A Proposal to Teach 3D Vector Operations in a Role-Playing Game

Hitoshi Nishizawa and Takayoshi Yoshioka
nisizawa@toyota-ct.ac.jp
Department of Electrical and Electronic Engineering
Toyota National College of Technology
Japan

Abstract: *In the college the authors work for, the students' achievement in linear algebra has been lower than the ones of other fields in mathematics. We suspect that one of the major reasons is rote learning of our students in linear algebra. Many students try just to memorize the formulas and solutions for certain problems expected at the examination but do not think about the mechanism nor relations to graphical objects, and forget the isolated knowledge in a short period. The lack of actual applications has worsen the situation.*
To improve this, we have introduced some graphical explanations of vector operations in 2D and 3D space with the help of dynamic graphic software. Each explanation appealed to some students but not to the others. We still need to develop an explanation that appeals to the majority of our students.
In this paper, we propose a role-playing game as a start-up application for 3D vector operations. We expect it helps our students to feel the reality in 3D vector space and to relate 3D graphic objects to the symbolic representations later in their lessons.

1. Background

With decreasing teaching hours of mathematics and science in primary and secondary schools, Japanese high schools and colleges are struggling to live up with the students who are not prepared to their education. Some college students in engineering departments haven't learnt physics, fundamental calculus, or trigonometric functions in high schools, and some high school students don't have sufficient skills of algebraic calculations.
Adding to that, rote learning has been spreading to the leaning of many subjects in Japan [1]. Even in high school mathematics, some students adopt black box approach. They memorize typical formulas and solving procedures to pass their examinations but forget them after the exams. Some adults even encourage the youths to do so as an *effective* learning method to pass the entrance examinations to high schools and colleges. As the result, some students believe in black box approach as the mighty tool, and they do not easily change their learning styles. Black box approach is most disastrous in learning linear algebra at the college we are working for.
Under the circumstances, the authors have conducted tailor-made paper-and-pencil exercises with the help of a Web-system [2]. The Web-system helped to serve specific exercises of algebra for each student according to his/her learning history stored on the system, and also served interactive online exercises of symbolic calculations or quadratic functions, where a student described the symbolic expressions of quadratic functions given as the graphs [3]. We have analyzed the error-patterns in students' symbolic calculations in the online exercises, and added the diagnosis of typical errors in the calculations of symbolic fractions [4]. As the result, our students have improved their symbolic manipulating skills and have become to relate symbolic expressions to the graphs more frequently in their learning of exponential or trigonometric functions.
But the tailor-made exercises had hardly improved their performance in liner algebra. In the exercises of linear algebra, they tend to give up quite easily leaving blank sheets as the answers

when they don't remember the formula or solution for just the problem in front of them. They don't seem to struggle or apply try-and-error in linear algebra.

2. Our Weakness in Teaching Linear Algebra

As a result, linear algebra was the worst topic in the first INCT (Institute of National College of Technology, Japan) achievement test of mathematics, which was held in January 2007 for all the third grade students (18 years old) of colleges of technology in Japan. Table 1 shows our students' average ratio of correct answers in each topic. The average ratio in linear algebra, especially three-dimensional (3D) vectors and matrices, was far lower than the ones in the other topics: numbers and polynomials, equations and inequalities, exponential and trigonometric functions, and fundamental calculus.

Table 1 Ratio of correct answers in topics at the INCT achievement test (Jan. 2007)

Topic	Numbers & Polynomials	Equations & Inequalities	Functions & Graphs	Fundamental Calculus	2D vectors	3D vectors & Matrices
Ratio of correct answers	90%	89%	92%	75%	81%	50%

The poor performance was suspected to be the direct result of black box approach because the students' answer sheets had more blank answers, fewer drawings, and fewer explanatory sentences in linear algebra.

A typical problem that many students feel difficult to solve in linear algebra is as follows:

(Problem) Calculate the distance *d* between a plane in 3D space expressed by an equation: $ax + by + cz + D = 0$ and a point located outside of the surface and has the coordinates: (x_0, y_0, z_0).

All the students who could write something tried to use the formula $d = \frac{|ax_0 + by_0 + cz_0 + D|}{\sqrt{a^2 + b^2 + c^2}}$ but many failed to write it down correctly. Instead, some of them wrote symbolic expressions similar but not equal to the formula. They complained that the formula was too complicated to memorize and looked similar to several other formulas. None of them could explain how the formula is related to the graphical objects nor could deduce the formula from the given geometric condition.

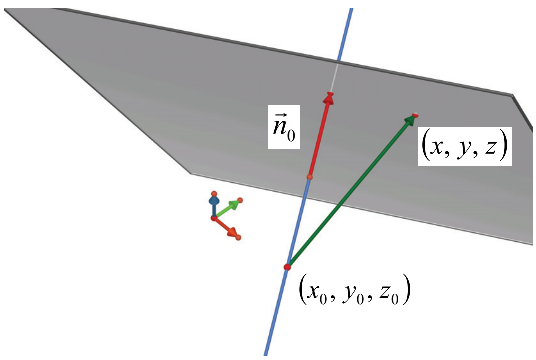

Figure 1 Distance of a plane and a point

One of the simplest explanations of the formula is a dot product of a unit perpendicular vector to the plane $\vec{n}_0 = \frac{(a \ \ b \ \ c)}{\sqrt{a^2 + b^2 + c^2}}$ and the vector drawn from the point $\vec{x}_0 = (x_0, y_0, z_0)$ to any point $\vec{x} = (x, y, z)$

on the plane: $\vec{x}-\vec{x_0}=(x-x_0, y-y_0, z-z_0)$ (Figure 1), but such an explanation seemed to be difficult to understand in a short time for them.
The graphical image of a vector equation and dot product, especially of a unit vector and another vector, are the keys to explain the mechanism of the formula. The explanation surprises some students when they finally understood it, because the mechanism is quite simple and reasonable once they understood it.
Through our teaching experiences, we state our students' weaknesses as follows:

1) Lack of Reality

Students are not motivated to learn linear algebra because they hardly feel the reality in it. Our textbooks start describing with the definitions of vectors and their operations, continue to explain how to use them as formulas, and finish with typical solutions of some problems. Traditional lectures follow the textbooks and handle problems solved only in the scope of mathematics. After the lectures, students tend to think the explanations just abstract ideas that are not related to the real world.
Lack of reality in linear algebra is obvious in comparison with calculus, which is taught as a mathematical model of Newton mechanics and electrical circuits.

2) Few links between symbolic expressions and graphic objects

The calculations in linear algebra are taught with hardly related to graphic objects, and thus become more abstract and harder to grasp the image for beginners, especially in three-dimension.
Adding to that, vector expressions in two-dimension, which are often adopted to avoid the complexity of drawing three-dimensional graphics, sometimes hinder the learning of linear algebra. Students tend to underestimate the necessity of vector expressions if they can solve the same problem in two-dimension with functional expressions, which are more familiar to them.

3) Few discussions on the mechanisms

If we do not refer to graphic objects, it is hard to plainly explain the mechanisms of vector expressions and operations. And fewer mechanisms naturally lead the students to black box approach.
For meaningful learning, the links between various representations: symbolic, graphical, and verbal (or descriptive) expressions must be learned and examined thoroughly [5].

3. Introducing a Role-Playing Game

To overcome these weaknesses, we have decided to start our lessons with a real-world application familiar to the students with full use of graphical objects representing the application [6]. Symbolic expressions are introduced after the application to explain certain features of the related graphic objects. The weakness of this approach lies in selecting an application because every application usually appeals to only some students but not to the others. We need to select an application that is familiar to most of the students, and it is a reason why we have selected a role-playing game as the startup application in our project. Although a role-playing game is not a real-world application, it appeals to most of younger Japanese students who have been playing video games regularly and felt certain reality in them.
Affinity to three or more dimensions is another benefit of a role-playing game as the startup application of linear algebra. When the application is expressed in three dimensions graphically and symbolically, students finally understand the need of 3D instead of 2D, which is far more familiar to them, and become ready to learn linear algebra.
In this paper, we propose to use a role-playing game as the base of a series of classroom activities rather than individual activity. It introduces 3D vectors and their operations before their definition or explanation of symbolic form. We hope it to become the common experience of our students, and

it will be referred by in later lessons as a popular example of 3D vectors.

3.1 A Tournament as the Role-playing Game

The role-playing game consists of a tournament, where a player or a team of players compete each other in different kind of battles at every stage of the tournament, for example, boxing or Sumo wrestling match, solving puzzles or chess match, or singing competitions or musical auditions. Every student in the class owns a player as his/her *avatar* in the tournament, and designs its characteristics before attending the tournament. The characteristics of a player is expressed by three independent elements; physical strength, thinking ability, and musical skills.

The tournament is fought between the teams, and all the students in the class are divided into one of the teams. Two teams compete each other in a battle, where the type of battle and the characteristics of battlefield differ from one to another. For example, if a tennis match is set at a stage of the tournament, three players selected each from the team play three games in the courts of slightly different condition. The team that wins two games or more proceeds to the next stage of the tournament.

The types of battle at all the stages are revealed after the team gathering but at the beginning of the tournament, so each team can decide its strategy based on the given information and the characteristics of the players in the team. In every stage, the teams decide detailed tactics in the battle along with the representing players, the battles are conducted one by one, and the winning teams proceed toward the final stage.

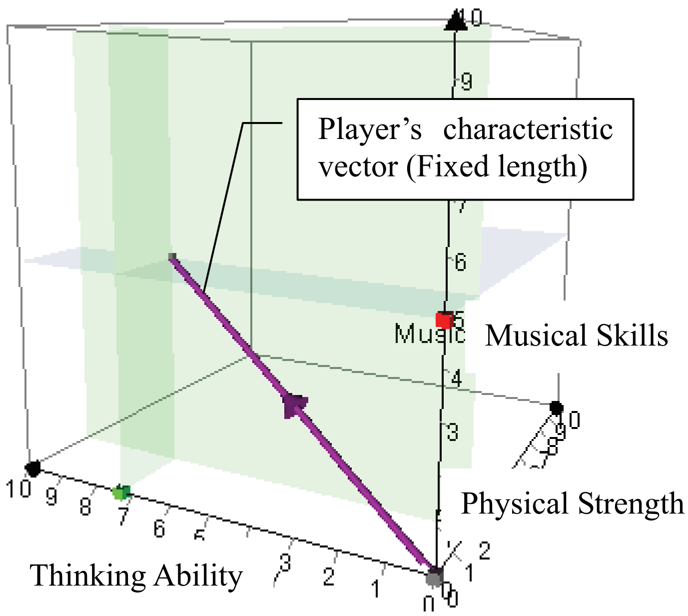

Figure 2 Designing a player in 3D space with the help of DGS

3.2 3D Vectors in the Game

A 3D vector expresses the characteristics of a player, where physical strength, thinking ability, and musical skills are three independent basic elements of a player. For example, a Sumo wrestler is to have high physical strength but modest thinking ability and musical skills as a player. An opera singer has high musical skills but modest physical strength. Each student designs his/her player by rotating the unit characteristic vector of the player in 3D space (Figure 2) before attending one of the teams in the tournament. The actual length of a player's vector is decided randomly according to

a normal probability distribution N(10, 1) just after the team-assembling to avoid the intentional occurrence of a super team.
The 3D space shows that the value of a basic element of the characteristic vector, for example the thinking ability, is expressed as the distance of the plane perpendicular to the "Thinking ability" axis and the origin. In Figure 2, it is the distance between the two points on the axis. It is also worthy to explain that the value is calculated by a dot product of the characteristic vector and a unit vector on the "Thinking ability" axis.
When the tournament plan with an explanation of the battlefields are shown, each team discuss the strategy and selects the participating players for the stages, for example 3 players out of 5 candidates in the team for the tennis match of semi-final stage.
A characteristic vector of a battlefield is also expressed as a 3D vector of the same elements but as a unit vector. Its direction is decided by the teacher to express the feature of each battlefield. For example, if it is a game of basketball, musical skills are of little use.
In a one to one battle, the strength of each player is calculated by the dot product of the player's characteristic vector and the battlefield's characteristic vector, which is a unit vector, and the winner of the battle is decided by comparing the strength of two competitors. In 3D space, a perpendicular plane to the battlefield vector helps to show the winner visually (Figure 3). The player whose vector stretches far beyond the plane from the origin is the winner, and it is player B in the case of Figure 3. 3D graphics with its capability to change the viewpoint help us to distinguish the winner clearly, telling the potential benefit of 3D vectors to our students.
In a battle of plural players in a team, the characteristic vector of a team is calculated as the vector sum of all the player's characteristic vectors, and the team vector is used instead of the player's vector in one-to-one battle.

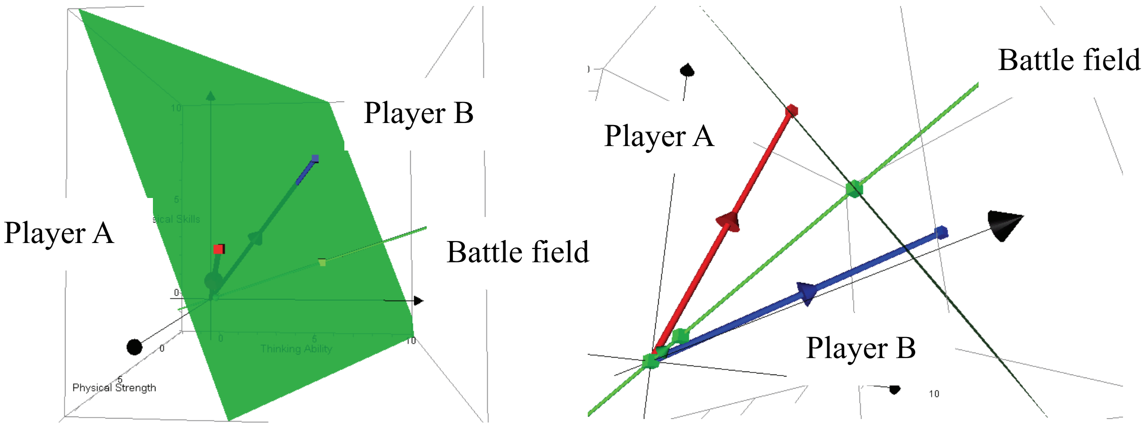

Figure 3 Displaying the winner of a battle graphically in 3D space

As the result of a battle, the winner grows 20% in the length of its characteristic vector, the loser shrinks 20%, and the waiting players stay the same length. In the case of team battle, all the players in the winning team grows 20% in length in their characteristic vectors.

3.3 Vector Operations in the Game

In the game, we use most of vector operations; dot product to calculate the value of each element of a characteristic vector and the strength of each player in a battle, vector addition to calculate the characteristic vector of a team in a team battle, and absolute value of a vector to set the strength of a player. A scalar multiplication is used to stretch / shrink the player's vectors as the result of a battle. These operations are shown graphically at first, and explained as their symbolic operations later.

4. Technologies that Support the Game

The essential technology for the game is DGS (dynamic graphics software) or CAS (Computer Algebra System) with 3D graphic capabilities. It displays the result of a battle as the 3D graphics like Figure 3, and is used by the students to design their players' characteristic vectors on their desktop like Figure 2. The software preferably displays the vectors symbolically in the same format as the one in the textbook.

A Web-system could help to collect and distribute the working data between the teacher and students quickly. It also has the advantage of involving students' homework into the result of the game. For example, if a training session of a player, given as the online exercises of vector calculation, increases the strength of his/her player's characteristic vector, some students are pleased to do their homework to get a better result at the tournament.

5. Conclusion

In teaching linear algebra of high school level, the authors proposed to use a role-playing game as the start-up application that has reality to our students, and getting more active participation of the students in classes. In the game, students use 3D vectors and their operations, mainly through the 3D graphics objects displayed by dynamic graphics software. Mathematical knowledge is to be introduced as the rule of the game at first, and experienced before the formal explanation.

In engineering education, we often use Project Based Learning approach successfully, and this is our first attempt to use the approach in the learning of fundamental mathematics at our college, where rote learning is the dominant learning strategy employed especially when the students learn 3D vectors and their operations. We expect the approach to improve the performance of our students in linear algebra, and we gain the evidence in the near future.

Acknowledgement

This paper is a result of the research project supported by Grant-in-Aid for Scientific Research of Japan Society for the Promotion of Science, No. (C) 20,500,767.

References

[1] S. Fujisawa, *Surface Learning* (in Japanese, Tokyo, Japan: Shin-yo-sha, 2002).

[2] H. Nishizawa, T. Saito, & S. Pohjolainen, An Implementation of a Hypermedia Learning Environment for a Small Group, *Transactions of Japanese Society for Information and Systems in Education, 15*(4), 1999, 249-253.

[3] T. Yoshioka, Y. Higuchi, & H. Nishizawa, A Web-based Interactive Exercise System for Learning Mathematical Functions, *Proceedings of the 8th International Conference on Computers in Education*, Taiwan, 2000, 1574-1576.

[4] H. Nishizawa, T. Yoshioka, K.J. Fuchs, & A. Dominik, A Knowledge-Sensitive On-line Exercise for Developing Algebraic Calculation Strategies, *Proceedings of the 8th Asian Technology Conference in Mathematics*, Taiwan, 2003, 118-126.

[5] L. Haapasalo, The conflict between conceptual and procedural knowledge: Should we need to understand in order to be able to do, or vice versa?, *Proceedings on the IXX Symposium of the Finnish Mathematics and Science Education Research Association, University of Joensuu, Bulletins of the Faculty of Education 86*, Joensuu, 2003, 1-20.

[6] H. Nishizawa & T. Yoshioka, Importance of Linguistic Explanation in Mathematics Education, and the Support from Technology, *Proceedings of the 11th Asian Technology Conference in Mathematics*, 2006.