

# Teaching-Learning Tool for Integral Calculus

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**Abstract:** *This paper presents an efficient Mathematics teaching-learning tool for integral calculus courses. The tool, consisting of a set of PowerPoint slides with Maple animation and interactive Maplets with Maple worksheets, is developed to help instructors to teach in class and to provide students with best opportunity for self-planned learning and self-assessment. More specifically, the PowerPoint slides with Maple animation help instructors to explain certain concepts and methods more effectively and clearly; while the interactive Maplets and Maple worksheets reinforce students' conceptual understanding of integral calculus.*

## 1. Introduction

Computer algebra systems (CAS) have been used widely as tools in Mathematics education. A number of symbolic packages for mathematics courses have been developed using MATHEMATICA, MATLAB or MAPLE over the last two decades. It is found that these symbolic packages enable students to achieve high level of logical-analytical reasoning by visually supporting the concepts and the proofs with graphics presented through the packages [4, 5, 15]. Many researchers have measured and evaluated student learning of mathematical concepts via computer generated dynamic visualizations [6, 7, 9, 11, 14, 17]. A CAS Maple has been used to teach double integration [13], number theory [2], graph sketching [10], mathematical analysis of electronic signals and circuits [16]. Recently, Cook developed a set of MAPLE graphing tools-*calcIIIplots* for the teaching of 3-dimensional calculus [3]. Man introduced CAS Maple and its applications in mathematics education at school level [12]. Students were provided with opportunities in using MAPLE to perform mathematics exploration or problem solving. Symbolic packages also have been used in teaching other subjects such as physics and engineering [1, 8, 19] and computation [18]. Tonkes *et al.* developed a learning model and designed a workbook for teaching computation using MATLAB to the first year students at university of Queensland. Throughout the learning model, students work through the workbook to cover all topics required for improving the proficiency in MATLAB. They found that their learning model together with

workbook improved learning outcomes over historical experiences. Hence, it is noted that the use of symbolic packages is essential in developing logical/analytical reasoning as well as for implementing habits for justifying the results.

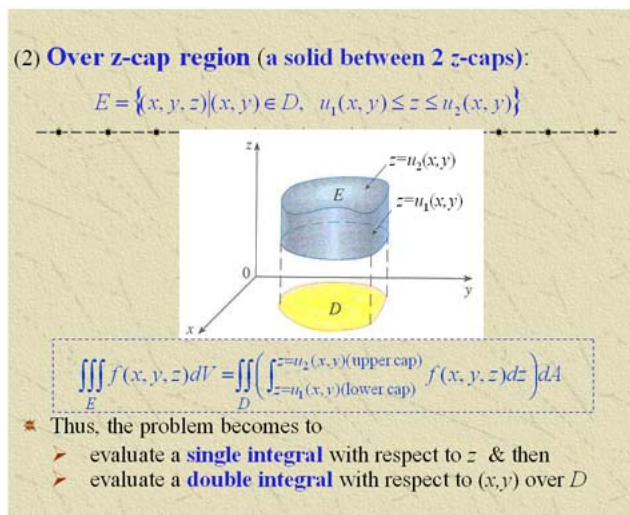
Based on a powerful scientific programming environments and libraries packages of special routines, Maple is one of the most popular systems because it is well-suited to aid college students to learn mathematics through verifying calculation and plotting complicated graphs, and also combines mathematical capabilities with a text editor. In this paper, we develop an efficient Mathematic teaching-learning tool to aid in teaching and learning Mathematics concepts of multivariate integral calculus for science and engineering students. The Mathematics tool consists of PowerPoint slides integrated with Maple animations, interactive Maplets and Maple worksheets. It provides students with the best opportunity to have self-motivation, self-planned learning and self-assessment. Interactive Maplets with Maple worksheets were designed to reinforce students' conceptual understanding of integral calculus.

## 2. Microsoft PowerPoint with Maple Animation

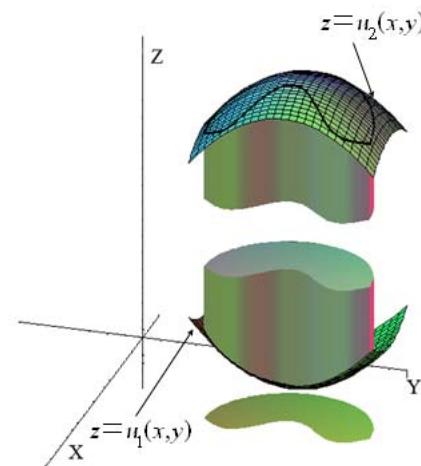
The PowerPoint presentation consists of a number of PowerPoint slides integrated with Maple animations. It has been designed for instructors to introduce Mathematics concepts and theories to students. Figures 2.1-2.2 show some PowerPoint slides we produced for our integral calculus course.

Figure 2.1 demonstrates the concept of  $z$ -cap region and shows how to evaluate a triple integral over a  $z$ -cap region.

Figure 2.2 presents a PowerPoint slide for evaluating a volume integral. To clearly identify the problem, double clicking the hyperlink figure opens a 3-dimension plot object of the solid laying above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$



(a)



(b)

Figure 2.1 Diagram showing how to evaluate triple integral over a  $z$ -cap region: (a) A PowerPoint slide showing the method of evaluation; (b) Maple animation showing the concept of a  $z$ -cap region

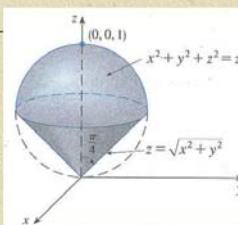
**Example:** Use spherical coordinates to find the volume of the solid that lies above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$

In spherical coordinates  
 $x = \rho \sin \phi \cos \theta$ ,  
 $y = \rho \sin \phi \sin \theta$ ,  
 $z = \rho \cos \phi$

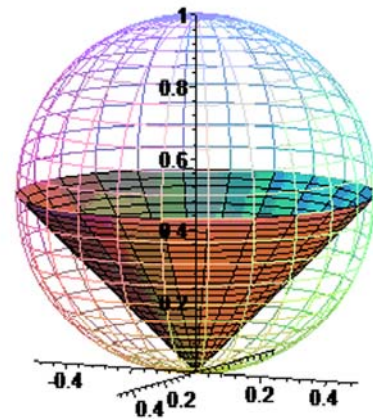
The sphere eq becomes:  $\rho^2 = \rho \cos \phi$   
 $\Rightarrow \rho = \cos \phi$

The cone eq becomes:  
 $\rho \cos \phi = \sqrt{(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2} = \rho \sin \phi$   
 $\Rightarrow \cos \phi = \sin \phi$   
 $\Rightarrow \phi = \pi / 4$

Hence  $E = \{(\rho, \theta, \phi) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi / 4, 0 \leq \rho \leq \cos \phi\}$  as shown



(a)



Now  $E = \{(\rho, \theta, \phi) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi / 4, 0 \leq \rho \leq \cos \phi\}$

So

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin \phi \left[ \frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos \phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin \phi \cos^3 \phi \, d\phi$$

$$= \frac{2\pi}{3} \left[ -\frac{\cos^4 \phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

(b)

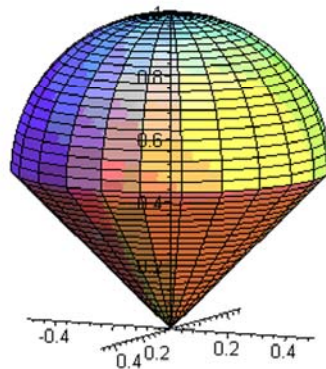


Figure 2.2 PowerPoint slides (a) and (b) presenting an application example of volume integral: find the volume of the solid that lie above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = z$

### 3. Maplets of Line/Surface/Volume Integrals

A symbolic package for integral calculus was developed by Maple. The symbolic package consists of three Maplets which are used to introduce the concepts of line integrals, surface integrals, and volume integrals. The package enables students to achieve high level of logicalanalytical reasoning by visually supporting the Mathematics concepts with graphics. Students learn Mathematics concepts step by step through these Maplets. In each Maplet, there are three main functions including a few input functions with help sections for defining a problem, a graphic visualization function, and an output function for showing the symbolic results. Included here are examples of using Maplets to check solutions of problems in integral calculus.

**Example 1** A wire takes the shape of the semicircle  $x^2 + y^2 = 1, y \geq 0$ , and is thicker near its base than near the top. Find (a) the mass of the wire and (b) the center of mass of the wire if the linear density at any point is proportional to its distance from the line  $y = 1$ .

*Solution* The semi circle can be parameterized by the parametric equations

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \pi$$

(a) *The mass of the wire.*

As the linear density is

$$\rho(x, y) = k(1 - y),$$

the mass of the wire can be determined by

$$m = \int_C k(1 - y) ds. \tag{3.1}$$

Figure 3.1 shows how to evaluate the above line integral with respect to arc length. Firstly, we select the integral type: line integral with respect to arc length. Then we enter the integrand which is the linear density function of the wire, and then we enter the parametric equations of the curve  $C$ . Finally, by clicking on the “Evaluate Integral” button, the mass of the wire is calculated step by step as shown on the result area yielding the result  $m = k\pi - 2k$ .

(b) *The center of mass of the wire.*

Let  $(\bar{x}, \bar{y})$  be the center of mass of the wire, then

$$\bar{x} = \frac{1}{m} \int_C x \rho(x, y) ds, \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) ds \tag{3.2}$$

By symmetry we see that  $\bar{x} = 0$ , so we only need to find  $\bar{y}$ .

$$\bar{y} = \frac{1}{m} \int_C y \rho(x, y) ds = \frac{1}{k\pi - 2k} \int_C yk(1 - y) ds. \tag{3.3}$$

By performing the same steps as above,  $\bar{y}$  can be determined as shown in the result area of Figure 3.2. Hence the center of mass is at the point with coordinates  $\left(0, -\frac{-4 + \pi}{2(\pi - 2)}\right) \approx (0, 0.38)$ .

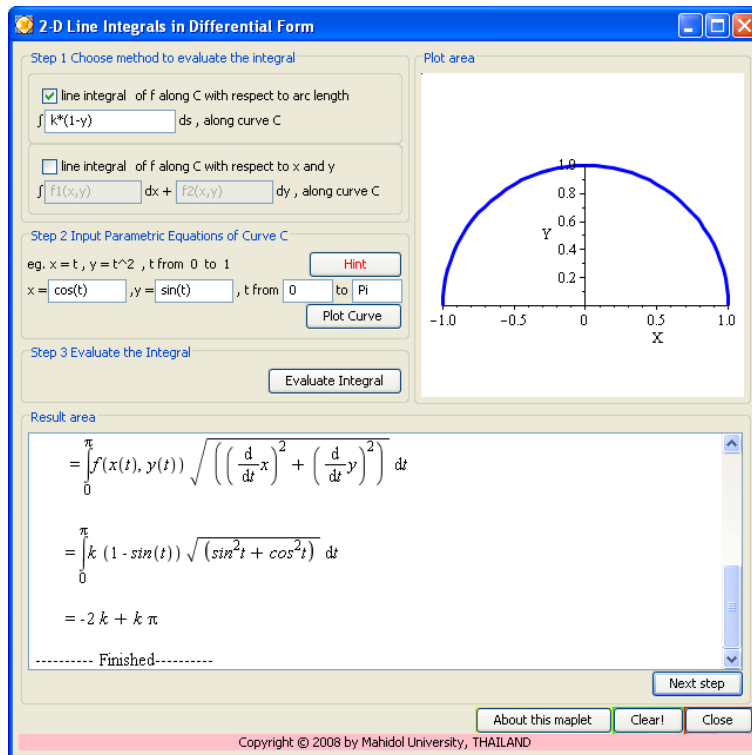


Figure 3.1 Maplet window for evaluation of two-dimensional line integrals

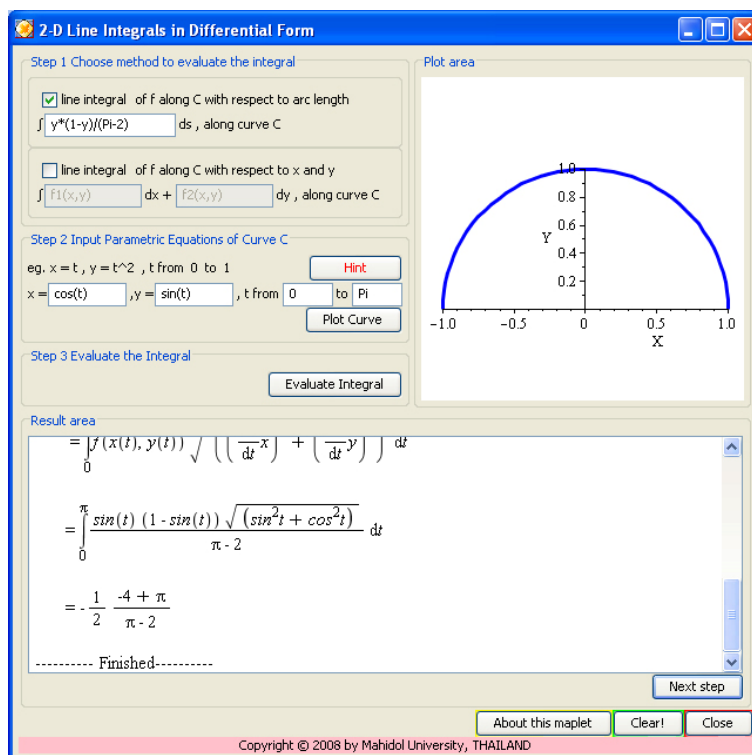


Figure 3.2 Maplet window for calculating the center of mass of the wire

**Example 2** Find the rate of flow through the unit sphere, where  $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}$  is a velocity field describing the flow of a fluid with density 1.

*Solution* The unit sphere  $x^2 + y^2 + z^2 = 1$  can be parameterized by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u,$$

where the parameter domain is

$$D = \{(u, v) \mid 0 \leq u \leq \pi, 0 \leq v \leq 2\pi\}.$$

The velocity field describing the flow of a fluid with density 1 is

$$\vec{F}(x, y, z) = z\vec{i} + y\vec{j} + x\vec{k}.$$

So, the rate of flow through the unit sphere is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS, \quad (3.4)$$

where the unit normal vector  $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ .

Figure 3.3 shows how to use the Maplet to evaluate the surface integrals. Firstly, define the parametric equations of the unit sphere, then the vector field  $\vec{F}(x, y, z)$ , and then click on the “Evaluate Integral” button. The rate of flow through the unit sphere will then be calculated step by step as shown on the result area and the visualization of the rate of flow through the unit sphere is shown on the plot area. The result obtained is  $\frac{4}{3}\pi$ .

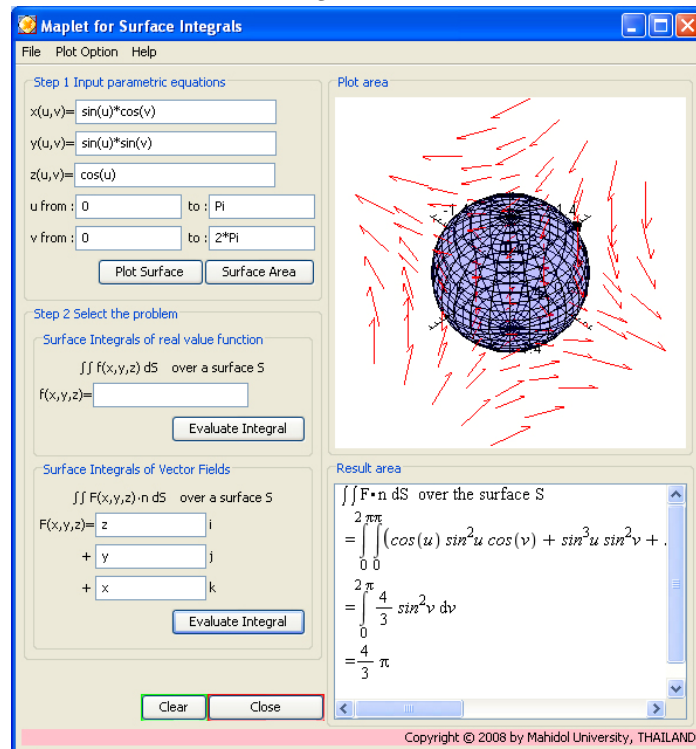




Figure 3.3 Maplet window for evaluating the surface integral representing the rate of flow through the unit sphere

#### 4. Assessment Process

Pre and post testing are used to measure students' achievement. All equations in pre and post testing including basic concepts of each topic and its applications are designed in parallel. To measure students' knowledge in the basic concepts, multiple choice tests and short answer tests are used. Writing test is used for the application part.

Instructors describe the objectives of learning for each topic and assign students to complete workbook which is designed to enhance students' learning outcomes. There are 3 main topics, line/surface/volume integrals, in this workbook. Figure 6 shows an example in the line integral topic which was produced to help instructor in measuring students' learning progress and help students in self-assessment.

 Evaluate  $\int_C x(1+4y)ds$  where  $C$  is the curve  $y = x^2$ , starting from a point  $(0,0)$  and ending at a point  $(1,1)$

<p> <b><u>Your solution</u></b></p>	<p>Plot area</p>
<p><input type="radio"/> Correct <input type="radio"/> Wrong</p>	
<p>Reasons (if wrong):</p>	


 **The solution using Maplet**

Figure 4.1 An example of workbook for line integral problems

## 5. Conclusion

An efficient Mathematics teaching-learning tool for integral calculus courses has been developed to enhance students' interest in mathematics. It enables students to have self-motivation, self-planned learning and self-assessment. Teacher and students have more time to cover a wider range of problems in class. Interactive Maplets of line/surface/volume integrals enable students to analyze mathematics concepts step by step. Students were encouraged to use Maplets with Maple worksheets on homework to check their answers. Maplets provide fast solutions with good visualization of applied mathematics problems. These can help students to identify patterns, see connections and allow students to deepen their knowledge of multivariable calculus.

## References

- [1] Beltzer, A. I. and Shenkman, A. W. (1995). Use of symbolic computation in engineering education. *IEEE Transactions on Education*, 38(2):177–184.
- [2] Cheung, Y. L. (1996). Learning number theory with a computer algebra system. *International Journal of Mathematical Education in Science and Technology*, 27(3):379–385.
- [3] Cook, D. (2006). Maple graphing tools for calculus iii. *Mathematics and Computer Education*, 40(1):36–41.
- [4] Drijvers, P. (2002). Learning mathematics in a computer algebra environment: obstacles are opportunities. *ZDM The International Journal on Mathematics Education*, 34(5):221–228.
- [5] Fuchs, K. J. (2001). Computer algebra systems in mathematics education. *ZDM The International Journal on Mathematics Education*, 35(1):20–23.
- [6] Hayden, M. B. and Lamagna, E. A. (1998). Newton: An interactive environment for exploring mathematics. *Journal of Symbolic Computation*, 25:195–212.
- [7] Heid, M. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal of Research in Mathematics*, 19(1):3–25.
- [8] Johnson, D. and Buege, J. (1995). Rethinking the way we teach undergraduate physics and engineering with mathematica. *Mathematics with Vision: Proceedings of the First International Mathematica Symposium*, pages 233–242.
- [9] Kendal, M. and Stacey, K. (2002). Teacher in transition: Moving towards CAS-supported classroom. *ZDM The International Journal on Mathematics Education*, 34(5):196–201.
- [10] Kong and Kwok (1999). An interactive teaching and learning environment for graph sketching. *Computers and education*, 32(1):1–17.
- [11] Kramarski, B. and Hirsch, C. (2003). Using computer algebra systems in mathematical classrooms. *Journal of Computer Assisted Learning*, 19:35–45.
- [12] Man, Y. K. (2007). Introducing computer algebra to school teachers of mathematics. *Teaching mathematics and its applications*, 20(1):23–26.
- [13] Mathews, J. H. (1990). Using a computer algebra system to teach double integration. *International Journal of Mathematical Education in Science and Technology*, 21(5):723–732.
- [14] Perjési, I. H. (2003). Application of cas for teaching of integral-transforming theorems. *ZDM The International Journal on Mathematics Education*, 35(2):43–47.
- [15] Peschek, W. and Schneider, E. (2002). Cas in general mathematics education. *ZDM The International Journal on Mathematics Education*, 34(5):189–195.
- [16] Røyrvik, O. (2002). Teaching electrical engineering using maple. *International Journal of Electrical Engineering*, 39(4):297–300.
- [17] Tall, D. (1991). Recent developments in the use of the computer to visualize and symbolize calculus concepts. *MAA Notes*, 20:15–25.



- [18] Tonkes, E. J., Loch, B. I., and Stace, A. W. (2005). An innovation learning model for computation in first year mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(7):751–758.
- [19] Ward, J. P. (2003). Modern mathematics for engineers and scientists. *Teaching mathematics and its applications*, 22(1):37–44.