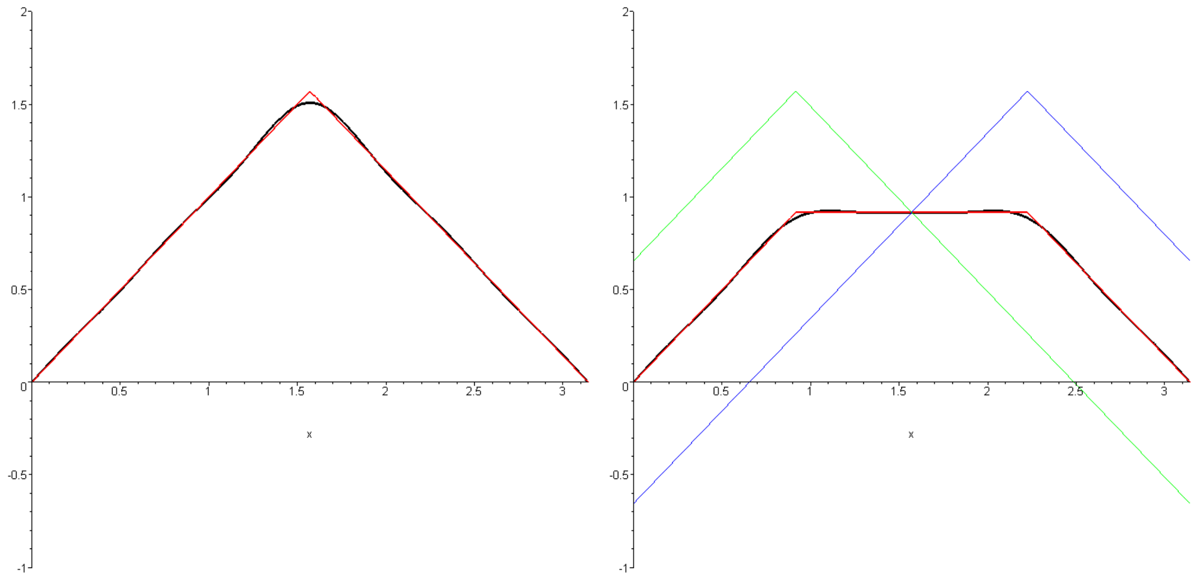

Figure 3: d'Alembert solution of the wave equation with nonsmooth initial data

At least, with modern technology we now always plot and animate any such solutions – and the corners in the pictures do raise some eyebrows among the students, and the less timid even ask what one means by a solution. This is indeed the question which opens the door to the completely rewritten modern theory of PDEs which is now routinely formulated in the sense of distributions, compare, e.g., [17].

The first class on PDEs cannot possibly go into the details of this modern theory – but it certainly is possibly to give a mathematically correct description conveys the principal ideas underlying the modern way of thinking. This is the kind of brain-food that the inquisitive students – whom we would love to attract to mathematics – are hungry for. Students love to come up with ideas for what one could consider a derivative of the absolute value function. Yet in our calculus classes we routinely cut them off, stifle all inquiry. Indeed, at our institution various related kinds set-valued derivatives (compare [4]) are routinely used in economics, but not taught in mathematics!

3.2 Approximate solutions

It is a routine exercise in any first course on PDEs to calculate formulas for the Fourier expansion of the initial condition such as above, and use this to obtain a closed form solution of the PDE in series form. For the above example this reads

$$u(x,t) = 4\sum_{k=0}^{\infty} \frac{(-)^{2k+1}}{(2k+1)^2} \sin((2k+1)x)\, \cos((2k+1)t). \tag{6}$$

Often some attention is paid to relate the quadratic rate of convergence the coefficients $b_{2k+1} \sim (2k+1)^{-2}$ to the continuity, but lack of differentiability of the initial condition. It is clear from general theorems that for sufficiently large values of N the partial sum of the first N terms is a good approximation of the true solution. However, before the advent of modern computers this was it. Little more than painful and dull practice of first year calculus integration techniques.

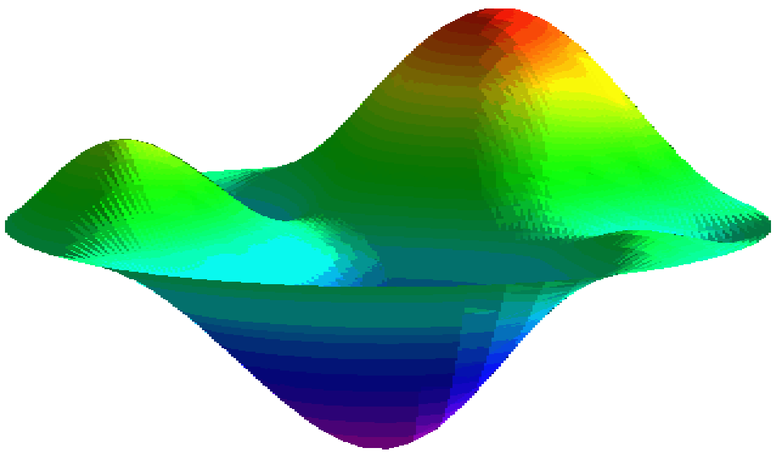

Figure 4: Frame of an animation of a nonperiodically vibrating drum

But now it is a child's play to graphically explore the quality of the approximations, and one may observe that $N = 3$ or $N = 5$ already are large enough to guarantee excellent matches. Indeed, this traditional area of our curriculum is one where computer algebra systems as well as numerical software packages together with the graphical capabilities can completely change the courses. Students are mesmerized, ask excellent questions, and quickly go on to their own projects, make intriguing observations and their own discoveries. This includes in particular non-rectangular domains which lead to eigenfunctions that are classical special functions, but which have recently found only little attention in our curricula. In general the vibrations are non-periodic and make for outstanding explorations, compare figure 4. Again for sample implementations into MAPLE and into MATLAB see the author's website [15]. But as exciting as this use of technology is, the instructor's attention should still be on how to channel the excitement towards current research issues – most likely, in this area, this involve more scientific computing as eigenfunction expansions for irregular shaped domains of interest will only rarely be accessible in closed form. A beautifully simple and tangible question asked by Kac in 1966[13] was only quite recently answered negatively [9]: *You can't always hear the shape of a drum* (see [25] for a quick description).

4 Ordinary differential equations: What acts on what?

4.1 Dynamical systems and flows

The first course on ordinary differential equations (ODEs) has seen some significant evolution over the last 20 years, compare [5]. Most prominent are the general emphasis on dynamical systems, their qualitative theory, and some attention to chaotic systems, and the routine use of some computing technology all at the expense of closed form solution techniques for special families of ODEs, many of them dating to the late 1800s. But this is not a reason to rest – indeed we need to continually reevaluate where we are, and strive to further connect undergraduate classes with cutting edge work.

One important innovation in the dynamical systems point of view is that instead of looking at a parameterized family of solution curves dynamical system, one considers the *flow* of the differential equation which is a function of both the initial condition and the time – with usual time considered the parameter, and the initial condition the primary input. For technical reason we may assume that all vector fields in the sequel are smooth and have compact support – else simply multiply the vector field by a smooth *cutoff* function that is constant equal to 0 outside some large ball. This assures that the flow Φ of a vector field f on $\mathbb{R}^n$ will be globally defined. It is the function $\Phi\colon \mathbb{R} \times R^n \mapsto \mathbb{R}^n$ that satisfies for all $x \in \mathbb{R}^n$, $\Phi_0(x) = x$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\frac{\partial}{\partial t}\Phi_t(x) = f(\Phi_t(x))$. Arguably the most important property is that Φ is an action of the abelian group $(\mathbb{R}, +)$ on $\mathbb{R}^n$, i.e. for all $s, t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$,

$$\Phi_{t+s}(x) = (\Phi_s \circ \Phi_t)(x). \tag{7}$$

Actions are one of the most important tools in advanced modern mathematics, and this is one of the most tangible examples – a great training ground if accompanied by a teacher who has a vision of where this is leading.

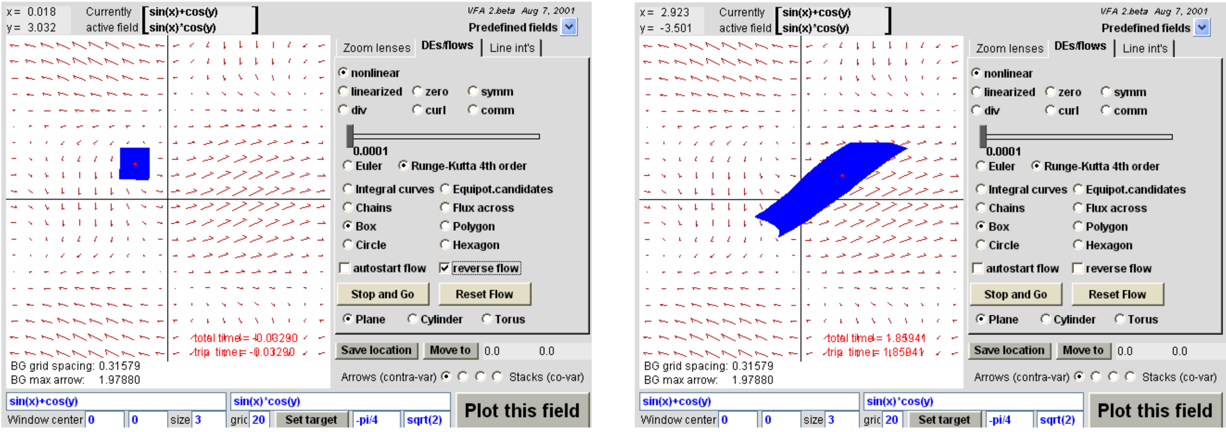

Figure 5: The flow of an ODE acting on regions

But much of the power of this construct is that Φ not only acts on individual initial conditions, but also on entire regions (subsets) of initial conditions – and it is this effect which gives tremendous insight into the dynamical system. The Vector Field Analyzer II [16] is a simple to use JAVA applet which allows for many novel innovative explorations of properties of flows, and thus of the corresponding differential equations, compare figure 5 for a snap-shot.

4.2 Action of flows on functions

Suppose that f is a smooth vector field on $\mathbb{R}^n$ and $F(0) = 0$, i.e. the origin is an equilibrium point. The differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$ with flow Φ is asymptotically stable if

- for every open neighborhood W of the origin and every $x \in \mathbb{R}^n$ there exists $T \in \mathbb{R}$ such that for all $t > T$, $\Phi_t(x) \in W$ and
- for every open neighborhood V of the origin there exists an open neighborhood U of the origin such that for every $x \in U$ and every $t > 0$, $\Phi_t(x) \in V$.

In plain English this mean that every solution curve eventually gets arbitrarily close to the origin (attractive) and every solution that starts sufficiently close, will stay close to the origin (stable). A routine tool for ascertaining asymptotic stability are Lyapunov functions. These are functions $V \colon \mathbb{R}^n \mapsto [0, \infty)$ that are continuously differentiable, are radially unbounded (or proper), satisfy $V(x) = 0$ if and only if $x = 0$, and for all $(t, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$

$$\frac{\partial}{\partial t} V(\Phi_t(x)) < 0. \tag{8}$$

Lyapunov's theorem ascertains that if such a function exists, then the system is asymptotically stable. The converse is true, too.

The traditional picture is that of solution curves of the system lifted to the graph of V will decrease in their last coordinate. Figure 6 is a typical such picture which illustrates what can go wrong if the function V is positive definite and monotonically increasing with increasing

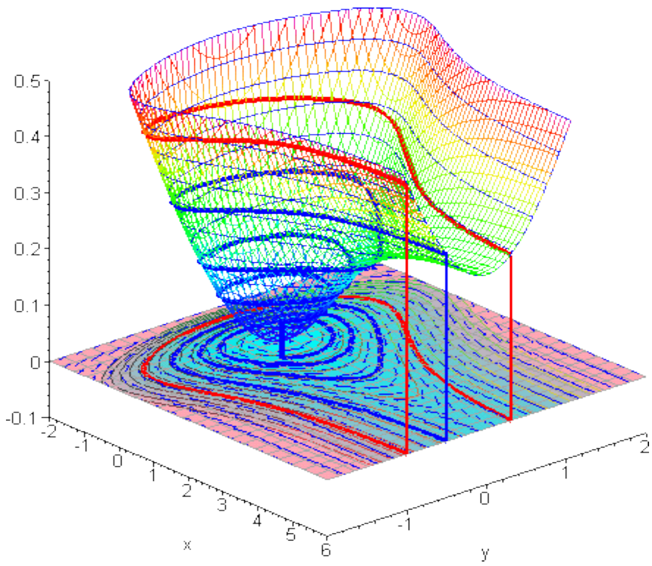

Figure 6: The traditional picture of a Lyapunov function and lifted trajectories

distance from the origin, but not being radially unbounded. Anyhow, this is all routine material in the first or second undergraduate course in ODEs.

Here we are after a different point of view. The traditional view focuses on the function $(t,x) \mapsto (V \circ \Phi_t)(x)$ which for every fixed initial condition is strictly decreasing. Here we look at the dual action – namely the flow Φ_t also acts on the algebra of smooth functions: For every $t \in \mathbb{R}$, Φ_t maps any function $V\colon \mathbb{R}^n \mapsto \mathbb{R}$ to the function $(V \circ \Phi_t)\colon \mathbb{R}^n \mapsto \mathbb{R}$. Indeed we would like to write $\Phi_t(V)$ which is perfectly justifiable.

A familiar starting point, that fits this picture, is to look for functions $V\colon \mathbb{R}^n \mapsto \mathbb{R}$ that are left invariant by the flow, i.e. for all t, $V \circ \Phi_t = V$. In classical terminology, these are called *fist integrals* – ingredients of implicit descriptions of solutions of ODEs. But here we want to go further, and leave it as an exercise to experiment what the flow of any simple, say linear vector field will do to simple functions V, e.g. start with linear and quadratic functions, not necessarily positive definite. Can you see what happens in the case of asymptotically stable systems? In the case that V is a Lyapunov function? Going a little further in this direction leads to the beautiful recent research paper [21]. Or paper article here cannot do justice to this action as the most impressive views demand animations of the family of graphs of $\Phi_t(V)$ (in our preferred notation), typically showing contours and using color coding by the last coordinate. A typical MAPLE command – this is just a first try! – is

```
V:=(x,y)->x^2+y^2;
Phi:=(t,x,y)->[exp(-t/10)*(cos(t)*x+sin(t)*y),
                exp(-t/10)*(sin(t)*x+cos(t)*y)];
display([seq(plot3d(V(op(Phi(log(tau),(x,y)))),
    x=-10..10,y=-10..10,style=patchcontour,shading=ZHUE),
    tau=1..500)],insequence=true,orientation=[-90,0]);
```

This will take some time to get comfortable with – but it is the beginning of a big story. Indeed, in many branches of advanced mathematics, instead of analyzing what happens to the points in a *space*, one instead analyzes the corresponding effects on the algebra of continuous or smooth functions which generally has better properties. This major innovation dates back to the 1930 when it was pioneered by the likes of I. Gelfand [8], M. Stone [19], and J. v. Neumann[18], and has now become the standard way of doing analysis and studying geometry. Even in applied areas such as control theory this approach has proven fruitful under the name of chronological calculus [1, 14]. Also in some of the leading abstract algebra textbooks subsequent editions have switched between preferring left and right actions – unsure whether to make it easy for the incoming novice, or help him/her gain the next level [11, 12].

As mentioned before, notation matters – it suggests a way of thinking. We observed that $\Phi_t(V)$ better reflected this point of view than the traditional $V \circ \Phi_t$. But if one was to keep this, and wanted to add in the x, it would have to be written on the left, leading to $x\Phi_t V$. Indeed this makes perfect sense, does not require any parentheses, and is the preferred language of many mathematicians. The key is that points x are dual to functions, and act on them on the left, while transformations (such as flows of ODEs) act on functions from the left and on points on the right. Function composition now appears in the natural order: xfg stands for the traditional $g(f(x)) = ((g \circ f)(x)$ which means that f is applied first! The next time a calculus student in your class makes the usual mistake, just smile, and encourage him/her that (s)he is just thinking like top mathematicians – *"however, in our class we have to live with the traditional cumbersome conventions ..."*

Another good place where to notice this duality of points and functions is when discussing linear independence of sets of functions $\{f_1, \ldots, f_n\}$, e.g. in the context of fundamental sets of solutions to systems of ODEs. One common tool is to use the Wronskian whose entries are successive derivatives evaluated at a point $W_{ij}(t_0) = f_j^{(i)}(t_0)$. But especially in the case that the formulas for the f_j involve products, computing these derivatives may be cumbersome. In many cases it is easier to choose n distinct points $t_1, \ldots, t_n$ and instead argue analogously in terms of the matrix with entries $f_j(t_i)$. In this case, each column contains the list of values of one of the functions at the sampled points – the format that is familiar from high school. But one may as well read the matrix along the rows – and in this case we get the results of each time, or point, t_i acting on a set of functions f_j – the format that opens the big doors.

Incidentally, this point of view of xf is not that unfamiliar: To evaluate e.g. $\sin(30^{\circ})$ on a basic scientific calculator one first enters 30, and then presses the `sin` button – this is the traditional reverse polish or postfix notation [20]. However, on advanced graphing calculators (with symbolic capabilities) the buttons are pressed in the reverse order, first type `pi / 6` and then press the `sin` button.

5 Conclusion

There are a lot of exciting developments and breakthroughs in cutting edge mathematics happening all the time. There are many mathematicians who are excited about their discoveries and who have a great time. But we have to do better, and can do better, to share this ex-

citement with our students at all levels and with the general populace. Emerging computing technologies that are becoming ever more pervasive in our classrooms provide unique opportunities to help bridge the gap, make the connection. There are many topics in our syllabi which in the big scheme are really not that important, are often dull and repetitive – let us be bold and err on the side of bringing in too much, rather than too little modern, exciting mathematics into our classrooms.
This requires innovative teachers who have an open eye to notice great opportunities, who have a solid knowledge of what is happening at the cutting edge, and feel comfortable with it. The teachers have to continue educating themselves, and more leading researchers have to become better at sharing their ways of thinking with the larger community.

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