Explorations and Reasoning in the Dynamic Geometry Environment

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Abstract: This article is to describe how the author took full advantage of the exploration feature of the Geometer's Sketchpad, a dynamic geometry software package, to help the preservice secondary school mathematics teachers develop their good learning habits such as making and verifying conjectures, as well as their mathematical reasoning and proof abilities. Three examples are presented to show the role of students' extensive GSP explorations – helping them discover important, interesting mathematical facts/ideas, which in turn became an impetus for generating proofs; and providing insights for them to come up with proof ideas.

Geometry is a weak spot in school mathematics education. Research (see [2] and [8]) has indicated that students entering high school have very little knowledge or experience of geometric properties and relationships; most are operating at the visual level of geometric thought. These students can do little more than recognize different geometric shapes. They do not realize that a square is also a rectangle and a rhombus, or that all three are also parallelograms. Many do not realize that a square must have four right angles or that all four sides are congruent. Most have never heard of a "line of symmetry", let alone understand why a square has four lines of symmetry whereas a non-square rectangle has only two (see [5]). The problems that students have with perceiving a need for proof are well known to all high-school teachers and have been identified without exception in all education research as a major problem in the teaching of proof (see [1]).

To improve this situation, the effective use of technology seems to be a solution.
Peressini & Knuth (see [6]) have formulated five primary ways in which technology is currently being used as a pedagogical tool in mathematics classrooms. After listing the four other roles technology plays in the mathematics classroom – as a management tool, as a communication role, as an evaluation tool, and as a motivational tool, they indicate,

> The fifth role, and perhaps the most important role from the perspective of school mathematics reform, is to harness technology in ways that help students better understand mathematical algorithms, procedures, concepts, and problem-solving situations. In this capacity, as a cognitive role, technology offers a unique means for supporting students' exploration of, and engagement with, mathematics. It affords "new ways of representing complex concepts, and makes available new means by which students [or teachers] can manipulate abstract entities in a 'hands-on' way" (p.280).

Dynamic Geometry is active, exploratory geometry carried out with interactive computer software. As Goldenberg & Cuoco (see [3]) point out, the term *dynamic geometry* has quickly entered the literature as a generic term because of its aptness at characterizing the feature that distinguished this geometry from other geometry: the continuous real-time transformation often called “dragging.” This feature allows users, after a construction is made, to move certain elements of a drawing freely and to observe other elements respond dynamically to the altered conditions. As these elements are moved smoothly over the continuous domain in which they exist, the software maintains all relationships that were specified as essential constraints of the original construction, and all relationships that are mathematical consequences of these. Hence the software allows a focus on the important geometrical idea of invariance.

The nature of the dynamic geometry software also makes it conducive to collaborative problem solving among small or even large groups of students. Several students gathered around a single computer are easily caught up in the conjecturing process as they watch the changes taking place on the computer screen and are quick to offer suggestions for further experimentation (see [5]).

This article is to describe how I took full advantage of the exploration feature of the Geometer’s Sketchpad (see [4]), a dynamic geometry software package, to help the preservice secondary school mathematics teachers develop their good learning habits such as making and verifying conjectures, as well as their mathematical reasoning and proof abilities.

My approach was to present problems to students not in a format like “Prove the following fact (or statement)” but in such a way that the students needed to construct the problem situation (generally a figurative representation of the given problem), observe the situation and manipulate the components of the situation (such as dragging a point or a line and doing some measurements), and answer questions such as “What do you notice?” and “Can you prove your findings?” My practice has shown that this inquiry-based, problem-solving approach can allow students to “manipulate abstract entities in a ‘hands-on’ way”, stimulate their curiosity of finding “why”, provide them with more flexibility to come up with proof insights, and help them achieve better, conceptual understanding.

1. Discoveries – Curiosity – Proof Insights

The first example is the following problem describing an old pirate parchment:

> The island where I buried my treasure contains a single palm tree. Find the tree. From the palm tree, walk directly to the Eagle-shaped rock. Count your paces as you walk. Turn a quarter circle to the right and go the same number of paces. When you reach the end, put a stick in the ground. Return to the palm tree and walk to the Owl-shaped rock, again counting your paces. Turn a quarter circle to the left and go the same number of paces. Put another stick in the ground. Connect the sticks with a rope and dig beneath its midpoint to find the treasure. If the two rocks remain but the palm tree has long since died, can the treasure still be unearthed?

I asked the students (the preservice teachers, the same hereafter) to create a Geometer’s Sketchpad (GSP) representation of the problem situation. Following the instructions written on the pirate parchment, the students quickly constructed a situation similar to the one shown in Figure 1:

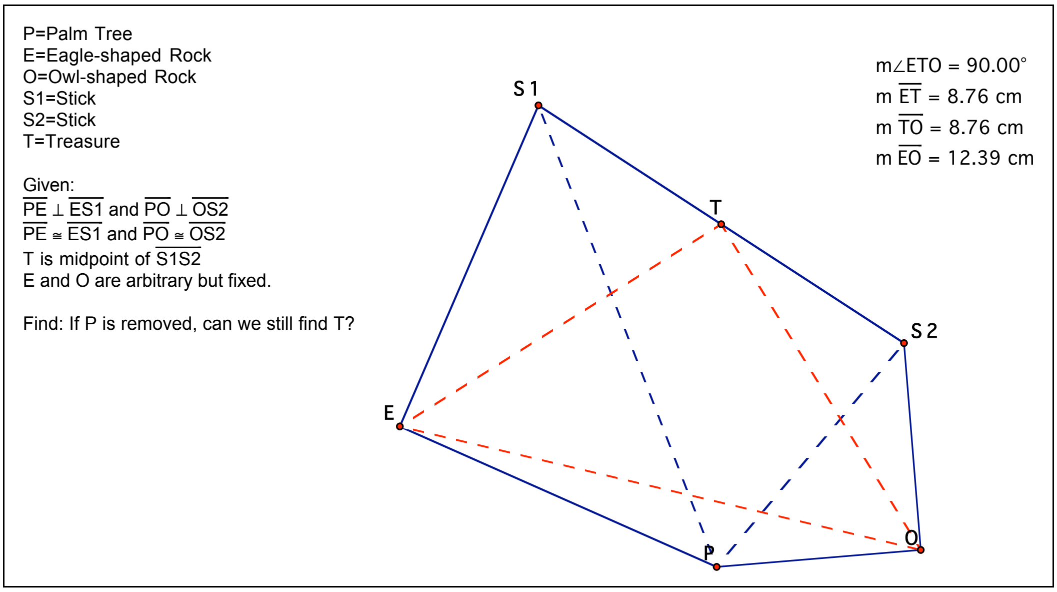

Figure 1. The GSP representation of the Pirate problem

While creating a GSP representation of the situation was quite straightforward, the most interesting part was the students' exploration with the representation. They dragged point P (the palm tree) all around. Even though the locations of S_1 and S_2 (two sticks) changed accordingly, the students discovered that T (the treasure) did not move at all. Therefore, the students recognized that the location of the treasure is independent of that of the palm tree. So they got to know there must exist a way to determine the location of the treasure with only the locations of E (the Eagle-shaped Rock) and O (the Owl-shaped Rock). At least one can start at any location, follow the instructions written on the pirate parchment (except "from the palm tree", of course), and find the treasure. They were just wondering whether there was a better method.

After continuing the explorations, students reported their new findings one after another, "I also see that Angle ETO = 90º and ET=TO (from measurement)." This indicated that the treasure is at the right angle vertex of isosceles right triangle ETO, which is a fixed point independent of the palm tree.

The students became very curious about the findings. They were very eager to find out why this was the case. However, it seemed difficult for them to come up with a logical explanation or proof. Many of them said, "I tried with the pirate proof, but was unable to tackle the problem." I told them that their explorations so far were very useful and it was important to be persistent. The students then continued to manipulate the situation constructed (like the one in Figure 1). They not only dragged points and did various measurements, but also played with transformations. One student realized that the whole process of "walking to the Owl-shaped rock form the palm tree, counting your paces, turning a quarter circle to the left, going the same number of paces, and putting a stick in the ground" was doing a -90º rotation from $\overline{PO}$ to $\overline{S_2O}$ around point O. So he did a -90º rotation of ΔEOP around point O to get ΔS_2OF (see Figure 2). The students who worked with him in the same group found (after $\overline{TF}$ was added to the situation) that points E, T, and F seemed collinear, ΔEOF is an isosceles right triangle, and T seemed to be the midpoint of segment EF. They used measurements to get all these conjectures verified.

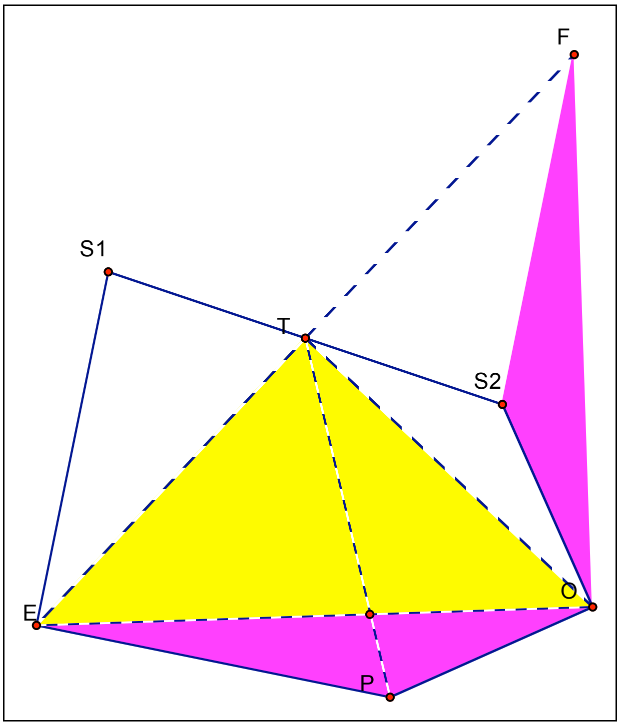

Figure 2. A rotation of ΔEOP around point O

When they presented these findings to the class, the whole class was very excited, because the students realized that all they needed to do at this point is to prove that T is the midpoint of $\overline{EF}$, the hypotenuse of right ΔEOF, using the constructions (the rotation of ΔEOP around point O and the construction of $\overline{TF}$) done so far.

They worked together toward this objective, asked and answered each other "why" questions in the reasoning process, and finally constructed their proof. Figure 3 below shows the proof written by one of the students (with points S and S' in Figure 3 being the same points S_1 and S_2 in Figure 2). Since the midpoint of $\overline{EF}$ (the hypotenuse of right triangle EOF) is T – the right angle vertex of isosceles right triangle EOT, the students completed their proof of their findings.

In an article published in the *Mathematics Teacher* journal, Scher (see [7]) indicates that various proofs of the pirate problem exists in the literature, and all are in some way algebraic, such as the one given by Gamow (1947) using imaginary numbers, the one given by Shilgalis (1998) with coordinate geometry, and the one by Flores (1998) with a vector-based technique. Scher himself developed a nonalgebraic proof by beginning with an interactive GSP construction, "tinkering with the model, examining special cases, and setting static figures into motion" (see [7]), p. 398). Benefited from a similar GSP exploration and the active thinking approach, our students came up with a geometric proof on their own. Both proofs (Scher's and that of our students) are interesting and original. When I mentioned this to the students, they were very pleased, and proud of their ownership of mathematical ideas.

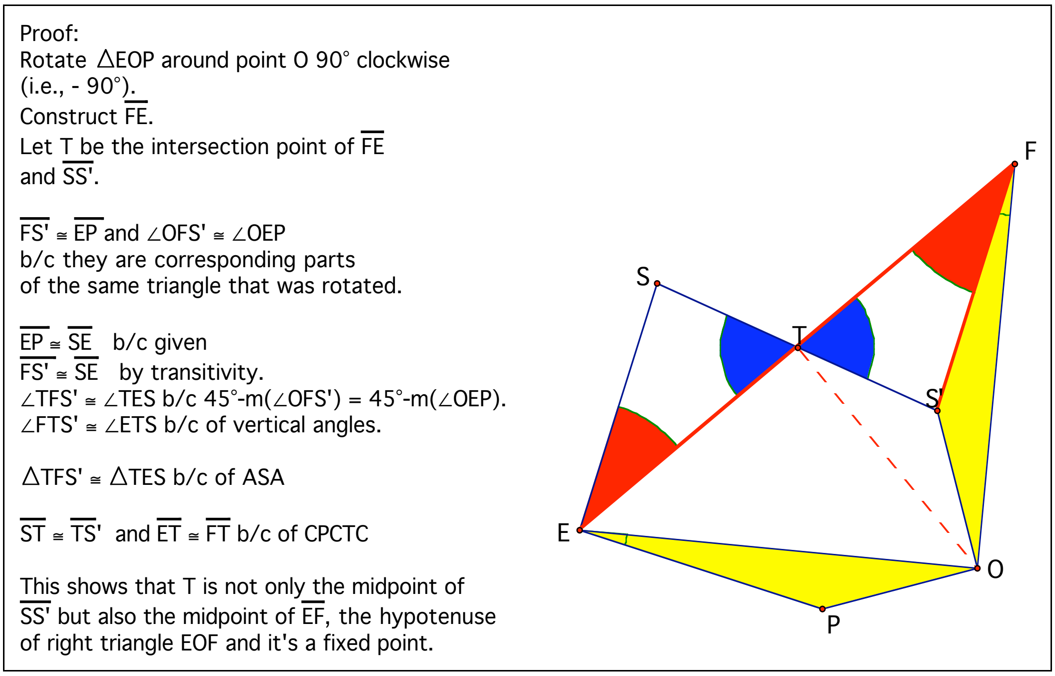

Figure 3. A student's proof of the Pirate problem

2. GSP Explorations and Indirect Proof

The second example is the problem students experienced after they had learned about cyclic quadrilateral, and explored its property – "The opposite angles of a cyclic quadrilateral are supplementary." I asked them, "What about its converse statement: If the opposite angles of a quadrilateral are supplementary, it is a cyclic quadrilateral?"

To explore this problem, the students first constructed an arbitrary convex quadrilateral, measured its two opposite angles, and found the sum of them (see Figure 4). While observing that the sum was not 180º, the students dragged point C around until the sum (from the measurement) became very close to 180º (179.99º in Figure 5).

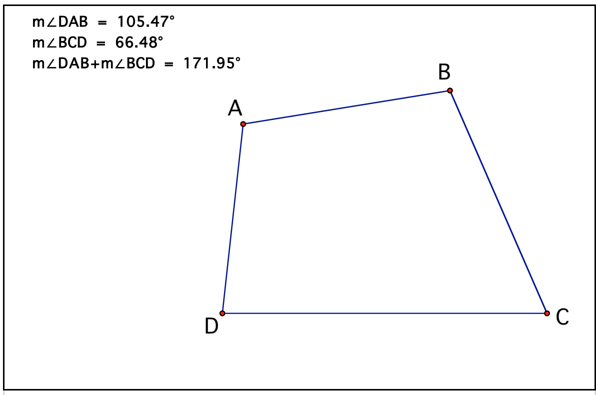

Figure 4. An arbitrary quadrilateral

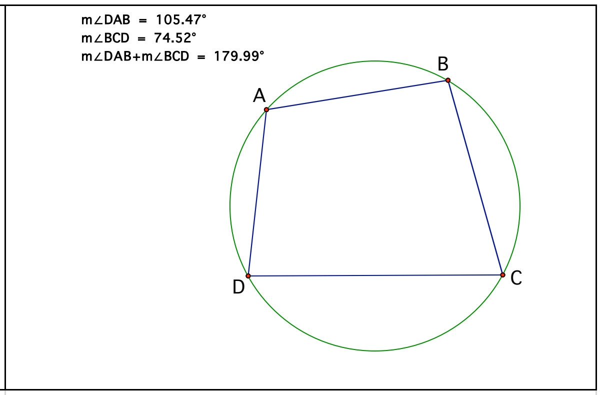

Figure 5. Opposite angles are supplementary

Could quadrilateral ABCD in Figure 5 inscribed in a circle? To test this, a circle needed to be constructed. The students discussed how to construct such a circle. They reviewed what they had learned – any triangle has a circumcircle, whose center is the circumcenter, the intersection point of the three perpendicular bisectors of the triangle. By constructing the perpendicular bisectors of $\overline{AB}$ and $\overline{DA}$ and getting their intersection point, the students used three of the four vertices (say D, A, and B) to construct a circle. They immediately observed that the fourth vertex C is also on the circle, which suggested that quadrilateral ABCD in Figure 5 could be inscribed in a circle.

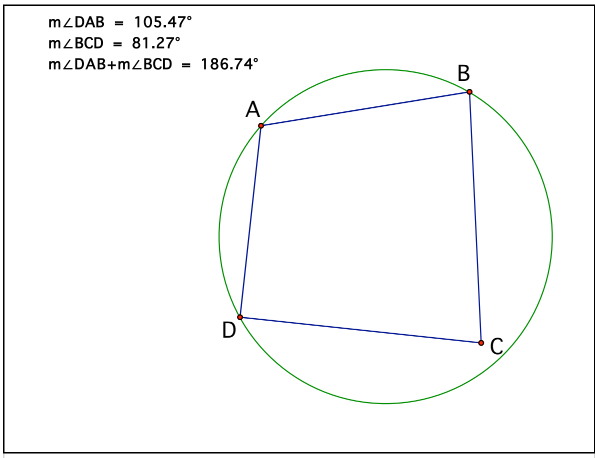

Figure 6. Point C is inside the circle

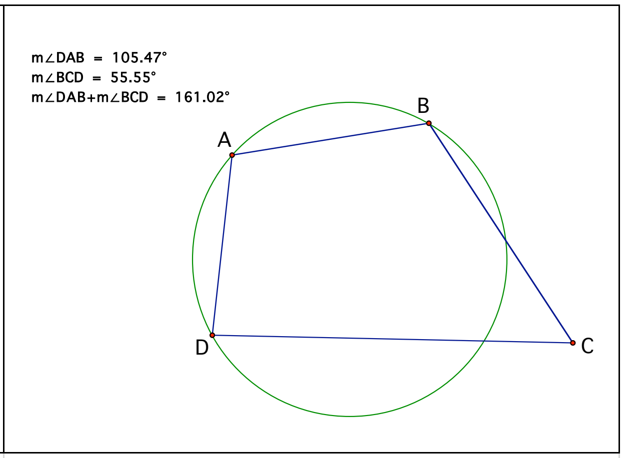

Figure 7. Point C is outside the circle

While the students were enjoying their finding, I asked them if they could explain why this was the case. They continued to explore the situation by dragging point C around and noticed that there were two cases other than the one shown in Figure 5. One of the two cases is shown is Figure 6, in which C is inside the circle and the sum of the two angles is >180º (186.74º specifically in the figure). The other case is shown in Figure 7, in which C is outside the circle and the sum of the two angles is < 180º (161.02º specifically in the figure).

I asked them to discuss these findings, and think about how these findings are connected to a proof. Their discussion revealed that these findings were actually saying that if the fourth vertex was not on the circle, then the sum of the opposite angles was either greater or smaller than 180º, which conflicted with the given condition. This just indicated that if the sum of the opposite angles was 180º, then the fourth vertex must be on the circle, and so quadrilateral ABCD was cyclic. Getting to this stage, the students were very excited, as they believed that they had already grasped the most important part of the proof and known the key ideas. They worked in groups, and completed their proofs. The following is the proof written by one of the groups:

Construct a circle using three of the four vertices of the quadrilateral (say D, A, and B), then there are three possibilities for vertex C: 1) C is inside the circle; 2) C is outside the circle; and 3) C in on the circle.

If C is inside the circle (see Figure 8), extend $\overline{DC}$ *to intersect the circle at E, and construct* $\overline{BE}$. *Since quadrilateral ABED is inscribed in the circle, m(Angle DAB) + M(Angle BED) = 180º. By the triangle exterior angle theorem, m(Angle BCD) > M(Angle BED). By substitution, m(Angle DAB) + M(Angle BCD) > 180º. This contradicts the given condition.*

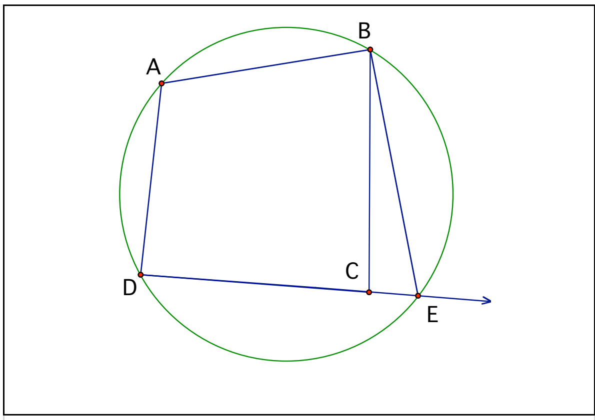

Figure 8. For showing the contradiction (1)

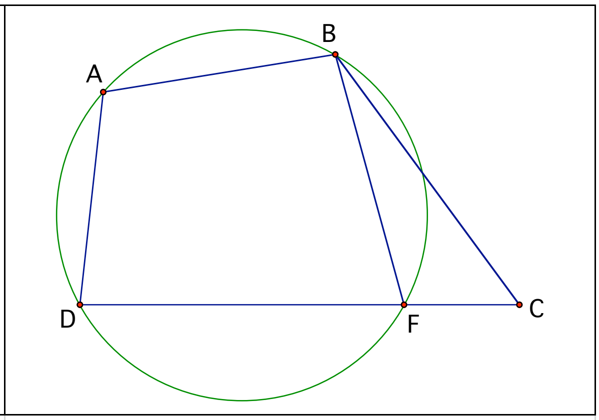

Figure 9. For showing the contradiction (2)

If C is outside the circle (see Figure 9), let F be the intersection point of $\overline{DC}$ *and the circle, and construct* $\overline{BF}$*. Since quadrilateral ABFD is inscribed in the circle, m(Angle DAB) + M(Angle BFD) = 180*^{o}*. By the triangle exterior angle theorem, m(Angle BCD) < M(Angle BFD). By substitution, m(Angle DAB) + M(Angle BCD) < 180*^{o}*. This contradicts the given condition, as well.*

Since both possibilities 1) and 2) are impossible as they contradict the given condition, the only possibility is possibility 3): C in on the circle. This proves that quadrilateral ABCD can be inscribed in a circle, and so is a cyclic quadrilateral.

In fact, the students used an indirect proof to explain their findings, giving an example that GSP explorations benefit the constructions of not only direct proofs, but also indirect proofs.

3. Necessary Prerequisite Knowledge

The third example is the following "Maximum Area" problem:

> Suppose a farmer has a fixed amount of fencing, and needs to build a triangular pen for his animals. How can you construct a triangle with the largest possible area?

I allowed the students time to experiment by drawing triangles either on paper or with GSP. While students felt that it was difficult to come up with a conjecture, I helped them to have a GSP sketch (like the one shown in Figure 10) constructed:

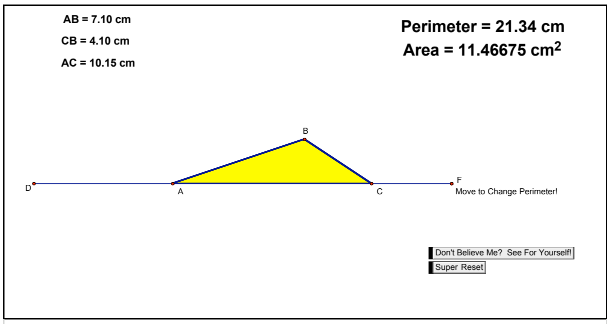

Figure 10. Triangle with constant perimeter

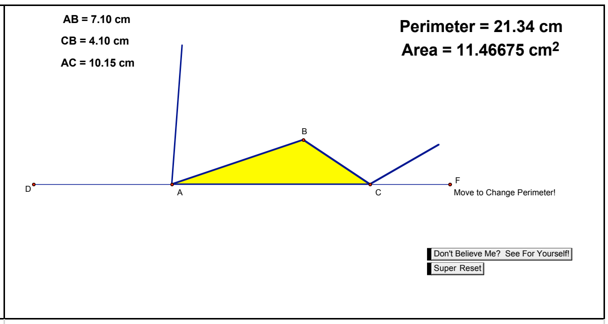

Figure 11. Showing the constant perimeter

In this sketch, A and C are two arbitrary points on $\overline{DF}$. B is the intersection point of two circles of which one has A as the center and AD as the radius, and the other has B as the center and BF as the radius. Therefore, the three sides of ΔABC are actually the three segments forming the longer segment $\overline{DF}$, indicating that the perimeter of ΔABC remains constant, equaling the length of $\overline{DF}$. If one has doubts about this, he or she can click the "Don't Believe Me? See For Yourself" button, and a rotation animation will show the fact (Figure 11).

The students measured the area of ΔABC. They then dragged points A and C along $\overline{DF}$ and noted how the area changed with the changing positions of A and C. They continued to drag points A and C until they found the maximum area (see Figure 12). By observing the triangle with the maximum area, students immediately conjectured that the area attained its maximum when the triangle was equilateral. By measuring the lengths of the three sides (or the sizes of the angles) of ΔABC they were able to verify that the conjecture was correct. In order to change the perimeter of the triangle to see whether the same finding can be obtained, one may move point F left or right and then continue a similar exploration.

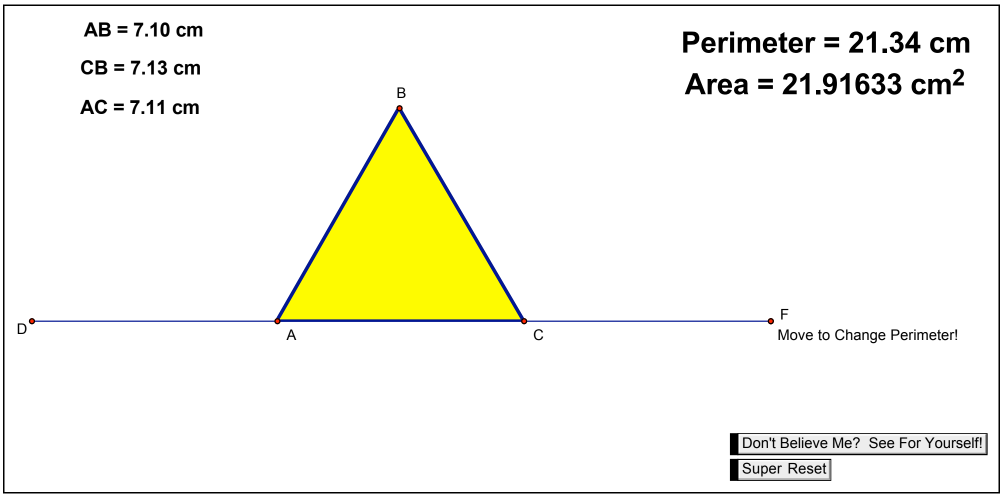

Figure 12. The area attains its maximum when the triangle is equilateral

I need to emphasize that this exploration process may not be easily done without the dynamic geometry software. When the students discovered that the equilateral triangle had the maximum area if the perimeter kept constant, they became very curious to find out why this was the case, just as they did in the explorations with the Pirate problem and the cyclic quadrilateral converse problem.

Doing proofs needs to have necessary pre-requisite knowledge. In most of the cases in high school geometry, the pre-requisite knowledge is nothing more than a general knowledge of triangles, quadrilaterals, circles, and basic transformations. However, this is not true sometimes. This problem belongs to the latter category. To prove the finding that students discovered from their GSP exploration – the maximum area of a triangular region with a given perimeter is attained when the triangle is equilateral, the students need to know the following two theorems:

1) The Inequality of arithmetic and geometric means: $(a_1 + ... + a_n)/N \geq (a_1 \cdot ... \cdot a_n)^{1/N}$, with N being a positive integer, $a_1, ..., a_N$ being positive real numbers, and equality if and only if $a_1 = ... = a_n$.

2) Heron's Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where A refers to the area of a triangle, a, b, and c are the lengths of the three sides of the triangle, and $s = (a + b + c)/2$.

While the two theorems had not been introduced in most students' previous courses, a few students did know these mathematical ideas. I asked these students to present and explain what they leaned and understood to the whole class. Discussions were organized so that all students made sense of the theorems. With the curiosity stimulated by their GSP exploration, students were eager to apply these ideas to explain why their finding was correct.

The following proof was constructed by one group of students, representing the effort made by the whole class:

By the Inequality of arithmetic and geometric means, *$[(s-a) + (s-b) + (s-c)]/3 \geq [(s-a)(s-b)(s-c)]^{1/3}$, with equality if and only if $s-a = s-b = s-c$ (i.e., $a = b = c$). This is to say that if and only if $a = b = c$, $[(s-a)(s-b)(s-c)]^{1/3}$ = its maximum value $[(s-a) + (s-b) + (s-c)]/3 = [(s-a) + (s-a) + (s-a)]/3 = s-a = (3a)/2 - a = (1/2)a$. So if and only if $a = b = c$, $(s-a)(s-b)(s-c)$ = its maximum value $(1/8)a^3$.*

By Heron's Formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$. If and only if $a = b = c$, A = its maximum value $\{[(3a)/2]*[(1/8)a^3]\}^{1/2} = (\sqrt{3}/4)\, a^2$.

This proves that in the given problem, if we construct an equilateral triangle with the fixed amount of fencing (perimeter), we can get the largest possible area.

4. Conclusion

Along with the widespread use of the dynamic geometry software, many people believe that the exploration feature of the software contributes significantly to the students' discovering mathematical facts/ideas and making thoughtful conjectures. However, fewer people think that students' extensive explorations with the software can also facilitate their mathematical reasoning and proof efforts. The three examples described above show that when the dynamic geometry software is used appropriately, the following things can happen: students' explorations help them discover important and interesting mathematical facts, which in turn becomes an impetus for the derivation of explanations/proofs; and their further explorations provide insights for them to come up with proof ideas. My many years' experience of using the dynamic geometry software with preservice mathematics teachers made me strongly believe that the exploration feature of the software can also benefit students when they develop their mathematical reasoning and proof abilities. I will continue to work in this area and try to provide more findings and insights of how to use technology in helping students do proofs to the mathematics education community.

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