Custom tools and the iteration process as a referent point for the construction of meanings in a DGS environment

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Abstract: *The paper draws on an experiment conducted in a secondary school mathematics classroom in Greece which aimed to explore ways in which students develop their intuition, their meanings construction and the proving process in the Geometer's Sketchpad v4 DGS environment, using a custom tool which combines the beautiful drawing and the figure with geometric properties. Custom tools and the iteration process can be shown to be a suitable and valuable way of enhancing the construction of mathematical meanings. By this way, the DGS environment offers openable windows which guide students in their study of infinite structures and convergent, divergent sequences in both numeric and graphic notations. Both the bridging of empirical and traditional methods by presenting the proofs using dynamic means and aesthetic development through the computer software enhance the user's interaction, instilling a 'digital proof' impulse.*

1. Introduction

The paper focuses on a very interesting study area in terms of dynamic geometry: the use of custom tools and their cognitive effect on students' mathematical learning. The paper draws on a number of ideas including cognitive chunking, instrumental genesis, reification and aesthetics and seeks to connect them. The researcher describes a study conducted in a dynamic geometry environment, with a focus on the construction and use of a custom tool, trying to show that the custom tool supports students' building of meanings of mathematical concepts. The paper describes how students use and perceive custom tools. Some questions the researcher had in mind when designing the problems in the DGS environment and when conducting research were the following:

- Are custom tools forms of chunking, or just a new set of tools that act more like black boxes?
- Do the students actually use custom tools and iteration process as objects, or continue to use them as procedures? Or, what would be evidence of the student treating something as an object and a procedure?

In other words, there are many open questions about the use of custom tools that deserve closer study, and that can be addressed using the research described in this paper.

This paper also investigates the development of proof of formulae in the dynamic geometry environment of Geometer's Sketchpad v4 (see [12]) using a custom tool which combines technology and aesthetics. The custom tool is a digital technology in the case of Sketchpad, but what does it mean that it is in combination with aesthetics? Is it that the tool itself is 'beautiful', or that it produces beautiful images? The Ancient Greeks, particularly the Pythagoreans, believed in an affinity between mathematics and beauty, as described by Aristotle "the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful" (XIII, 3.107b quoted in [30]). According to Sinclair (see [30, p.262]) many "mathematicians (see [9],[24] and [25] quoted in [30]), as well as mathematics educators (see [2],

[11], [21] and [29] quoted in [30]) have drawn attention to some more process-oriented, personal, psychological, cognitive and even sociocultural roles that the aesthetic plays in the development of mathematical knowledge". Sinclair (ibid.) declares that "they associate the aesthetic *with mathematical interest, pleasure, and insight*, *and thus with important affective structures*…".

In the paper the role of aesthetics is also important. Perhaps is developed an expectation that the results of the study will demonstrate how the work with DGE and its aesthetics will motivate students to build meanings of mathematical concepts. This study does not focus on motivation stemming from the improved aesthetics obtained through dynamic geometry constructions. In this study is demonstrated how the images that are produced in the DGE environment are not simply beautiful but they have other aesthetic qualities. In other words they can, as digital material entities, be more than beautiful "drawings'; they can be 'figures' with geometric properties referring to the theoretical object (see [18]). And these properties are perceived by the students as they interplay with it. This seems pertinent in terms of the theory of instrumental genesis reported in the section on theoretical underpinning below, and is used in the study.

Firstly it is important to draw out the connections between "chunking" and custom tools. The resemblance is evident, but the process of using a custom tool seems to be different from the act of creating one, which may or may not have chunking as its goal. It does not mean that any act of creating a custom tool coincides with the act of chunking, but it cannot just be a faster way of doing or repeating things, helping our memory with a difficult construction.

2. Cognitive chunking and custom tools

Research in cognitive psychology asserts that chunking is an essential mechanism supporting learning of mathematics and thus reducing the complexity of a problem solving process. Chunking "supports and facilitates cognitive processes involved in encoding, extracting, remembering, and understanding information" (see [41] and [8] quoted in [28]).

Dynamic geometry environments such as Geometer's Sketchpad (see [12]) and Cabri Geometry II (see [17]) share three main features, drag mode, locus of points, and the ability to define and use macros/scripts-custom tools, a kind of 'technical chunking'. Macros/scripts can help students effectively to structure a geometrical construction by condensing a complicated sequence of construction steps into one single command (see [14],[15] and [33]). Dealing with scripts/custom tools rather than simple sketches can alleviate the complexity of the construction by reducing the steps of construction, reducing in that way the amount of information "that must be stored since chunks (in our case custom tools) are conceptually treated as single elements" (see [28]). Custom tools allow one "to encapsulate constructions into new commands, -constructing figures of arbitrary complexity- as well as to create entire microworlds with yours own tools using them an unlimited number of times in an unlimited number of sketches" (see [13]).

To create a new tool, we begin by building the general construction we want to define as a tool. This construction will also serve as the 'definition' during the tool creation process. For instance, thanks to the manner in which it is constructed and saved as a custom tool, an equilateral triangle can be defined, and thus categorized, as a geometric object with given properties. If, for example, the shape is constructed using the rotation command, then every side is the product of the adjacent side rotated by 60 degrees about the vertex–center common to the two sides. The student can thus categorize the equilateral triangle in his mind with the following properties: "a triangle with sides of equal length and angles equal to 60 degrees".

In this sense, through the custom tool, the equilateral triangle --object-- acquires a conceptual categorization as well as the meaning of the archetypal / primitive object. Davis & Tall (see [3] p.151) declare that "The fact that we can talk about chairs, or a chair, without referring to or pointing at a particular world-thing is a result of a process of perceptual categorization. The concept ‘chair’ is a mental concept, and not a corporeal world-thing.” Paraphrazing last expression “the concept ‘custom tool-equilateral’ becomes a mental concept and not a corporeal artefact” in the screen.

The building of the custom tool mentioned in the paper, takes into account the theories referred to in the next section relating to knowledge, learning and teaching during its design/construction and implementation of the activities.

3. Theoretical underpinning

A learning environment is something which is co-built in the teaching and learning activity by participants of the activity, and it evolves during the development of the activity. The design of activities in the learning environment (the software) as a part of the instruction thus has a crucial role to play in the comprehension of mathematical meanings. The ongoing study is underpinned by the theory of instrumental genesis (see [38]). From Trouche’s point of view, “instrumental geneses are individual processes, developing inside and outside classrooms, but including of course social aspects” (see Figure 3) (personal e-mail correspondence with Luc Trouche on April 4, 2008 quoted in[23]). An artefact, with which the interaction takes place during the mathematical activity, is transformated in an instrument. The use of the tool is directly linked to the use of the artefact (see for example [7], [27], [36] and [37]). The students accomodate the tool to their needs, it can therefore execute a particular operation and it can potentially be functionally extended. This means that a student can use the custom tool to discover the properties of the more complex shape (which is an extension of the structure) which therefore requires the student to act on the tool (external use of the construction) thus “the tool is shaped by the user during the instrumentalization process” while the artefact simultaneously acts upon the subject (internal use of the structure) and “the tool affects and shapes the users’ thought during the instrumentation process” (see for example [1] , [7] ,[27],[36] and [37]).

Consequently, in the first case, we have the accommodation of the user to the tool/ object, and in the second we have the assimilation of the tool acting upon the subject. During the instrumental genesis both the phases (instrumentation and instrumentalization) coexist and interact. Then the user structures that Rabardel (see [27]) calls utilization schemes of the tool/artefact. This process leads to the development or appropriation of *usage schemes* and *schemes of instrumented action*. As the discussion session of the Mind and Machine group coordinated by Drijvers (see [5]) makes clear: usage schemes are oriented to handle/manipulate tools, containing gestures (activity in the technological environment) and knowledge. Schemes of instrumented action are comprehensive chains of usage schemes anticipating a specific goal.

In the present case, the user shapes schemes of instrumented action, which can be defined as coherent and meaningful mental schemes for using the technological tool to solve a specific type of problem. Consequently, the ‘instrumented action’ scheme, which is based on the construction and use of the custom tool, leads us to construct the mental object which is based on our mode of construction in the software, and consequently on our actions. In the sequence of mental activities a student follows through the instrumented action schemes, mathematical knowledge and knowledge of the tool are combined. The shaping of the custom tools /scripts can help students mentally categorize a construction, initially in terms of their perceptions as a schematic entity, and then lead them through various stages to more abstract levels of cognitive perception. This also agrees with Edelman’s viewpoint (see [6] quoted in [3]): “in forming concepts,...the brain areas responsible for concept formation contain structures that categorize, discriminate, and recombine the various brain activities occurring in different kinds of global mappings”. This means that custom tools can serve as structural units of knowledge, and hence as schemes, too, including the structure and function of the collection.

Consequently, *custom tools operate as a referent point for organizing, pursuing, and retrieving information, and thus facilitating the reusing and handling of the schemes in a wide range of situations*. According to Trouche (see [36]) a scheme of instrumented action includes theorems-in-action ("propositions believed to be true" see [39]) and concepts-in-action ('concepts implicitly believed to be relevant' see [39]). It may help to analyze the students' work and to decompose the problem solving strategy. But we will begin in the next paragraph by examining the construction of the activity and how it is possible to construct supplementary concepts using the two different modes of constructing the tool.

4. The construction and operation of the custom tool

The current paper is concerned with a detailed description of the design process of the custom tools used for the construction of activities in the linked multiple pages facilitated by Geometer's Sketchpad v4 software. The resulting interlinked successive pages could be compared with a vivid, living section of a textbook. The researcher took into account the theories referred to in the previous section relating to knowledge, teaching and learning during the design /construction and implementation of the activity.

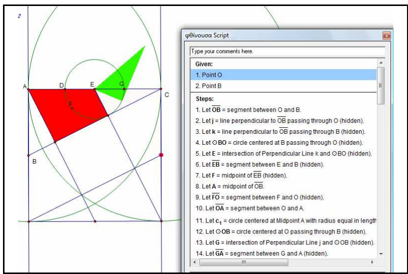

Figure 4.1

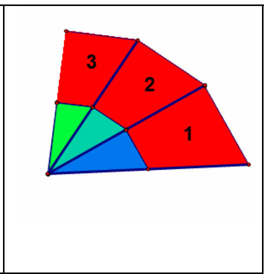

Figure 4.2

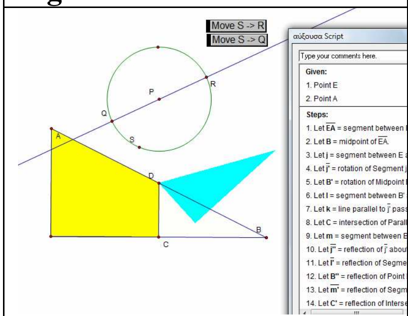

Figure 4.3

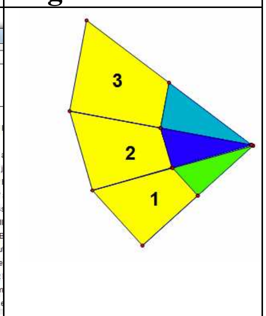

Figure 4.4

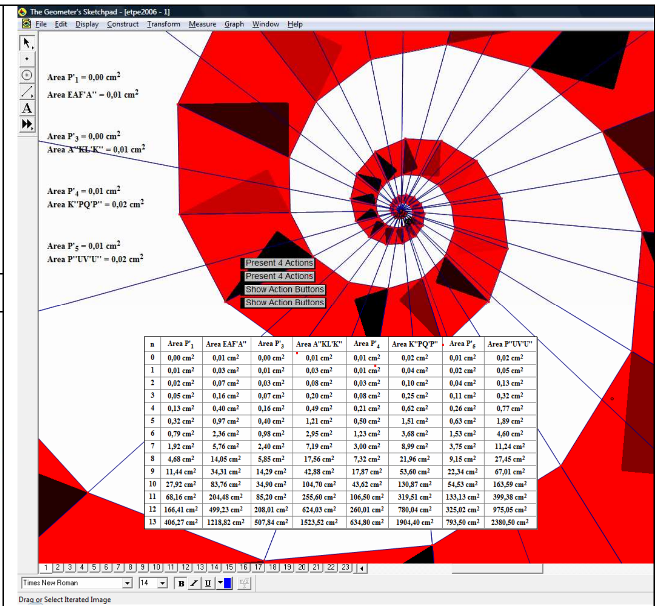

Figure 4.5

For example: Figures 4.1, 4.3 illustrate two different methods of constructing the original /initial right triangles, and include transformations which will subsequently be used to construct the custom tools. In Fig. 4.1, the right triangle with vertical sides in a ratio of 2:1, has resulted from joining the vertices with the middle points of the opposite sides of the square. In Fig. 4.3, a triangle with vertical sides in a ratio of 2:1 has been constructed directly. In both constructions all the lines that are not essential have been hidden to the operation of the custom tool and subsequently applied the transformation in the manner illustrated. The rearrangement demonstration occurs on the right triangle whose vertical sides are proportional to the original right triangles' sides in a ratio of 2:1. The application of the transformation to the vertex "acts as a hinge allowing the learner to rotate one of the connected triangles by direct manipulation. This process can be repeated resulting in the formation of a combination of shapes, while the area of all these shapes remains the same" (see [28]). Rearranging the construction, students could be helped as new information are highlighted otherwise difficult to understand (see [32] quoted in [28]). Prior to constructing the tool, the researcher also measured the areas and lengths of the sides of the initial construction. Although the final result of the two methods for constructing the initial right triangle including the rearrangement appear identical, they lead to ways of constructing a custom tool whose application provides different results in both computational and constructional (scheme) terms.

For example applying the tools three times in succession produces the results in figures 4.2, 4.4. This means that as we can see in the illustration, the areas of the shapes steadily decrease (figure 4.2) or increase (figure 4.4). Concretely, applying the tool using the appropriate method for constructing it, we take different constructional, representational results:

1. In method A, the longer vertical side of the initial triangle becomes the hypotenuse of the next right triangle in the sequence. Meaning the sequence of the measurements and calculations that emerges is descending.
2. In method B, the hypotenuse of the initial triangle becomes the longer vertical side of the next right triangle in the sequence. Meaning the sequence of measurements and calculations that emerges is ascending.

If we iterate the initial points of the construction of the tool we can take different results relating to the construction the measurements and the calculations. As it is well known for someone who uses the Sketchpad software the result of the process of iteration (see [34] and [13]) is the construction of the tables that repeat the process of initial measurements and calculations in dynamic connection with the shape, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table, the sequence of the natural numbers is presented (see [22]).

In that way through this operation, the environment of the software promotes the exploration of the sequences. The iteration process by functioning thus has integrated or embodied the meaning of sequence while there is a direct connection between the user's perception and the abstract mathematical meaning. As a result of the construction and application of the custom tool as much as the process of *iteration* the direct perception of the user is attained in regard to the steps in the development of the construction pertaining to (see [22]):

- the repetitions in the measurements or calculations of the areas of initial shapes
- the developmental way of the construction of the shape and
- its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file.

The process of animation can produce the changes in the tabulated measurements (calculations) that allow the user to examine the dynamic process. Fig. 4.5 illustrates the construction of the tables that repeat the process of initial measurements and calculations of the ascending sequence in dynamic connection with the shape. In the software, via the process of iteration we have the potential of the constructions thus becoming more complex being in theory rendered inductively to infinity. This function of the software also constitutes a certain crucial and essential particularity, while the construction with a compass and a straightedge as static tools of geometry has a beginning and an end.

The students in the experimental group came to understand this process, as discussed in the next section.

5. Research methodology

The qualitative study (see [23]) was conducted in a class at a public high school in Athens, Greece and involved twenty eight volunteers, students aged 15-16, randomly divided between the 'experimental' and the 'control' teams, with 14 students in each. The researcher ensured that both teams consisted of equal number of boys and girls, equal distributed to their achievement in mathematics. The students were friends, which fostered group discussion. The methodology of the class experiment discussed in this paper includes the exploration of the methods A, B as open problems. For example the first problem is the following: C*onstruct sequential right triangles with vertical sides in a ratio of 2:1, so that the longest vertical side takes the place of the hypotenuse in every new repetition* (figure 4.2). *Calculate the hypotenuse in every new repetition.*

And the second problem is the following: C*onstruct sequential right triangles with vertical sides in a ratio of 2:1, so that the hypotenuse takes the place of the longest vertical side in every new repetition* (figure 4.4). *Calculate the hypotenuse in every new repetition.*

Students of the experimental team had only worked on the Sketchpad v4 figures reported in the previous section using the pre-constructed custom tools, and later applying the iteration process. The researcher did not consider it necessary "to separate the 'teaching of mathematics' from the 'teaching of the tool', preferring to integrate the appropriation of the functioning of a tool with the learning of the mathematics" (see [20]). The experimental team discussion was designed to record how students would react while interacting with the tools. The researcher's aim was in this way to adapt the experiment to real classroom conditions. It provides an example of the interplay between the students and the researcher and reflects Kaput's (see [16]) writing on the importance of technology in mathematics education, concerning the feasibility of innovative practices emanating from technological advents, which were otherwise impracticable. The researcher had hooked a projector to her computer, and the pupils could participate individually or as a group during the session by interacting with the activity. This raised two research questions: 1) How do iteration process and custom tools affect on students development of meanings? 2) Do the students actually use iteration process and custom tools as objects, or continue to use them as procedures and what would be evidence of the student treating something as an object and a procedure?

The experimental sessions were videotaped. The analysis of the results that follows is based on observations in class and of the video. Later sections present the sessions of the solution process the students undertook.

5.1 The solution process with the students of the control team

The group was initially tested to see whether they could produce a shape with the properties we want. Facing the problem 1 or 2 the students were unable to go on. The repetition of the procedure with the assistance of geometrical tools like compass and ruler (or straightedge) is difficult since students have to calculate a segment y with side 2a as its hypotenuse, and then a segment z with side 2y as its hypotenuse and so on. Their difficulty was partly down to technical reasons, but also due to the students being unable to dynamically transform the shape in their minds. If nothing else, it is more time-consuming and impedes students from developing their imaginations.

5.2 A discussion with the students of the experimental team

The group was initially tested to produce a shape with the properties we want. For instance, in method A, the students processed problem 1. Constructing the tool using method A produces a descending sequence of measurements and calculations (relating to lengths of sides or to areas), while the use and application of method B results in an ascending sequence. In both cases, the ratio of the areas is stable and equal to 1.25 (figure 5.2.1). The students began by experimenting with the 'rearrange the figure' action button. They realized, though did not see the hidden circle on which the point of triangle's vertex is moved, that "there is a circle through which the vertex of the triangle moves while turning through 180 degrees".

The use of the custom tool aroused the students' *interest* as they played with it. The students had not learned or even heard about the concept of the sequence or the limit. Initially, the students verified the relationships between the areas of the sequential triangles or trapeziums by applying the tool several times over. Then, knowing what would ensue, the researcher suggested applying the tool to the sequential vertical sides of the shape (figure 5.2.1).

The application of method A, for instance, is dealt with below. Applying it to the long vertical sides resulted in figure 5.2.1. The students wanted to experiment with the custom tool because they didn't believe it worked on a small shape. Student M_2 points at a small, deeply-positioned triangle.

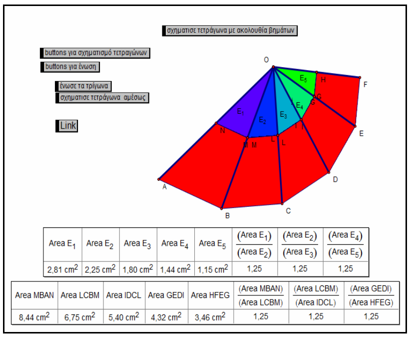

Figure 5.2.1

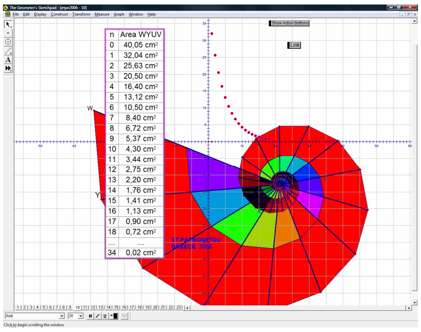

Figure 5.2.2

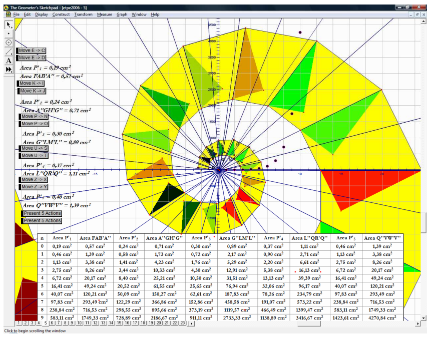

Figure 5.2.3

341. M_2: *here on the smaller triangle, can I apply it here***?** And notes, surprised like the others, that the construction is repeated even if the triangle is very small. The students noted that the values of the areas of the emerging descending (or ascending) sequence depended on the construction of the initial custom tool. When they applied the iteration process, the results were stupendous (figure 5.2.2, 5.2.3): the students could press the presentation buttons, rearranging the triangles and they stated:

345. Students: *the figure becomes a spiral. It resembles the nautilus scheme … anyone can believe that the shape does not finish*. (Figure 5.2.2)

The tables that resulted from the application of the iteration process led them to note that the ratios of the areas remained stable as the sides changed in line with a pattern.

347. Students: *all the numbers (in the same column in the table)* ***are related*** *in the same way, which is why they have the* ***same proportions*** *in the table.*

348. M_2: *But we cannot be sure what is happening in its depths* and she pointed into the spiral that had formed. Student M_2 speaks pointing to the spiral.

349. Researcher: *what do the areas do when the values of the sequence increase?*

350. M_4: *they get smaller… they go down to zero*.

351. M_1: *then we wouldn't have any geometric shapes at all.*

352. M_2: *the values* ***tend towards zero.***

The researcher asked them to use the dilate tool to increase the size of the shape.

355. Students: *Our initial triangle (be mentally) becomes as large as the room*; *it can become infinite.* (Figure 5.2.3)

356. Students: ***Once again****, the ratio remains constant.*

The students experimented with the table of measurements, plotting the points in a graph. Fig. 5.2.2, and 5.2.3 illustrate the plotting points of the descending and the ascending sequence.
382. Students: *The areas become so small they* ***tend to 0*** (Figure 5.2.2)**.**
383. Researcher: *what is the domain of the sequence*?
384. M_2: *isn't N the Natural numbers*?
385. M_1: *as N (natural numbers) increases, E (the area) grows ever smaller* (Figure 5.2.2) and as N increases, E grows ever larger (figure 5.2.3).
386. M_8: *yes, but there* ***might be a second domain*** *(pointing to the values of N)* ***from which point forward the sequence tends to zero.***
The students consequently had an environment of multiple representations in which the shape of the fractal had been linked with the table of the measurements via the functional process of iteration, which continuously could be linked with the graphic representation of the sequence.
388. Researcher: *What do you think will happen to the area if we increase the number of iterations?*
389. M_6: *because we are dealing with the area of triangles rather than the area of a point,* ***it is impossible for the area to reach zero****; we can't depict something of zero value.*

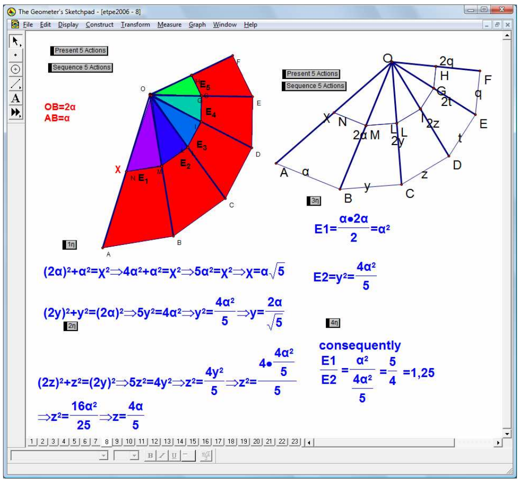

Figure 5.1.4

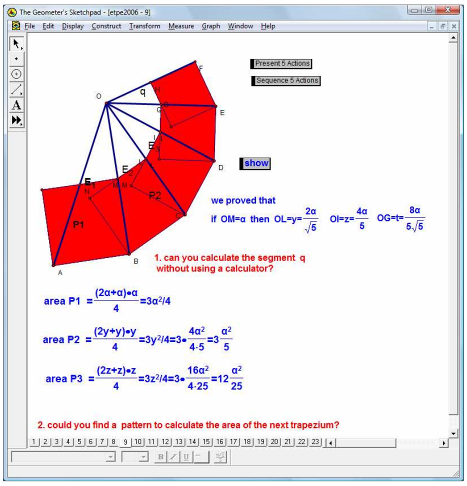

Figure 5.1.5

The students see a point on the screen, but their intuition tells them the point is a triangle with an area other than zero. This means they have intuitively formed the concept of the *infinitesimal* quantity of the area of the triangle. This is another point at which we can discern the degree to which the student's awareness has been heightened through their experimentation using the software. The connection of the *concept image* with the *concept definition* of the meaning (see [40]) was developed through the environment of the software. The custom tool process derived from method B helped the students understand that the values of the sequence increase steadily. The graphic representation of the sequence of areas displayed / showed the isolated points as they appear on screen (Figure 5.2.3).
417. Researcher: *How can we work out the length of this side*? I pointed the hypotenuse out to them.
418. M_3: ***If*** *we assume that the vertical side one is a,* ***then*** *the next vertical side is 2a, since it is double its length. By applying Pythagoras' theorem we can work out the length of the next side.*
Their initial observations were confirmed using the proofs of the formulae which could be hidden/shown in the software, which motivated the student to move on to the next step while gradually leading them empirically and by means of formalistic processes to the construction of the meaning of the sequence and of geometrical progress. They reached conclusions relating to the way in which the tool is used, and hence about meaning including limit, a sequence of values and

geometric progression. Having worked out the formulae and hidden them using the hide/show buttons, I made sure the students clicked to reveal the proof that confirms their efforts.

When they had noted that the relationship between the sides and areas of the sequential shapes was constant, I asked them:

426. Researcher: *How did this result (1.25) come about? (*Figure 5.1.4)

427. Students: *It is the same ratio we saw earlier in the tables. The calculations of the areas produce the same result. The results **confirm the correctness** of the tables.*

428. M_8: ***Every number** is the previous number multiplied by 1.25.*

429. Researcher: *How does the number 1.25 result?*

430. M_7: ***It's a constant value** which results from **every division** of the two continuous areas of the triangles (or the trapeziums) **….**each triangle area, results from the previous area when divided by 1.25.*

Through an in-depth analysis of the students' discussions, an explanatory framework identified the key elements in the development of their intuition, their meanings construction and the proving process. It is discussed in the next section.

5.3 Discussion and analysis of the experimental team's process

To understand why a maths activity for students built with ICT is powerful needs from us an objective analysis. The analysis of student discussions includes the following objectives: the construction of meanings, recognition of instrumented action schemes through the emergence of theorems and concepts-in-action. We noted the following in student-student and student-researcher utterances that are marked in bold in the paper dialogues:

[341-345]: use of tools (custom tool and iteration command) for the construction of a beautiful image.

[347-383]: construction of the meanings of the sequence, of the geometric progression and the intuitive construction of the limit of the areas which tend to zero.

[383-386]: students using multiple representations in order to bridge procedure and object

[388-389]: intuiting the meaning of infinitesimal quantity.

[417-418]: the emergence of a theorem-in-action

[426-430]: the custom tool and the iteration process as forms of chunking and the emergence of a concept-in-action

During the session, the students formed a usage scheme for using the custom tool and the iteration process. The custom tool led students using it to discover the properties of the more complex shape which therefore required the subject to act on the tool (instrumentation of the tool) while the artefact simultaneously interacted upon them (instrumentalization of the tool). Consequently, in the first case, we had the accommodation of the students to the tool, and in the second we had the assimilation of the tool and the construction of utilization schemes that resulted the construction of the meanings such as sequence and limit mentioned above. The different way in which the tool was constructed encouraged the students to build the meaning for the concept of the ascending and descending sequence. They correlated the construction of the nautilus spiral with the shape on the screen and they were led by the tables to construct the meanings of the limit and the infinitesimal.

Students through the instrumental genesis are able to build up instrumentation schemes that combine technical and conceptual aspects. The mental activities render meaningful the enacting on the above technical actions. As Drijvers (see [4]) writes "The instrumentation schemes integrate technical skills and conceptual insights". As we "cannot look inside the heads of the students to observe the mental schemes" Drijvers (ibid.), "we focused on the techniques, which can be considered as the observable parts of the instrumentation schemes". For example, the process of rotating a triangle by specifying a mark angle of 180^o and marking point F as the centre results in a congruent triangle as the original triangle but rotated through 180^o. Any effort to modify the lengths of the sides of the original triangle by dragging its vertex will result in an equivalent modification of the dependant rotated triangle due to transformation. The rotation of the triangle in the software shapes an instrumented action scheme which leads the students to conceptually grasp the meaning of congruent triangles, having a significant impact: the student structures a utilization scheme of the tool, and consequently a mental image of the functional/operational process of rotation, since any modification/ transformation of the initial triangle (input) results in the modification/transformation of the final triangle (output) (see [23]). In such mental schemes technical and conceptual aspects are interwoven (see [4]). The instrumentation process proceeding through the tools affects and shapes the way the user thinks, while the student also exerts an affect on the tools and acts by formulating his thoughts through the instrumentalization process. The student structures a usage schema in order to use the custom tool while simultaneously organizing his activity through the tool's own utilization schema. On the other hand through the application of the custom tool the possibility is given to the user to acquire an inductive way of thinking for the finite steps of the construction but the generalisation with regard to the constructional result can be achieved from the process of iteration which inductively renders the construction theoretically to infinity (see [22]). For example students formulate the sentence "***Every number** is the previous number multiplied by 1.25".* This sentence includes a generalization.

Students' utterances that are marked in bold in the paper such as "*all the numbers **are related in the same way", "**the values **tend towards zero", "**The areas become so small they **tend to 0",*** contain concepts-in-action and theorems-in-action. Vergnaud (see [39]) insists that "theorems-in-action cannot exist without concepts-in-action, as theorems cannot exist without concepts-in-action, and vice-versa". What is important to note is that the student M_1 thus guided to formulate the theorem-in- action "as N (natural numbers) increases, E (the area) grows ever smaller..." was not generally successful at Maths. The formulation of the student's thoughts includes a hidden/implicit "if... then" expression. M_1 does not express his thoughts exactly; his phrase is incomplete, but he seems to understand the process. He means: "**if** N (natural numbers) increases, **then** E (the area) grows ever smaller...". According to Vergnaud (see [39]) "when operational invariants are expressed and involved in systems of concepts and symbols, their cognitive status changes, up to the point that schemes can sometimes become algorithms. When the relevant properties of the mathematical objects and operations involved in action are made explicit, it becomes possible to analyse their connections, and eventually to demonstrate that a certain set of rules, for a certain class of situations, is effective".

The environment also helped them discover that "*it is impossible for the area to reach zero" and "**might be a second domain** (pointing to the values of N) **from which point forward** the sequence tends to zero"*. This marks the start of a transition to a level of rigour which emerged with the building of the concepts in the dynamic geometry environment.

6. Conclusions

"Advanced mathematical thinking today involves using cognitive structures produced by a wide range of mathematical activities to construct new ideas that build on and extend an ever-growing system of established theorems. The cognitive growth from elementary to advanced mathematical thinking in the individual may therefore be hypothesised to start from "perception of" and "action on" objects in the external world, building through two parallel developments—one visuospatial to verbal-deductive, the other successive process-to-concept encapsulations using manipulable symbols—leading to a use of all of this to inspire creative thinking based on formally defined objects and systematic proof" (see [35]). Sfard (see [31]) speaks of *reification* of process as object and claims that an interaction between a process and an object is indispensable for a deep understanding of mathematics whatever the definition of "understanding" may be. For this the understanding of mathematical meanings is connected with the potential to conceive them simultaneously as objects and as processes. In the present method through the complex construction of *instrumental genesis* (see [38]) students have been assisted intuitively develop their understanding of infinity and they are led to approximate the meaning of limit, but simultaneously as resulting from the function of the process of iteration and through the tables that contain vivid digital measurements and calculations, appreciate the numerical value of the limit as the result of an infinite approximating process. Consequently: The proposed approach leads the students to consider the limit as an object but they (do not) loose the sense of the process that lies beneath the object itself (see [19]). It appears that the custom tools and the software iteration process lead to the bridging of *procedures and object* during the construction of the meaning of the limit (see [22]).

Descriptively the students are urged to develop a theoretical way of thinking and despite the fact that the observations were limited to numerical relations the students answered to qualitative questions concerning the dynamic behaviours of mathematical objects, since the process as it is being built progressively guides them to examine more parameters. With respect to the affordances offered by the dynamic geometry environment, was developed the proving process. This uses the proof process to verify the on-screen 'visual theorem' which was argued visually. The students constructed their arguments by forming the visual representations as well as the symbolic representations derived from the formulae. They concluded the properties from the geometric shape and ended up with formulations of algebraic rules. In a DGS environment, the hide/show action buttons act as a tool which supports focusing on the proving process. In direct connection with the shape, hiding and showing the formulae for parts of the proof leads to a different form of proof: a combination of traditional and digital means which could be called a '*digital proof*'.

The custom tool with which the interaction takes place during the mathematical activity is directly linked to the construction of the artefact-spiral. The students accommodate the tool to their needs. This means that a student organizes his/her actions by mental schemes and uses the custom tool to discover the properties of the more complex shape (which is an extension of the structure), as the tool acts on the student in order to extend his/her mental *scheme*. As a consequence, the use of the custom tool which includes the mental scheme of its use organizes the activity of the subject and leads to a more complex structure by creating a new mental scheme for the new object with its properties. In our case, a new mental scheme is for example the ascending/descending sequence or the meaning of limit for the triangles' areas. Meaning that, apart from guiding the student to more complex constructs, the appropriately constructed custom tool also renders the mental and dialogic processes more fruitful, and thus leads to higher levels of abstraction. The constructed spiral allows the students to grasp the sense of infinity, and is imbued with numerous mathematical properties.

Some maths can be beautiful for the designer and awful for the user and vice versa. Maths can be extremely ugly at crucial moments, but nevertheless we can say because of that, "beautiful" when the property has been discovered. The spiral combines beauty and maths but as Hardy (see [10] quoted in [30]) argues "Beauty is the first test: there is no permanent place in this world for ugly mathematics."

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