Animations that Help Students Construct Knowledge in Mathematics

Darel W Hardy
Colorado State University
hardy@math.colostate.edu

Abstract

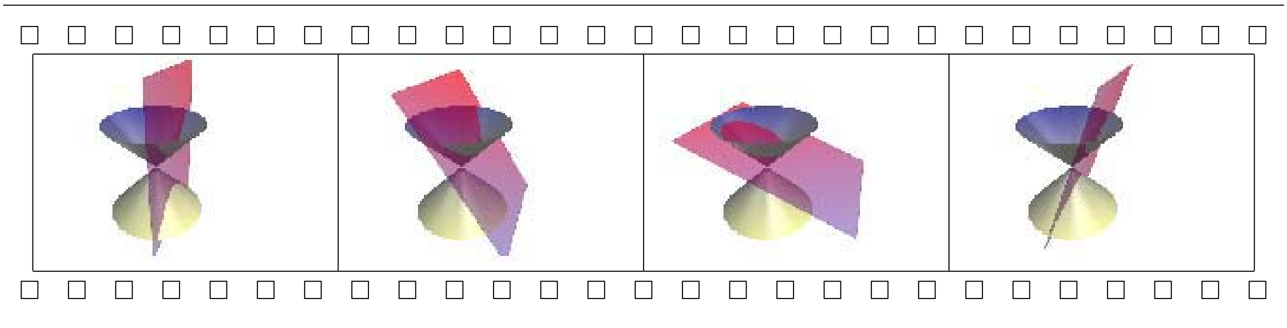
Traditionally mathematics teachers lecture and students learn by listening. Students develop a narrow set of skills, which quickly fade. Active approaches to learning show that students can indeed develop deep understanding that does not fade over time. Computer animations actively engage students in the learning process. If a picture is worth a thousand words, then pictures that move must be worth a fortune. Wouldn't it be great if students could create their own animations? Unfortunately, most computer animations currently used for mathematics instruction are written in languages such as Java, Maple, or Mathematica. Creating such animations requires special syntax and knowledge of the underlying system. Scientific Notebook makes it easy to generate animations. To watch a hyperbolic paraboloid transform into an elliptic paraboloid, enter $x^2 + ty^2$, choose **Plot 3D Animated** *from a menu, and change the t-interval to $-1 \leq t \leq 1$. Multiple formulas can be dragged to the plot frame one at a time to create sophisticated animations. In this talk, a variety of novel animations demonstrate fundamental concepts from algebra, geometry, trigonometry, and calculus. These animations can be used by the instructor as classroom demonstrations or assigned as lab exercises for the students to create for themselves.*

Introduction

The following are some of my favorite animations. I have used them in the classroom and have given them as assignments to my students. Although some of the animations appear to be complicated, the individual curves and surfaces can be added one at a time with Scientific Notebook, using standard mathematical notation and choices from menus. Students learn to work very quickly in this environment, and have no need to learn about special programming or syntax. The talk includes additional examples and real animations, rather than merely film strips. As you read this, try to visualize the missing frames in each of the following series of snapshots.

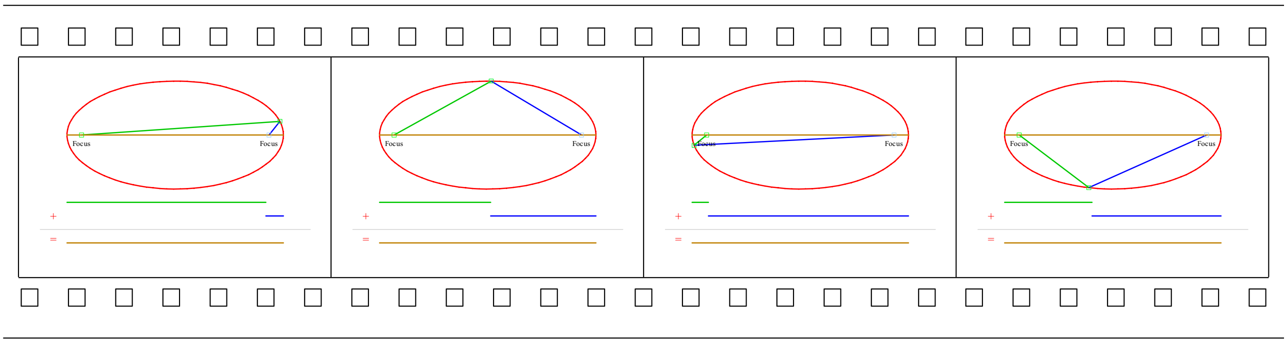
Curves

A conic section can be generated by intersecting a cone with a plane. Here is what happens when a plane rotates so that it cuts a cone at different angles.



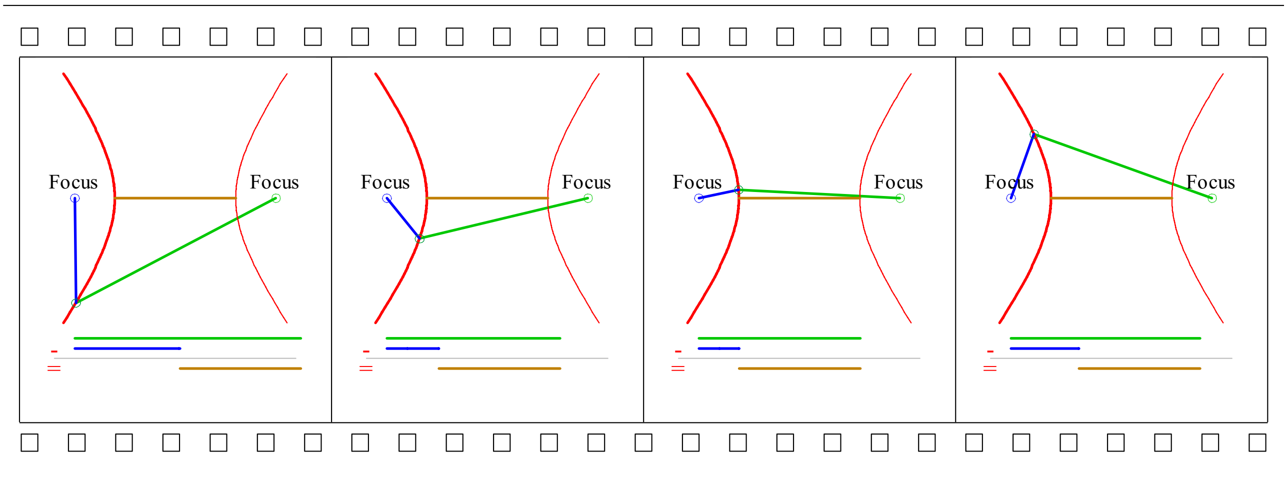
A vertical plane cuts the cone in a hyperbola. At one particular angle, the plane cuts the cone in a parabola. Rotate the plane a bit more and you see an ellipse, then back to a hyperbola. The four views do not do justice to the live animation, which allows the user to zoom in and out, or rotate the whole picture horizontally or vertically while the plane rotates about a fixed axis.

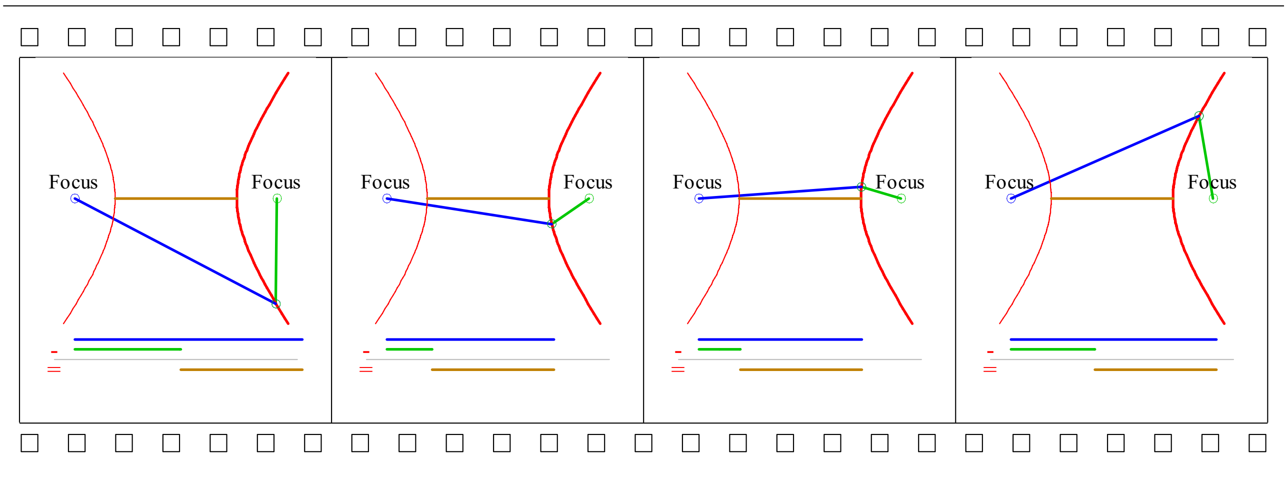
An *ellipse* consists of all the points in a plane, the sum of whose distances to two fixed points (the foci) is constant. This definition can take on meaning by using an animation, as suggested in the following film strip.



Note that as the point on the ellipse moves, the sum of the lengths of the green and blue line segments is always the same.

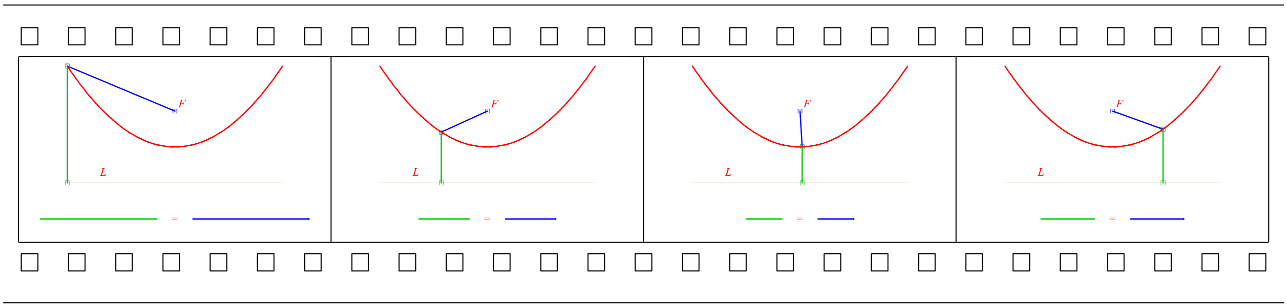
A *hyperbola* consists of all the points in a plane whose distances from two fixed points (the *foci*) differ by a constant. This definition is much easier to understand by viewing an animation. Here are two series of snapshots that show the two branches of a hyperbola.





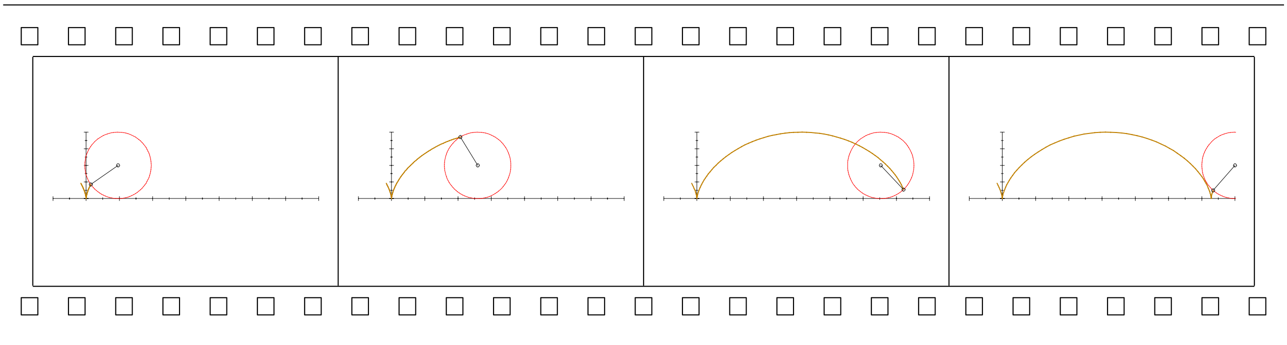
Note that the difference in the lengths of the green and blue line segments is always the same. This common difference is also the (minimum) distance between the two branches.

A *parabola* is a curve in the plane determined by a point F, its focus, and a line L, its directrix. A point is on the parabola if its distance to F is the same as its distance to L. This definition comes alive using an animation. The following film strip shows a few snapshots out of such an animation.

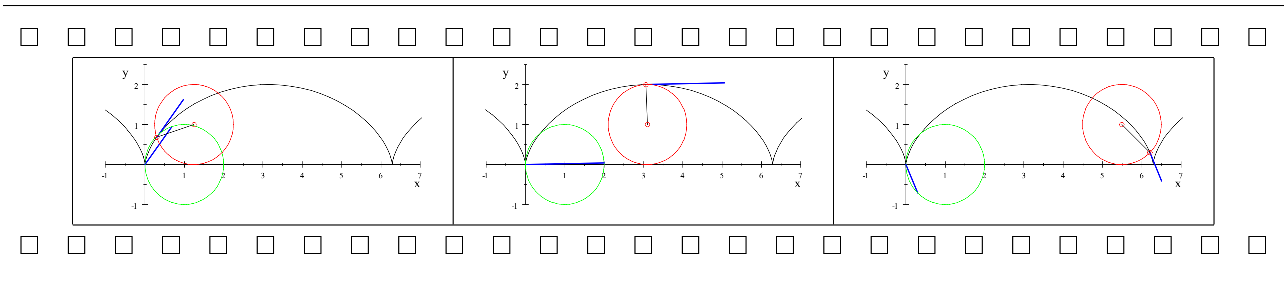


Note that the green and blue line segments always have the same length.

A *cycloid* is the path of a point on the edge of a circular wheel as the wheel rolls along a straight line.

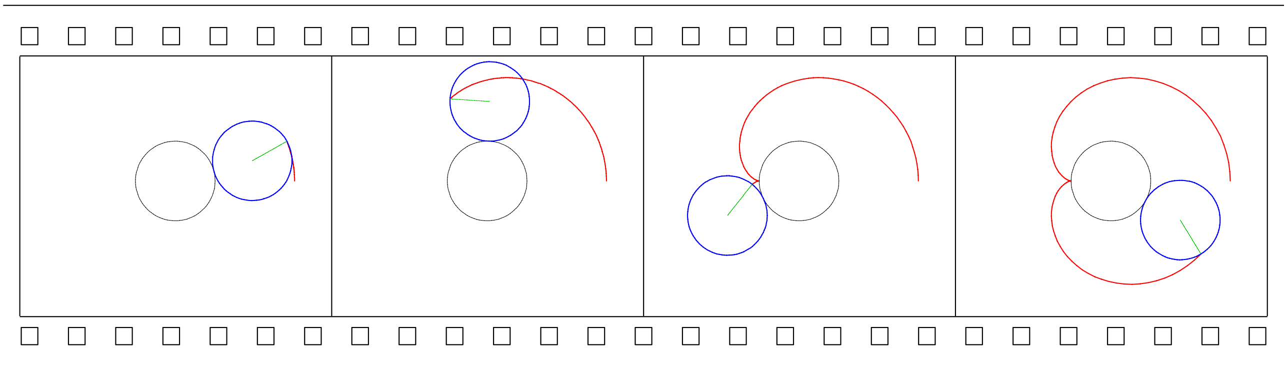


The following film strip illustrates the velocity of a point on the circular wheel, assuming the wheel rolls along at a constant speed.

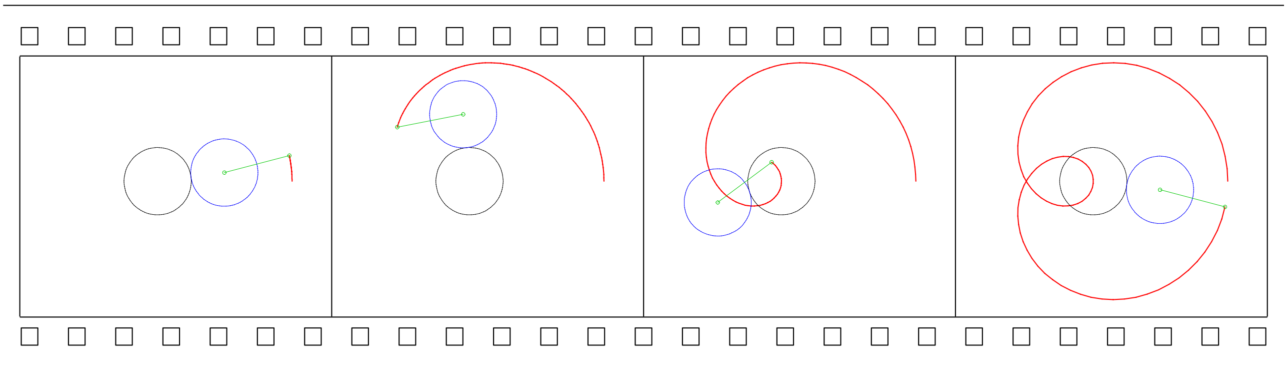


The blue line segment represents the speed and direction of the point at each instant. Note that the blue line segment can also be generated by moving a point along the circumference of the fixed green circle. The point at the top of the wheel has the greatest velocity. The point where the rubber meets the pavement has zero velocity.

Other interesting curves can be constructed by tracing a point on the edge of a circular wheel as the wheels rolls along some other curve. If the second curve is also a circle with the same radius, you get a *cardioid*.



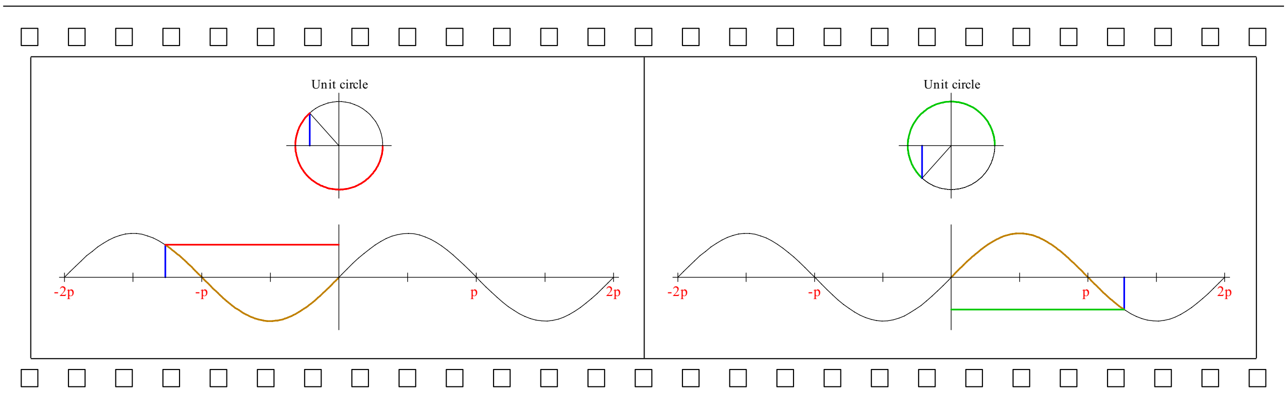
Other curves can be generated by extending the radius of the moving circle beyond the circle. Here is a typical example of such a curve, called a *Limaçon.*



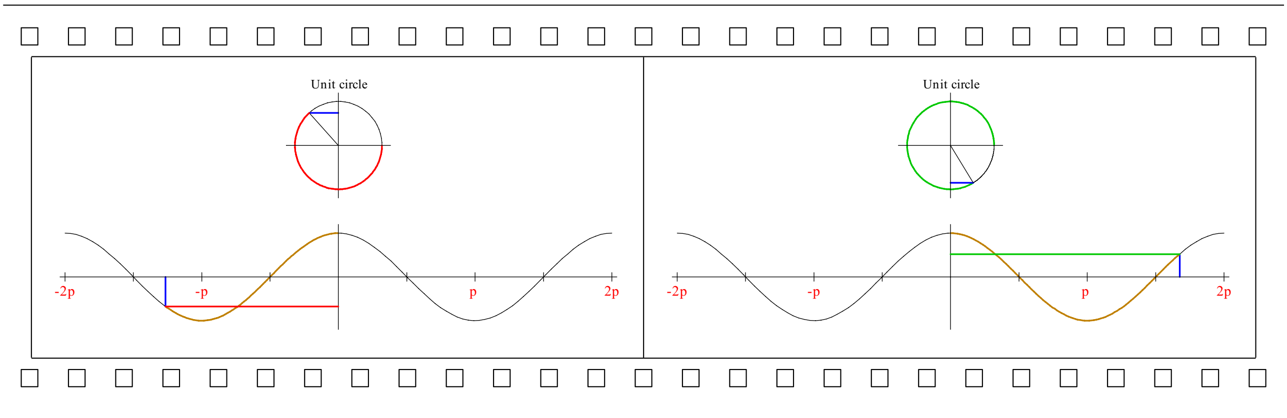
Functions

Animations can also help explain the definitions of some standard functions, such as the trigonometric or circular functions. To define the trigonometric functions, look at points on the circle C of radius 1 around the origin. A point (x, y) is on C if $x^2 + y^2 = 1$. For each real number θ, we define a point $P(\theta)$ on C. If θ is positive, we start at the point $(1, 0)$ on C and travel a distance θ from $(1, 0)$ counterclockwise along C. If θ is negative, we go clockwise. The point on C where we end up is $P(\theta)$ and is called the *trigonometric point corresponding to* θ.

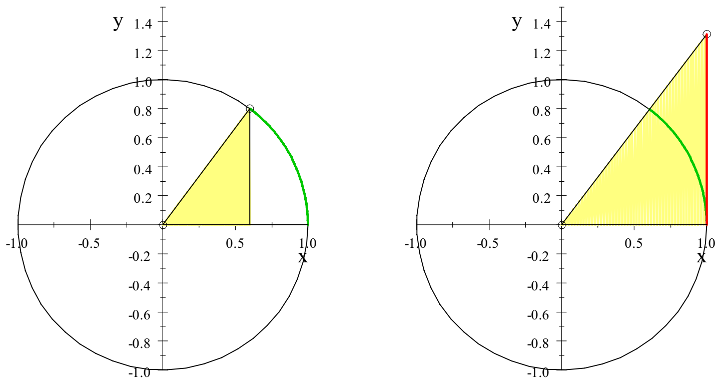
The *cosine* and *sine* functions $\cos\theta$ and $\sin\theta$ are defined by $(\cos\theta, \sin\theta) = P(\theta)$. Here are two snapshots of an animation that illustrates the definition of the sine function.



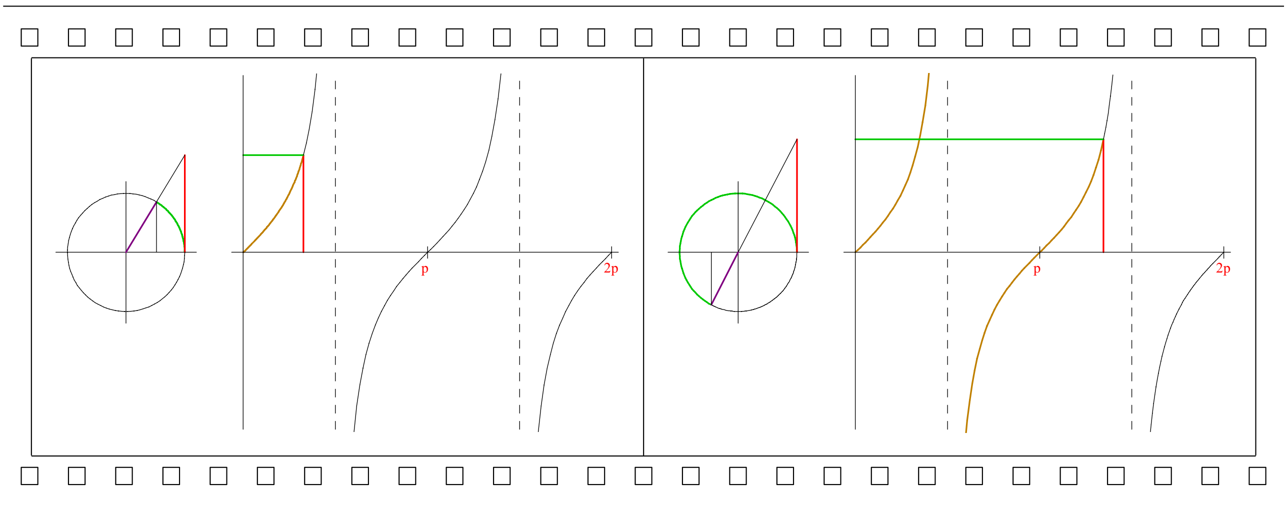
Green corresponds to positive θ and red to negative θ. The two vertical blue line segments are equal. Following are two snapshots that illustrate the definition of the cosine function.



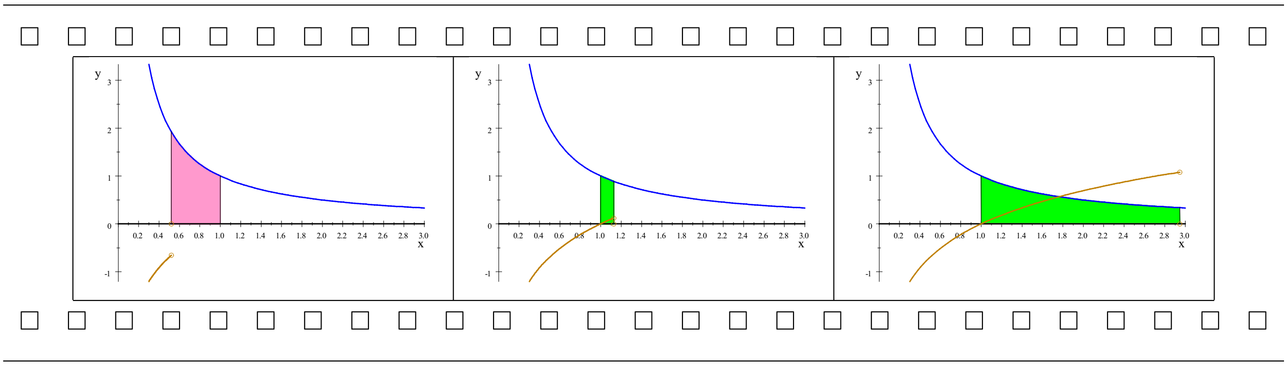
The tangent is the ratio of the sine divided by the cosine. By considering similar triangles, the tangent can be understood by viewing an animation that shows the tangent as a fraction with denominator 1.



Here are two snapshots from such an animation.



The natural logarithm $\ln a$ is defined as the signed area of the region under the graph of $y = 1/x$ and above the x-axis between the vertical lines $x = a$ and $x = 1$. If $a > 1$, then $\ln a$ is positive and if $a < 1$, then $\ln a$ is negative.

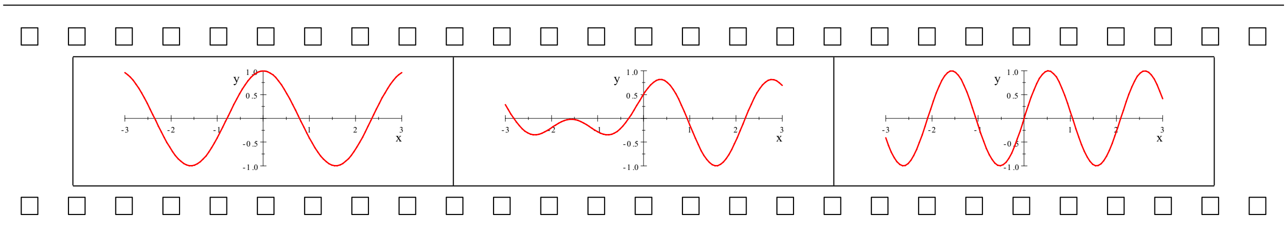


In the snapshots, a red region indicates a negative value of the natural logarithm. A green region indicates a positive value of the natural logarithm. Note in particular that the logarithm of 1 is equal to 0.

Morphing

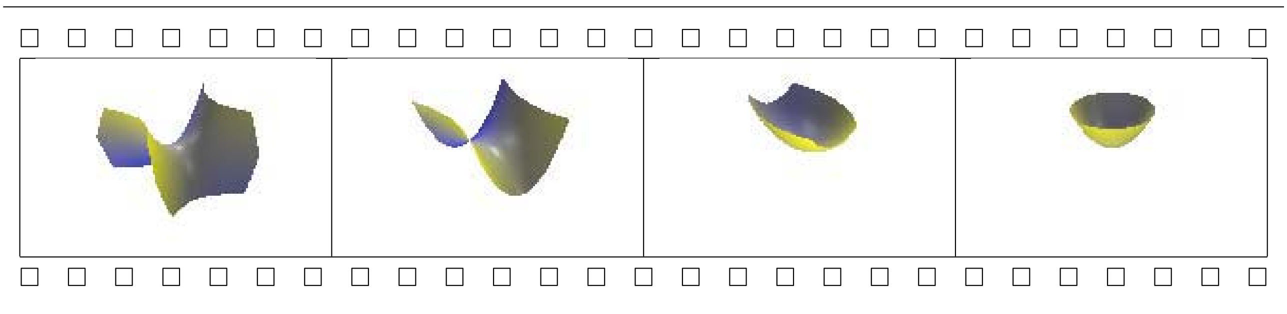
How can you make a motion picture that shows a person changing into a werewolf? Mathematically the problem is rather simple. Let M represent the man and W the werewolf. Let t be a number between 0 and 1. Then $(1-t)\,M + tW$ looks like M when $t = 0$ and looks like W when $t = 1$. For t somewhere in between, you see something in between M and W.

Here is a simple example of how to transform $\cos 2x$ into $\sin 3x$. The film strip shows three views of the morph from $\cos 2x$ to $\sin 3x$.

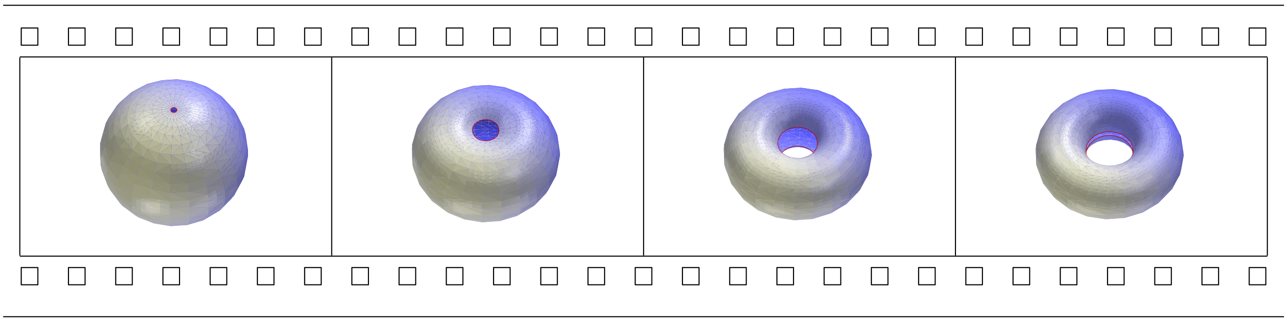


The snapshot on the left shows $y = \cos 2x$ and the one on the far right shows $y = \sin 3x$. In between you see the creation of an extra hump in the smooth transition from $\cos 2x$ to $\sin 3x$.

Morphing is even more fun in three dimensions. Here are a few snapshots that illustrate an animation that transforms the hyperbolic paraboloid $z = x^2 - y^2$ into the elliptic paraboloid $z = x^2 + y^2$.



Here is an animation that shows how to transform a sphere into a donut.



Simply cut holes in the top and bottom of the sphere, stretch the holes into larger and larger circles, and tie the two circles together.

References

1. Diane Davis, Darel Hardy, Fred Richman, Carol Walker, and Robert Wisner, *Hardy Calculus Library; Calculus for Advanced Placement, Business Calculus, Precalculus*, Hardy Calculus LLC, 2008

2. Darel Hardy and Carol Walker, *Doing Calculus with Scientific Notebook*, Brooks/Cole, Pacific Grove, 1997

3. Darel Hardy and Carol Walker, *Doing Mathematics with Scientific WorkPlace and Scientific Notebook*, Version 5.0, MacKichan Software, Inc., 2003

4. Darel Hardy Fred Richman, Carol Walker, and Robert Wisner, *Calculus: Understanding Its Concepts and Methods*, MacKichan Software, Inc., 2006