On strong positive realness of a system that contains a parameter

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Abstract

Compute algebra systems are getting more and more attention from the society of engineering and industry, because of its ability to handle a symbolic parameter. This ability is particularly advantageous for a controller design of a system that contains a parameter, since it is difficult to apply conventional numerical methods to such a system directly.

Given a square transfer function matrix $G(s)$, $G(s)$ is said to be strongly positive real if and only if $G(i\omega)+G(-i\omega)^T$ is a positive definite matrix for all $\omega \in \mathbf{R}\cup\{\pm\infty\}$.

Strongly positive real systems, in short, correspond to systems made up of passive elements such as resistance, inductance and capacitance. The concept of strongly positive real functions has been used in, for example, stability analysis of nonlinear systems, adaptive control, and so on. Although algorithms to check strong positive realness has been already reported, the algorithms are numerical and can not be applied directly to a system that contains a parameter.

This paper focuses on a system that contains a parameter. In this case, the system is strongly positive real for certain values of k, i.e. there exists a range Ω of real numbers such that "$k \in \Omega \Leftrightarrow$ the given system is strongly positive real". In this paper, we present a method to compute such range a Ω of the parameter k. The method utilizes the properties of a certain ARE (Algebraic Riccati Equation) that has close relationships with strong positive realness of a system.

1 Introduction

In control engineering, numerical packages such as MATLAB, Octave, Scilab have been used to design and analyze a control system. These packages provide easy access to advanced modern control theory such as H_∞ control, μ analysis and controller designs based on LMI (Linear Matrix Inequality). However, since these numerical packages can not handle a symbol, they can not be directly applied to control systems that contain a parameter. Computer algebra system (CAS) provides one approach to solve the problem, and applications of computer algebra to the design and analysis of control system is increasing. For example, references [1] and [2] apply QE (Quantifier elimination) technique to the design and analysis of control systems. References [3] and [4] treat the H_∞ problem, presenting algorithms to compute the H_∞ norm of a system that contains a parameter. References [5] and [6] also treat H_∞ problem for a system that contains a parameter, where H_∞ optimal control problem is discussed (in H_∞ optimal control problem, we compute the minimum H_∞ norm achievable by static or output feedback controllers). Reference [7] focuses on an H_2 optimal problem for a system that contains a parameter.

In this paper, we focus on strong positive realness of a system that contains a parameter. In this case, the system is strongly positive real for certain values of k, i.e. there exists a range Ω of real numbers such that "$k \in \Omega \Leftrightarrow$ the given system is strongly positive real". This paper presents a method to compute such a range Ω of parameter k.

In the rest of the paper, we use the following notations:

i : The imaginary unit, i.e., $i^2 = -1$.

E : The unit matrix in appropriate size.

$\mathbf{R}$: The set of real numbers.

$\mathbf{Det}(M)$: Determinant of matrix M.

M^T : Transpose of matrix M.

$M > 0$: Matrix M is positive definite (i.e. symmetric and has real positive eigenvalues).

$\mathrm{Res}_x(r_1(x), r_2(x))$: Resultant of polynomials $r_1(x)$ and $r_2(x)$ with respect to x.

2 Problem formulation

Let $G(s)$ be a given square transfer function matrix of a system, and A, B, C, D be its state-space realization, i.e., $G(s) = C(sE - A)^{-1}B + D$.

Definition 1 *$G(s)$ is strongly positive real if and only if*

$$G(i\omega) + G(-i\omega)^T \text{ is a positive definite matrix for all } \omega \in \mathbf{R} \cup \{\pm\infty\}. \tag{1}$$

Suppose that we are asked the question "Is a given system strongly positive real or not ?". When the given system contains no parameters, the answer is yes or no (if condition (1) is satisfied, then "yes", and otherwise "no"). However, when a given system contains a parameter k, the answer is more complicated. In general, the answer depends on the value of parameter k, and there may exist a range Ω of real numbers such that

$$k \in \Omega \Leftrightarrow G(s) \text{ is strongly positive real.} \tag{2}$$

In this paper, we present algorithms to compute such a range Ω. First, we note the following lemma:

Lemma 2 *If* $G(s)$ $(= C(sE-A)^{-1})B + D$ *is strongly positive real, then* $D + D^T$ *is a positive definite matrix.*

Proof

Since we have $G(s) = C(sE-A)^{-1}B + D$ and $C(sE-A)^{-1}B$ is strictly proper (degree of the denominator with respect to s is greater than that of the numerator), we have $G(i\omega) \to D \quad (\omega \to \infty)$. Thus, $G(i\omega) + G(-i\omega)^T \to D + D^T \quad (\omega \to \infty)$. This and the assumption of the theorem complete the proof. □

The above lemma implies that $D + D^T > 0$ is a necessary condition for a system to be strongly positive real. Hence, in the rest of the paper, we assume that $D + D^T > 0$ for all parameter values of k. We also assume that a given system $G(s)$ is stable for all parameter values of k (otherwise, we limit the parameter values to the range satisfying these conditions).

Let us focus on condition (1), which is difficult to check directly, because it is necessary to check positive definiteness of $G(i\omega) + G(-i\omega)^T$ for infinitely many ω. However, we have the theorems outlined below:

Theorem 3 $G(s)$ *is strongly positive real if and only if there exists positive definite matrix* P *satisfying*

$$\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}'\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix} < 0. \tag{3}$$

Theorem 4 $G(s)$ *is strongly positive real if and only if the algebraic Riccati equation (ARE)*

$$X(A - BR^{-1}C) + (A - BR^{-1}C)^T X + XBR^{-1}B^T X + C^T R^{-1} C = 0 \tag{4}$$

has a stabilizing solution X*, where* $R = D + D^T$.

Theorem 3 presents a LMI condition that is equivalent to (1), and **Theorem 4** presents another equivalent condition based on an ARE. The two conditions are independent, and we can apply each condition separately for the computation of a range Ω that satisfies (1). In fact, LMI condition (3) in **Theorem 3** can be formulated as a proposition with quantifiers $(\forall, \exists)$ if we let each entry of symmetric matrix P be variables, i.e. $P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \vdots & \vdots \\ p_{1,n} & \cdots & p_{n,n} \end{bmatrix}$. This implies that it is possible to compute a range Ω in (1) with QE (Quantifier elimination) technique. However, since the number of variables $p_{i,j}$ increase quadratically with respect to n, this method has difficulties in computational complexities. Hence, in the next section, we present another algorithm to compute that utilizes **Theorem 4**.

3 Algorithm

First, we briefly explain a standard algorithm to solve ARE equation

$$X\hat{A} + \hat{A}^T X + X\hat{R}X + \hat{Q} = 0. \tag{5}$$

See standard textbook such as [8] for details.

Algorithm 1 ([8])

Input: ARE (5)
Output: Stabilizing solution X of (5)

⟨1⟩ Let matrix H be

$$H = \begin{bmatrix} \hat{A} & \hat{R} \\ -\hat{Q} & -\hat{A}^T \end{bmatrix}. \tag{6}$$

⟨2⟩ Let λ_i $(i = 1, \cdots, n)$ be eigenvalues of H that have negative real parts, and let v_i be the eigenvector corresponding to eigenvalue λ_i.

⟨3⟩ Let $n \times n$ matrices X_1, X_2 be

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}, \tag{7}$$

and output $X = X_2 X_1^{-1}$.

From the above algorithm, we see that there exists stabilizing solution X of (5) if and only if the following two conditions are satisfied:

(C1) There exist n eigenvalues of H that have negative real parts.

(C2) X_1 in (7) is not singular.

Therefore, Ω is a range such that

$$k \in \Omega \Leftrightarrow \text{Both of the conditions (C1) and (C2) are satisfied.} \tag{8}$$

This implies that Ω can be written as $\Omega = \Omega_1 \cap \Omega_2$, where Ω_1 and Ω_2 are defined by

$$\Omega_1 = \{ k \mid \text{Condition (C1) is satisfied}\}, \quad \Omega_2 = \{ k \mid \text{Condition (C2) is satisfied}\}. \tag{9}$$

First, let us consider Ω_1. It is well-known that if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H. Hence,

$$\text{eigenvalues of } H \text{ are given by } \lambda_1, \cdots, \lambda_n, -\lambda_1, \cdots, -\lambda_n. \tag{10}$$

This implies that the above condition (C1) is equivalent to

(C1') Matrix H has no pure imaginary eigenvalues.

Let $\underline{\omega}_j$ $(j = 1, \cdots, \underline{m})$ be real numbers such that for small enough real number ε (> 0), condition (C1') is not satisfied at $k = \underline{\omega}_j - \varepsilon$, while the same condition is satisfied at $k = \underline{\omega}_j + \varepsilon$. Similarly, let $\overline{\omega}_j$ $(j = 1, \cdots, \overline{m})$ be real numbers such that for small enough real number ε (> 0), condition (C1') is satisfied at $k = \overline{\omega}_j - \varepsilon$, while the same condition is not satisfied at $k = \overline{\omega}_j + \varepsilon$. Then, from condition (C1') and (10), we see the following

(A1) H has multiple eigenvalues ri $(r \in \mathbf{R})$ at $k = \underline{\omega}_j, \overline{\omega}_j$.

(A2) Region Ω_1 can be written as

$$\Omega_1 = \Omega_{1,1} \cup \Omega_{1,2} \cup \cdots \cup \Omega_{1,u}, \quad \Omega_{1,j} = \{\ x \in \mathbf{R}\ \mid\ \underline{\omega}_j < x < \overline{\omega}_j\ \}, \tag{11}$$

where $\underline{\omega}_j \in \{-\infty, \mu_1, \cdots, \mu_{\underline{m}}\}$ and $\overline{\omega}_j \in \{\mu_1, \cdots, \mu_{\overline{m}}, \infty\}$.

From the properties of the resultant we have (for details of the resultant, see [11])

$$\begin{aligned} H \text{ has multiple eigevalues at } k = k_0 \quad &\Leftrightarrow \quad \chi(x) = \mathrm{Det}(xE - H) \text{ has multiple roots at } k = k_0 \\ &\Leftrightarrow \quad \mathrm{Res}_x(\chi(x), \chi'(x))|_{k=k_0} = 0. \end{aligned} \tag{12}$$

This and (A1) imply that $\underline{\omega}_j$ and $\overline{\omega}_j$ are roots of $\mathrm{Res}_x(\chi(x), \chi'(x)) = 0$ with respect to k. From this and (A2), we obtain the following algorithm to compute region Ω_1 in the form of (11) (similar algorithm is used to solve H_∞ problem in [4]).

Algorithm 2

Input: Matrix H
Output: Range Ω_1 of real numbers in (9)

⟨1⟩ Compute the characteristic polynomial $\chi(x) = \mathrm{Det}(xE - H)$ of H.

⟨2⟩ Compute the real roots α_j $(\alpha_1 < \alpha_2 < \cdots < \alpha_m)$ of $\mathrm{Res}_x(\chi(x), \chi'(x)) = 0$.

⟨3⟩ Determine $\underline{\omega}_j, \overline{\omega}_j$, computing the roots of $\chi(x)$ at $k = \alpha_1 - 1, (\alpha_j + \alpha_{j+1})/2$ $(j = 1, \cdots, m-1), \alpha_m + 1,\ \alpha_j$ $(j = 1, \cdots, m)$ (note that $\underline{\omega}_j \in \{-\infty, \mu_1, \cdots, \mu_{\underline{m}}\}$ and $\overline{\omega}_j \in \{\mu_1, \cdots, \mu_{\overline{m}}, \infty\}$).

Next, let us consider the computation of Ω_2. In this case, we can use the algorithm in [5] and [6], which we will explain briefly below (for details, refer to [5] and [6]). First, we compute the eigenvector $v(\lambda)$ corresponding to an eigenvalue λ of H with the following algorithm, leaving λ as an indeterminate:

Algorithm 3 ([5],[6])

Input: Matrix H
Output: Eigenvector $v(\lambda)$ corresponding to an eigenvalue λ of H
Remark: λ is an indeterminate

⟨1⟩ Let x be $x = [\ x_1\ \cdots\ x_{2n}\ ]$ and compose $2n$ linear equations $(\lambda E - H)x = 0$ (each entry of $(\lambda E - H)x = 0$ forms a linear equation).

⟨2⟩ Select $(2n-1)$ linear equations from the above $2n$ equations. Then, solve the equation with respect to $x_1, \cdots, x_{2n-1}$.

⟨3⟩ Substitute the solution of $x_1, \cdots, x_{2n-1}$ into x, and multiply an adequate polynomial so that each entry of x becomes a polynomial in λ.

⟨4⟩ Let $v(\lambda) \leftarrow x/x_{2n}$ and output $v(\lambda)$.

The above algorithm outputs eigenvector $v(\lambda)$ corresponding to eigenvalue λ of H. Hence, matrices X_1 and X_2 in **Algorithm 1** can be written as

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} v(\lambda_1) & \cdots & v(\lambda_n) \end{bmatrix}, \tag{13}$$

where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of H with negative real parts. Let us define $n \times n$ matrices $\Lambda_1(y_1, \ldots, y_n)$ and $\Lambda_2(y_1, \ldots, y_n)$ by

$$\begin{bmatrix} \Lambda_1(y_1, \ldots, y_n) \\ \Lambda_2(y_1, \ldots, y_n) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} v(y_1) & \cdots & v(y_n) \end{bmatrix}, \tag{14}$$

where $y_1, \ldots, y_n$ are variables. Then, from (13) and (14), we see that $X_1 = \Lambda_1(\lambda_1, \cdots, \lambda_n)$. Since $\mathrm{Det}(\Lambda_1(y_1, \ldots, y_n))$ is an alternating polynomial in $y_1, \ldots, y_n$, $\frac{\mathrm{Det}(\Lambda_1(y_1,\ldots,y_n))}{\prod_{l<m}(y_l - y_m)}$ is a symmetric polynomial in $y_1, \ldots, y_n$. Therefore, we obtain the following;

(P1) $\prod_{s_l=\pm 1} \left\{ \frac{\mathrm{Det}(\Lambda_1(y_1,\ldots,y_n))}{\prod_{l<m}(y_l-y_m)} \right\}\Big|_{(y_1,\ldots,y_n)=(s_1\lambda_1,\cdots,s_n\lambda_n)}$ is a symmetric polynomial $\lambda_1^2, \cdots, \lambda_n^2$.

(P2) $|X_1| = \mathrm{Det}\left(\Lambda_1(\lambda_1, \cdots, \lambda_n)\right) = 0 \Rightarrow \prod_{s_l=\pm 1} \left\{ \frac{\mathrm{Det}(\Lambda_1(y_1,\ldots,y_n))}{\prod_{l<m}(y_l-y_m)} \right\}\Big|_{(y_1,\ldots,y_n)=(s_1\lambda_1,\cdots,s_n\lambda_n)} = 0$.

Since we have

$$|xE - H| = (x^2 - \lambda_1^2) \cdots (x^2 - \lambda_n^2) = x^{2n} + g_{2n-2}(k)x^{2n-2} + \cdots + g_0(k), \tag{15}$$

fundamental symmetric polynomials of $\lambda_1^2, \cdots, \lambda_n^2$ are given by $g_{2n-2}(k), \cdots, g_0(k)$. This and the above (P1) imply that there exists polynomial $\xi(k)$ in k satisfying

$$\xi(k) = \prod_{s_l=\pm 1} \left\{ \frac{\mathrm{Det}(\Lambda_1(y_1, \ldots, y_n))}{\prod_{l<m}(y_l - y_m)} \right\}\Bigg|_{(y_1,\ldots,y_n)=(s_1\lambda_1,\cdots,s_n\lambda_n)} \tag{16}$$

that can be computed by the algorithm outlined below:

Algorithm 4 ([5],[6])

Input: Matrix H
Output: Polynomial $\xi(k)$ in (16)

⟨1⟩ Compute eigenvector $v(\lambda)$ of H with **Algorithm 3**.

$\langle 2\rangle$ Let $\Lambda_1(y_1,\ldots,y_n)$ be matrix defined by (14) and compute

$$\prod_{s_l=\pm 1}\left\{\frac{\mathrm{Det}\,(\Lambda_1(y_1,\ldots,y_n))}{\prod_{l<m}(y_l-y_m)}\right\}\Bigg|_{(y_1,\ldots,y_n)=(s_1\lambda_1,\cdots,s_n\lambda_n)} \tag{17}$$

as a symmetric polynomial in $\lambda_1^2,\cdots,\lambda_n^2$.

$\langle 3\rangle$ Compute the characteristic polynomial $|xE-H|$ of H and compute fundamental symmetric polynomial $g_{2r}(k)$ $(r=0,\cdots,n-1)$ of $\lambda_1^2,\cdots,\lambda_n^2$.

$\langle 4\rangle$ Eliminate $\lambda_1^2,\cdots,\lambda_n^2$ in (17) in $\langle 3\rangle$, computing Groebner basis (see [9] and [10]) of

$$\left\{\text{Polynomial (17)},\ g_0(k)-(-1)^n\lambda_1^2\cdots\lambda_n^2,\ \cdots,\ g_{2(n-1)}(k)-(-1)(\lambda_1^2+\cdots+\lambda_n^2)\right\}. \tag{18}$$

with lexicographic order $\lambda_1,\cdots,\lambda_n\succ k$, and output the result.

The above (P2) and (16) imply that

$$|X_1|=0\Rightarrow\xi(k)=0. \tag{19}$$

Therefore, we obtain such that

$$\text{Condition (C2) is not satisfied at } k=k_0\ (\in\mathbf{R})\Rightarrow\xi(k_0)=0. \tag{20}$$

In other words, $k_0\in\mathbf{R}$ satisfying condition (C2) is a root of polynomial $\xi(k)$ (note that the converse of the statement may not be true, i.e. a root of $\phi(k)$ is not necessarily value of k where condition (C2) is not satisfied). Thus, finally, we obtain the following algorithm to compute Ω_2.

Algorithm 5

Input: Matrix H
Output: Range Ω_2 of real numbers in (9)

$\langle 1\rangle$ Compute polynomial $\xi(k)$ in (20) by **Algorithm 4**.

$\langle 2\rangle$ Compute the real roots α_j $(j=1,\cdots,m)$ of $\xi(k)$, and let $\Gamma=\{\}$ (empty set).

$\langle 3\rangle$ Check condition (C2) for $k=\alpha_j$ $(j=1,\cdots,m)$. If the condition is not satisfied (i.e. X_1 is singular at $k=\alpha_j$), then add α_j into Γ (i.e. let $\Gamma\to\Gamma\cup\{\alpha_j\}$).

$\langle 4\rangle$ Output $\Omega_2=\Gamma^c=\{\ x\in\mathbf{R}\mid x\notin\Gamma\ \}$.

Thus, we obtain the following algorithm to compute Ω.

Algorithm 6

Input: Matrix H
Output: Range Ω of real numbers in (8)

$\langle 1\rangle$ Compute polynomial Ω_1 by **Algorithm 2**.

$\langle 2\rangle$ Compute polynomial Ω_2 by **Algorithm 5**.

$\langle 3\rangle$ Output $\Omega=\Omega_1\cap\Omega_2$.

4 Numerical example

Let us consider the system where A, B, C, D is given by

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ k \end{bmatrix},\ C = \begin{bmatrix} k+1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 \end{bmatrix}. \tag{21}$$

In this case, eigenvalues of A are $-1,\ -2$, and the system is stable for any $k \in \mathbf{R}$. Transfer function $G(s)$ and matrix H in (6) are given by

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B + D = \frac{s^2+(2k+4)s+k^2+4k+4}{s^2+3s+2}, \\ H &= \begin{bmatrix} \hat{A} & \hat{R} \\ -\hat{Q} & -\hat{A}^T \end{bmatrix} = \begin{bmatrix} A-BR^{-1}C & BR^{-1}B^T \\ -C^TR^{-1}C & -A^T+C^TR^{-1}B^T \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -k-3 & 1 & 1 & k \\ -k(k+1) & -k-4 & k & k^2 \\ -(k+1)^2 & -k-1 & k+3 & k(k+1) \\ -k-1 & -1 & -1 & k+4 \end{bmatrix}. \end{aligned} \tag{22}$$

We will compute range Ω in (2) by **Algorithm 6**. First, we compute Ω_1 by **Algorithm 2**.

⟨1⟩ The characteristic polynomial $\chi(x)$ of H is computed to be

$$\chi(x) = \mathrm{Det}(xE-H) = x^4 + (k^2-2k-6)x^2 + 2k^2 + 8k + 8. \tag{23}$$

⟨2⟩ Resultant $\mathrm{Res}_x(\chi(x), \chi'(x))$ is computed to be

$$\mathrm{Res}_x(\chi(x), \chi'(x)) = 32(k+2)^2(k^4-4k^3-16k^2-8k+4)^2, \tag{24}$$

whose real roots are

$$\alpha_1 = -2,\ \alpha_2 = -1.76733,\ \alpha_3 = -1.13165,\ \alpha_4 = 0.303225,\ \alpha_5 = 6.59575, \tag{25}$$

where the values of $\alpha_1, \ldots, \alpha_5$ are given to six significant figures.

⟨3⟩ Computing the roots of $\chi(x)$ at $k = \alpha_1 - 1,\ (\alpha_j + \alpha_{j+1})/2\ (j = 1, \cdots, 4),\ \alpha_5 + 1$, we see that $\chi(x)$ has no pure imaginary roots at $k = (\alpha_2+\alpha_3)/2,\ (\alpha_3+\alpha_4)/2,\ (\alpha_4+\alpha_5)/2,\ \alpha_3,\ \alpha_4$. Thus, we obtain $\underline{\omega}_1 = \alpha_2 = -1.76733,\ \overline{\omega}_1 = \alpha_5 = 6.59575$ and

$$\Omega_1 = \Omega_{1,1},\ \Omega_{1,1} = \{\ x \in \mathbf{R}\ \mid\ -1.76733 < x < 6.59575\ \}. \tag{26}$$

Next, we will compute Ω_2 with **Algorithm 5**. Polynomial $\xi(k)$ in (20) is computed by **Algorithm 4** as follows:

⟨1⟩ Eigenvector $v(\lambda)$ of H is computed to be

$$v(\lambda) = \begin{bmatrix} k(\lambda-1)(\lambda+k+2) \\ k^2(\lambda-1)(\lambda+1) \\ k(k+1)(\lambda+1)(\lambda+2) \\ 2\lambda^3+(k+4)\lambda^2+(k^2-k-4)\lambda-k^2-6k-8 \end{bmatrix}. \tag{27}$$

⟨2⟩ From the above $v(\lambda)$, we obtain

$$\prod_{s_l=\pm 1} \left\{ \frac{\mathrm{Det}\,(\Lambda_1(y_1,\ldots,y_n))}{\prod_{l<m}(y_l-y_m)} \right\} \Bigg|_{(y_1,\ldots,y_n)=(s_1\lambda_1,\cdots,s_n\lambda_n)} = 63336k^4+\cdots+16k^8\lambda_1^8\lambda_2^8. \tag{28}$$

⟨3⟩ Since the characteristic polynomial $\xi(x)$ of H is given by (23), we have $g_0(k) = 2k^2+8k+8,\ g_1(k) = k^2-2k-6$.

⟨4⟩ Computing Groebner basis of

$$\left\{63336k^4+\cdots+16k^8\lambda_1^8\lambda_2^8,\ 2k^2+8k+8-\lambda_1^2\lambda_2^2,\ k^2-2k-6+(\lambda_1^2+\lambda_2^2)\right\}, \tag{29}$$

we obtain $\xi(k) = 9k^{12}(k+1)^8$.

Therefore, Ω_2 is computed by **Algorithm 6** as follows:

⟨1⟩ As is shown above, $\xi(k) = 9k^{12}(k+1)^8$.

⟨2⟩ The real roots of $\xi(k)$ are $\alpha_1 = -1,\ \alpha_2 = 0$.

⟨3⟩ $X_1|_{k=\alpha_1}$ and $X_1|_{k=\alpha_2}$ are given by

$$X_1|_{k=\alpha_1} = \begin{bmatrix} -3-2\sqrt{2} & 1 \\ 3+2\sqrt{2} & 0 \end{bmatrix},\ X_1|_{k=\alpha_2} = \begin{bmatrix} -1 & 3+2\sqrt{2} \\ 1 & 0 \end{bmatrix}, \tag{30}$$

neither of which is singular. Thus, $\Gamma = \{\}$.

⟨4⟩ Therefore, $\Omega_2 = \Gamma^c = \mathbf{R}$ (the set of real numbers).

From the above results, we obtain $\Omega = \Omega_1 \cap \Omega_2 = \{\ x \in \mathbf{R}\ \mid\ -1.76733 < x < 6.59575\ \}$.

5 Conclusion

Let $G(s)$ be a given square transfer function matrix that contains a parameter k. This paper presents a method to compute the range Ω of parameter k where $G(s)$ is strongly positive real, i.e.

$$k \in \Omega \quad \Leftrightarrow \quad \forall\omega \in \mathbf{R} \cup \{\pm\infty\},\quad G(i\omega)+G(-i\omega)^T > 0. \tag{31}$$

Two theorems (**Theorem 3** and **Theorem 4**) are presented to compute the range Ω. **Theorem 3** describes the relation between strong positive realness of a system and the solution of LMI, while **Theorem 4** describes the relation between strong positive realness of a system and ARE.

Although it is possible to computes Ω if we apply QE technique to **Theorem 3**, the method based on QE has difficulties in computational complexities (QE is known to be computationally heavy), and can not be applied for a practical problem. Hence, in this paper, we present a method based on **Theorem 4**. We also present an example to illustrate the method.

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