The Paradox of Twins Described in a Three-dimensional Space-time Frame

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Abstract : *After traveling from outer space, the elder twin looks younger than his brother waiting on the earth based on the calculation from the formula of Special Relativity. There is no absolute reference coordinate in the universe. Theoretically, the younger twin could describe the motion of his brother flying forward, vice versa, the elder twin could describe the motion of his brother flying backward. If the younger twin felt his brother looking younger, the older twin should also felt his brother looking younger. It is hard to comprehend that in reality the elder twin actually looks younger than the younger twin. The motion of an object described in a three-dimension space-time frame by embedding time into space could be drawn using the graph-command of MATLAB. It leads a better understanding the paradox of twins based on the calculation from the graphic method in the 3-d s-t frame.*

1. Introduction

In classical physics, the concepts of time and space are absolute. We can talk about space without specifying time; for example, measuring the length of an object in geometry without referring to time. We can also talk about time without specifying space; for example, describing two events happening simultaneously without referring the location of an observer. Since space and time can be separated, space and time are independent.

We use position and time to describe the motion of an object where position is a function of time. Although space and time are discussed together, there is no constraint binding space and time. Space is composed of three independent dimensions, and time is composed of one independent dimension in classical physics.

For example: The movement of an object in a space with 0 sec. at (0 m, 0 m, 0 m), 1 sec. at (0.5 m, 1.5 m, 1 m), 2 sec. at (1 m, 3 m, 2 m), 3 sec. at (1 .5 m, 4.5 m, 3 m), and ...etc. can be decomposed into (0 sec., 0 m), (1 sec., 0.5 m), (2 sec. 1 m), (3 sec., 1.5m), forward or backward along x-axis; (0 sec., 0 m), (1 sec., 1.5 m), (2 sec. 3 m), (3 sec., 4.5 m), rightward or leftward along y-axis; (0 sec., 0 m), (1 sec., 1 m), (2 sec. 2 m), (3 sec., 3 m), upward or downward along z-axis. The motion of this object can be described in the Fig.1a, Fig.1b, and Fig.1c.

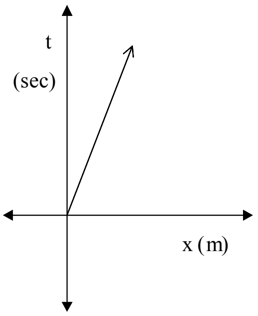

Fig. 1a: The motion of the object is decomposed along x-axis.

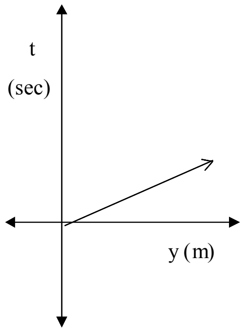

Fig. 1b: The motion of the object is decomposed along y-axis.

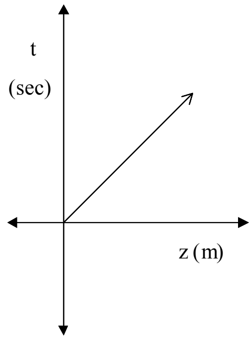

Fig. 1c: The motion of the object is decomposed along z-axis.

Einstein demonstrated that space and time are not separable by using the experiment of emitting light from the middle of a car in a moving train [1]. An observer on the train sees the two pulses hit the rear and front walls of the car simultaneously. An observer on the road sees one pulse hit the rear wall before the other hit the front. This experiment shows that we cannot talk about the time of an event without linking it to the position of an observer in the space. Similarly, we cannot talk about the position of an object in space without linking it to time. This inseparability of space and time shows that space and time are dependant.

Einstein further derived Lorentz transformation. Frame S' moves to the right with the velocity, v, with respect to Frame S. At the point where the origins of both frmes coincide, let t=t'=0. At this time, a beam of light was fired from the origin O'. The beam reaches a receiver at the point P' in Frame S'. In the Fig.2, the coordinates of point P' are (x', y', z', t') with respect to Frame S', where the beam travels along the path O'P' and the coordinates of point P' are (x, y, z, t) with respect to Frame S.

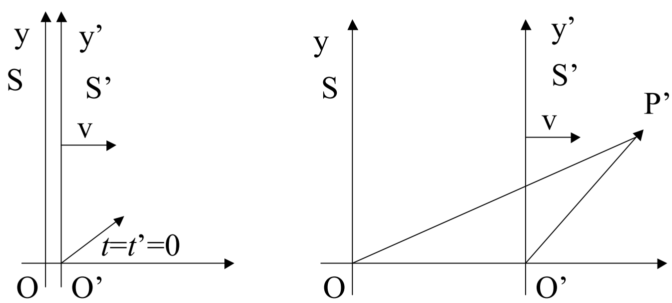

Fig.2: When O and O' coincide, sends a beam from O' and receives the beam at the point P' in Frame S'. The coordinates of P'are (x',y',z',t') in Frame S' and (x,y,z,t) in Frame S.

The velocity of light measured from Frame S' will be

$$c = O'P / t' = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{t'} \tag{1}$$

and the velocity of light measured from Frame S will be

$$c = OP / t = \frac{\sqrt{x^2 + y^2 + z^2}}{t} \tag{2}$$

Based on the principle of the invariance of speed of light, the velocity of light measured from two different inertial frames remains the same.

Minkowski adopted a four-dimensional space-time (4-d s-t) frame shown as Fig.3 to describe the motion of an object. There are three independent dimensions in space (x-axis, y-axis, z-axis), which are perpendicular to one another, and one independent time dimension (t-axis), which is perpendicular to all three dimensions in space [2]. To reveal indirectly the dependency of space and time, he added a constraint by combining Eq.(1) and Eq.(2)

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2 = const. \tag{3}$$

The constraint is called the invariance of an event interval between two inertial frames.

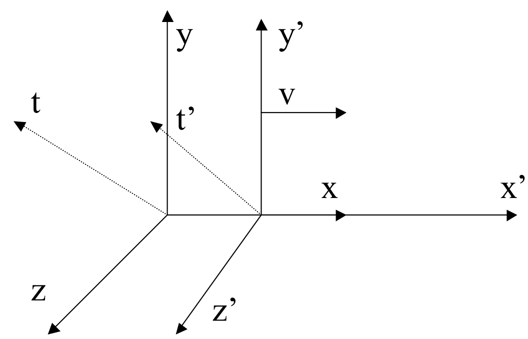

Fig.3: Two 4-d s-t inertial frames were constructed by Minkowski to describe the motion of an object moving in space.

The way to describe the motion of an object in a 4-d s-t frame is the same as in frame of classical physics except the constraint binding space and time. With this additional constraint, it raieses up the difficulty of finding mathematical solutions for coordinate transformation problem between two different frames.

2. Time Dilation and Length Contraction

Before discussing the paradox of twins, we need review time dilation and length contraction in Special Relativity. Lets define some terms first. If two frames have a constant relative velocity between them, two frames are said to be inertial frames to one another. For this discussion, we will have a train passing by a station platform at constant velocity. Theoretically, we are allowed to choose any one frame of the two frames to be the **stationary inertial frame** and the other frame to be the **moving inertial frame**. For convenience, we construct a stationary frame S on the platform and a moving frame S' on the train.

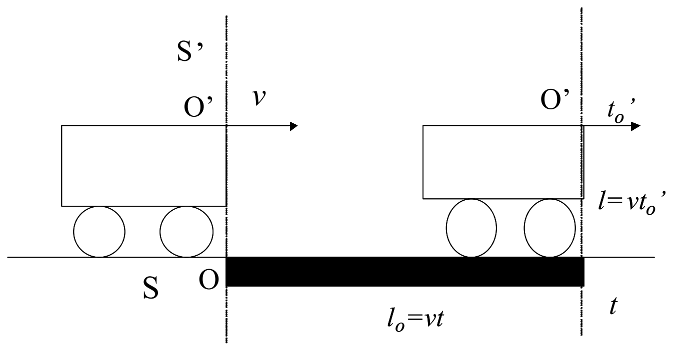

Fig.4: A stationary rod, laid along the side of the platform, by a moving train.

In Fig.4, a rod is laid along the side of the station platform. There is an observer on the platform and another observer on the train and both measure the rod's length using a sensor attached to the front of the train, i.e. the origin O' of the moving frame S'. The length of the rod as measured by an observer in the stationary frame S, is defined as **proper length,** l_0, while the length of the rod as measured by an observer in the moving frame S', is defined as **regular length,** l'. When the sensor touches the left edge of the rod, the time is recorded as 0 for both observers. When the sensor touches the right edge of the rod, the time is recorded t for the observer in the stationary frame S and $t_0^{'}$ for the observer in the moving frame S'. The event where the sensor moves from one end of the rod to the other can be described by the two different observers. This event occurs at the same location for the observer in the moving frame S', then the period of the event as measured by this observer is defined as the **proper time,** $t_0^{'}$. This event happens at different locations for the observer in the stationary frame S, then the period of the event measured by this observer is defined as the **regular time**, t.

The proper length of the rod is calculated by multiplying the train's velocity by regular time,

$$l_0 = vt , \tag{4}$$

and regular length is calculated by multiplying the train's velocity by proper time,

$$l' = vt_0^{'} . \tag{5}$$

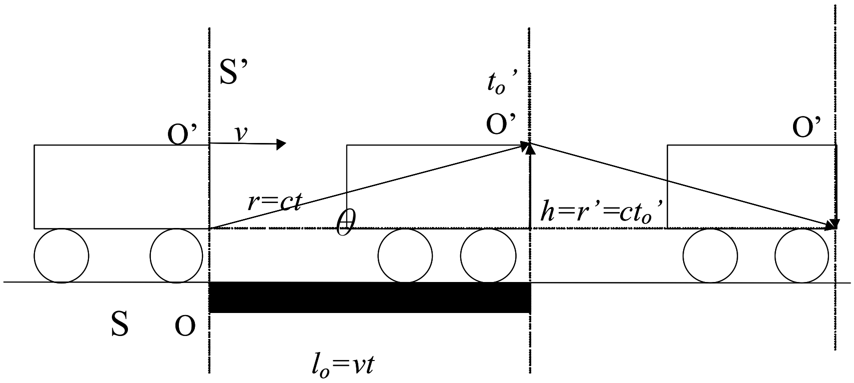

Fig.5: The height of the ceiling is adjustable. The pulse of light reaches the ceiling at the same time as the sensor touches the right side of the rod.

At the same time the sensor touches the left end of the rod, the observer in the moving frame S' sends a pulse of light towards the ceiling of the car, where a mirror is placed. To the observer in the moving frame S', the light travels vertically up towards the ceiling and is then reflected vertically down. The ceiling height of the boxcar is adjustable, such that the pulse of light reaches the ceiling at the same time that the sensor reaches the right end of the rod. In Fig.5, if it takes the proper time $t_0^{'}$ for light to reach the ceiling then the height of ceiling is equal to the distance traveled by light is $h = r' = ct_0^{'}$, as measured by the observer in the moving frame O'. To the observer in the stationary frame O, the light travels diagonally upwards to the ceiling and is then reflected diagonally downwards. If it takes the time t for light to reach the ceiling then the distance traveled by light on each diagonal leg is $r = ct$, as measured by the observer in the stationary frame O, where

$$h = \sqrt{r^2 - l^2} = \sqrt{(ct)^2 - (vt)^2} \qquad (6)$$

From Fig.5, we can derived the following property for θ, where

$$\sin\theta = h/r = \sqrt{r^2 - l^2}\,/r = \sqrt{(ct)^2 - (vt)^2}\,/ct = \sqrt{1-(v/c)^2} \qquad (7)$$

From the previous discussion, we know that $r = ct$ and $h = r' = ct_0^{'}$, then

$$\sin\theta = h/r = r'/r = ct_0^{'}/ct = t_0^{'}/t \quad \text{and} \quad t = t_0^{'}/\sqrt{1-(v/c)^2}\ . \qquad (8)$$

Therefore, $t \geq t_0^{'}$. This equation shows that the regular time, t, is larger than or equal to the proper time, $t_0^{'}$. This result says that the time interval measured by the observer in the stationary frame is longer than that measured by the observer in the moving frame. This difference is referred to as **time dilation**. Since $\sin\theta = t_0^{'} / t$, then

$$\sin\theta = t_0^{'} / t = vt_0^{'} / vt = l' / l_0 \quad \text{and} \quad l' = l_0 \sqrt{1-(v/c)^2} \, . \tag{9}$$

Therefore, $l' \le l_0$. This equation shows that the regular length, l', is less than or equal to the proper length, l_0. This result says that the length of a rod measured by the observer in the moving frame is shorter than that measured by the observer in the stationary frame. This difference is referred to as **length contraction**.

3. Constructing a 3-d s-t Coordinate System

Traditionally speaking, length and time are fundamental quantities and velocity is a derived quantity. As a matter of fact, if no object in the universe exhibits any motion; we would have no concept of space or time. Because there was motion of an object, the occupation of space could be observed and the order of time (the process from the beginning to the ending of an event) could be recorded. Hence, velocity should be treated as a fundamental quantity and length and time should be treated as derived quantities. Periodic motions are the most regular type of motion and light waves are the steadiest type of periodic motion. The velocity of light is constant for all observers and independent from wavelengths. Hence, a three-dimensional space-time (3-d s-t) frame is constructed with waves of light moving along x-axis, y-axis, z-axis. Based on Einstein's thought experiment about two simultaneous events on a moving train and invariant velocity of light, space and time are dependent. We postulate that there is a constraint in a 3-d s-t frame by embedding time into space for describing the motion of an object [3].

The waves of light along the x-axis, y-axis, z-axis are chosen as the foundation to construct a 3-d s-t frame. The units of the dimensions of space and time are constrained by the velocity of light:

$$\lambda / T = \lambda' / T' = c \tag{10}$$

The constraint is the invariance of the ratio of a unit of length to a unit of time. Waves along x-axis, y-axis, z-axis are chosen from the same type of light waves, because they have the same wavelength and period. MATLAB has extensive facilities for displaying vectors and matrices as graphs. It includes high-level function for two-dimensional data and three-dimensional data to describe the motion of an object moving in space in graphs.

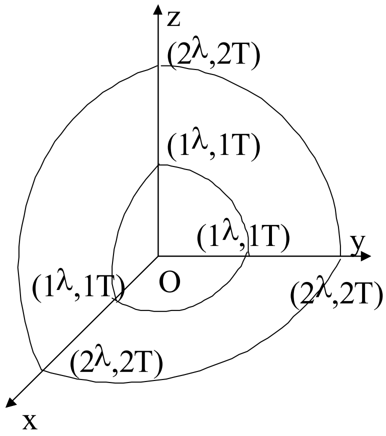

Fig.6a: A 3-d s-t frame is composed of x,y,z axes which are perpendicular to one another and spherical surfaces of different radii which represents the proceeding of time.

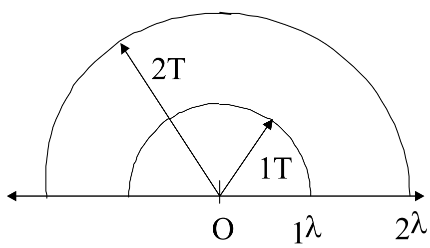

Fig.6b: For any dimension of a 3-d s-t frame, λ is chosen to be the unit of axis of space, and T is chosen to be the unit of radii of time such that they satisfy the constraint λ/T= c.

Spherical surfaces of different radii are used to represent the proceeding of time having the unit of a wave period, *T*, of light. Three axes are used to represent the distance in space, each with the unit of a wavelength, λ , of light. The coordinates of the intercepts among spheres of time and three dimensions of space are $(1\lambda,1T)_x$, $(2\lambda,2T)_x$, ... $(n\lambda,nT)_x$ on the *x*-axis, $(1\lambda,1T)_y$, $(2\lambda,2T)_y$, ... $(n\lambda,nT)_y$, ... $(n\lambda,nT)_y$ on the *y*-axis, $(1\lambda,1T)_z$, $(2\lambda,2T)_z$, ... $(n\lambda,nT)_z$ on the *z*-axis. The 3-d s-t frame is shown on Fig.6.

This 3-d s-t frame is another representation of space-time coordinate system. A 3-d s-t frame not only reveals direct dependency of space and time but also has the simplest constraint. Though the space coordinates are bi-directional, time only has one outgoing direction in this 3-d s-t frame. Since a 3-d s-t frame by embedding time into space can be comprehended easily, it is chosen as the coordinate system to describe motions of an object in this paper.

4. The Motions of the Elder Twin Described in Two 3-d s-t Frames

The younger twin was in frame O and the elder twin was in the frame O'. If the elder twin in the frame O' flied to the star from the earth with 80% of the velocity of light, then the velocity, *v*, is (4/5)*c*. After reaching the star, then the elder twin on the frame O' flied back with the same velocity to meet the younger twin in the frame O staying on the earth. Assuming the time of the velocity accelerated from zero to (4/5)*c* or decelerated from (4/5)*c* to zero is very short. There is no absolute motion in the universe, so there was only relative motion between the frame O and the frame O'. Because of the reversed direction of motion due to the frame O' flying back, only the frame O can be treated as a stationary inertial frame and only the frame O' can be treated as a moving inertial frame [4].

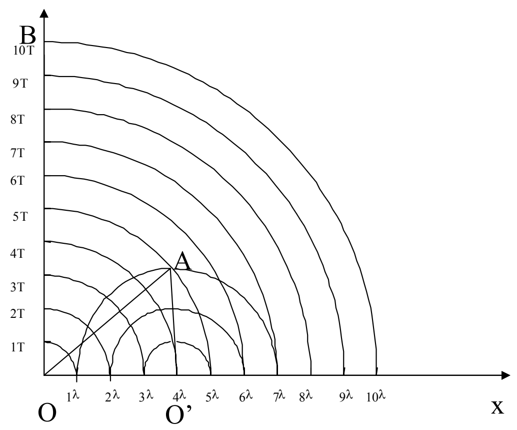

Fig. 7: The motion of the elder twin on the frame O' flied from the earth to the star with the velocity v=(4/5)c was described as the line OA by an observer on the frame O. The motion of the elder twin was described as the line O'A by an observer on the frame O.

From Fig.7, the motion of the elder twin in the frame O' flying to the star with the velocity $v = (4/5)c$ could be described as the line OA by an observer in a 3-d s-t frame O and the line O'A by an observer in the 3-d s-t frame O'. The ratio of OO'/OA is always 4/5 for any right triangle similar to the right triangle OAO'.

The time, *t'*, which he took, measured by the observer in the frame O' is called proper time. The motion of the elder twin in the frame O' flying to the star could be described as the line OA by an observer in a 3-d s-t frame O. The time, *t*, which he took, measured by the observer in the frame O is called regular time. If the elder twin traveled with 80% velocity of light, then O'A could be calculated as *3* by Eq.(11) from the right triangle OAO' of the graph where $OA = 5$, $OO' = 4$.

$$O'A = \sqrt{OA^2 - OO'^2} = \sqrt{5^2 - 4^2} = 3 \tag{11}$$

If it took the period of proper time *t'*, measured from the observer in the frame O', for the elder twin flied from the earth to the star, then it took the period of related time *t*, measured from the observer in the frame O. The period of regular time *t* and the period of proper time *t'* have the following relation:

$$t' : t = 3 : 5 . \tag{12}$$

It can be rewritten as

$$t' = (3/5)t . \tag{13}$$

For example: If it took 10 years for the elder twin to fly from the earth to the star measured on the frame O, then it took 6 years measured on the frame O'. This result using graphic method in a 3-d s-t frame is the same result as calculated by Eq.(8) of Special Relativity.

4.1 The Motion of Elder Twin Described by an Observer on the Earth

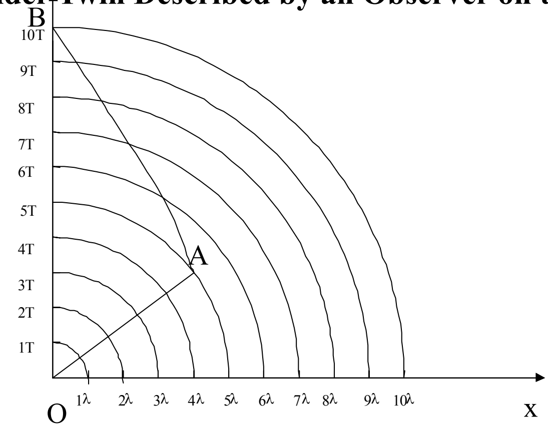

Fig. 8: The motion of the elder twin in the frame O’ flied to the star with a velocity v=(4/5)c was described as the line OA and after reaching the star, he flied back with the same velocity to meet his brother described as the line AB in a 3-d s-t frame by an observer in the frame O.

From Fig.8, the motion of the elder twin in the frame O’ could be described by the line OA flying to the star and the line AB flying back to the earth in a 3-d s-t frame by an observer in the frame O. Because the velocity of flying back was the same as the velocity of flying out, the period time of the round trip for the elder twin was 10 years measured by the observer in the frame O.

4.2 The Motion of the Elder Twin by an Observer on Space Shelter

From Fig.9, the motion of the elder twin in the frame O’ could be described by the line O’A flying to the star and the line AC flying back to the earth in a 3-d s-t frame by an observer in the frame O’. Because the velocity of flying back was same as flying out, the round trip for the elder twin was 6 years measured by the observer in the frame O’.

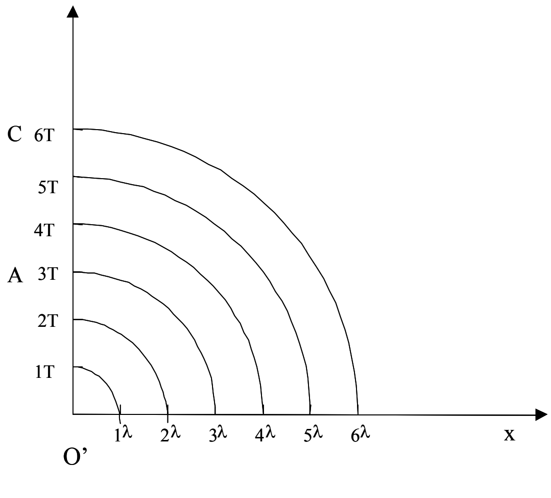

Fig. 9: The motion of the elder twin in the frame O’ flied to the star with a velocity v=(4/5)c was described as the line O’A and after reaching the star, he flied back with the same velocity to meet his brother described as the line AC in a 3-d s-t frame by an observer in the frame O’.

5. Conclusion

By the same graphic method using an adequate right triangle for the frame O' moving with different percentage of the velocity of light, we are able to calculate the proper time measured by an observer in the frame O' related to the regular time measured by an observer in the frame O. The results are listed in the following table:

Velocity	10%c	20%c	30%c	40%c	50%c	60%c	70%c	80%c	90%c	95%c	99%c
Regular time	10 years	10 years	10 years	10 years	10 years	10 years	10 years	10 years	10 years	10 years	10 years
Proper time	9.94 years	9.8 years	9.54 years	9.16 years	8.66 years	8 years	7.14 years	6 years	4.36 years	3.12 years	1.42 years

The motion of an object described in a 3-d s-t frame leads an easier and better understanding that after traveling from outer space. The elder twin looks younger than his brother waiting on the earth based on the calculation from the graphic method in the 3-d s-t frame. These results are the same results calculated from the formula of Special Relativity.

6. Acknowledgements

We would like to thank Bo Liu for a fruitful collaboration.

7. References

[1] A. Einstein, The Special and General Theory (Random House, Incorporated, 1977).

[2] J. J. Callahan, The Geometry of Space Time: An Introduction to Special and General Relativity (Springer Verlag, 2000).

[3] T. Chen and Z. Chen, Motion of Particles Described in a Three-Dimensional Space-Time Frame, the Proceedings of the 10th Asian Technology Conference in Mathematics, December 12-16, 2005.

[4] T. Chen and Z. Chen, A Pair of Inertial Frames versus an Inertial Frame, Volume V(2008), Number 3, the Journal of Concepts of Physics, www.uni.lodz.pl/concepts.