Linking Visual Active Representations and the van Hiele model of geometrical thinking

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Abstract:*The present study presents the different modes of LVAR which can be constructed in Geometer's Sketchpad v4 dynamic geometry software. The paper posits an explanation of the correlation between the five phases in the apprenticeship/learning process proposed by van Hiele and the developing theory on LVAR. A few examples of the different modes of LVAR are presented, including the answers of the pupils participants in the didactic experiment conducted. We can thus conclude that transformations through LVAR lead students to structure mental transformations relative to the development of their van Hiele level.*

1. Introduction

The paper is about secondary school students development of geometrical reasoning in an environment based on the interaction between the students and the different modes of linking representations, facilitated by Geometer's Sketchpad v4 (see [16]) dynamic geometry software. The paper touches on a research area which is still debatable in the mathematics education community: the effect that external representations (for example software tools) and human interaction (for example teacher guidance or classroom discourse) can have on student's cognitive development, considering both social and cognitive dimensions in the study of the problem solving process in the dynamic geometry context. Some questions the researcher had in mind when conducting research and when designing the problems in the DGS environment were the following: How is the idea of Euclidean proof correlated with the van Hiele model of geometrical thinking? How can we transfer this idea to the dynamic geometry environment, exploiting the role that dynamic representations play in the development of students' geometrical reasoning?
Concretely: During a didactic experiment conducted in Greece supported by the Geometer's Sketchpad v4 dynamic geometry software, student participants followed a 4-phase research-based curriculum proposed by the researcher. The researcher during the last phase of her qualitative study (see [29]), conducted a didactic experiment with (semi) pre-designed multiple-page sketches detailing the sequential phases of the solution to the problem using rigorous proof, and in so doing transferring her classroom teaching style into the software problem-design, drawing on the chain questioning method of Socrates, which aim to stimulate interaction. For this reason, she linked all the software functions/actions using the interaction techniques supported /facilitated by the Geometer's Sketchpad v4 (DGS) environment to better allow students to discover solution paths and to reason by rigorous proof. This mode of design and the results of the experimental use of the software with students led to the need to define two new concepts: the meanings of Linking Visual Active Representations (LVAR) and Reflective Visual Reaction (RVR). Linking Visual Active Representations, and Reflective Visual Reaction during a dynamic geometry problem solving session, are defined as follows (see [29] and [30]):

Linking Visual Active Representations are the successive phases of the dynamic representations of the problem which link together the problem's constructional, transformated representational steps in order to reveal an ever increasing constructive complexity; since the representations build on what has come before, each one is more complex, and more integrated than in previous stages, due to the student's (or teacher's, in a semi-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or exist in their mind).

Reflective Visual Reaction is that reaction which is based on a reflective mode of thought, derived from interaction with LVAR in the software, thus complementing and adding to the student's pre-exesting knowledge or facilitating comprehension and integration of new mathematical meanings.

The results of the research can be illustrated as follows (see [29]): LVAR motivated the students to answer rapidly and spontaneously; the researcher kept her questions coming fast, which meant students did not have time to use paper and pencil. The researcher's classroom observations reveal that the same pupils did not always display the same spontaneous reflex reactions. The LVAR spread over multiple pages helped the students to react instantaneously and to articulate their thoughts; The LVAR helped the students to operate in a auxiliary/ complementary manner, assimilating /accommodating their prior knowledge, or as a confirmation of the pupil's thought processes /mental approach; The students' RVR occured at many points during the didactic experiment thanks to the use of interaction techniques. As a result the pupils constructed mental schemes for mathematical meanings and were "starting to develop longer sequences of statements and beginning to understand the significance of deduction" (see [8]). LVAR helped the students form rigorous Euclidean proofs and they reached conclusions on the problem by correlating the theorems they already know. This is to say that LVAR assisted students to develop their van Hiele level.

In the present study the different modes of LVAR are presented correlated with the phases developed by Dina van Hiele-Geldolf.

2. The van Hiele model and the five phases of apprenticeship/learning process

Pierre van Hiele and his wife Dina van Hiele–Geldof developed a theoretical model involving five discrete levels of thought development in geometry (see [10]). According to Dina van Hiele Hiele-Geldof (see [10]) the didactic experiments that she discussed had the objective "to investigate the improvement of learning performance by a change in the learning method". She investigated in her study if "it was possible to use didactics as a way of presenting material, so that the visual thinking of a child is developed into abstract thinking in a continous process, something that is requisite for logical thinking in geometry". The five levels of thinking reflect on students' progress and increasing development in the way in which they are able to reason about geometrical objects and their relationships and focuses "on the role of instruction in teaching geometry and the role of instruction in helping students move from one level to the next" (see [10]). Central to this model, is the description of that, which are: Level 1 (Recognition), Level 2 (Analysis), Level 3 (Informal deduction) Level 4 (Formal deduction) and Level 5 (Rigor). Another important aspect of the van Hiele model is the five phases it specifies in the apprenticeship/learning process, which are, in brief: 1) information (inquiry), 2) directed orientation, 3) explication, 4) free orientation, 5) integration. Instruction that takes this sequence into account promotes the acquisition of a higher level of thought. Concretely according to Pierre van Hiele (see[10]):

The first phase is one of inquiry/information: "The student learns to know the field under investigation by means of the material which is presented to him. This material leads him to discover a certain structure. One could say that the basis of human knowledge consists of this: mankind is characterized by the relevation of structure in any material, however disorganized it may be, and this structure is experienced in the same way by several people which results in a conversation that they can have about this subject."

In the second phase, that of directed orientation , "The student explores the field of investigation by means of the material. He already knows in what direction the study is directed; the material is chosen in such a way that the characteristic structures appear to him gradually".

In the course of the third phase, explicitation takes place. "Acquired experience is linked to exact linguistic symbols and the students learn to express their opinions about the structures observed during discussions in class. The teacher takes care that these discussions use the habitual terms.It is during this third phase that the system of relations is partially formed."

The fourth phase is that of free orientation. "The field of investigation is for the most part known, but the student must still be able to find his way ther rapidly.This is brought about by giving tasks which can be completed in different ways. All sorts of signposts are placed in the field of investigation: they show the path towards symbols"

The fifth phase is that of integartion: "The student has oriented himself , but he must still acquire an overview of all the methods which are at his disposal. Thus he tries to condence into one whole the domain that his thought has explored. At this point the teacher can aid this work by furnishing global surveys. It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows."

3. Relative research studies

Battista (see [5]) found that spatial visualization and logical reasoning were important determinants of geometry achievement. Clements, Battista, and Sarama (see[7]) designed a research Logo-based curriculum Geometry to look at how elementary students learn geometric concepts and characterize how Logo might facilitate students' learning. Clements, Battista and Sarama concluded that "compared to the traditional curriculum, the research Logo-based curriculum had only moderate positive effects on the students' ability ...". The experimental group outperformed the control group in identifying better lines of symmetry for a given figure, justifying why pairs of figures were congruent, and better understanding of slides (translations), turns (rotations) or flips (reflections). Sedig, Klawe, and Westrom (see [32] quoted in [33]) conducted an empirical study and they found that "adding scaffolding to direct manipulation of representations of transformation geometry concepts significantly improved student learning".

Dynamic geometry systems such as the Geometer's Sketchpad (see [16]) or Cabri II (see[21]), or any other DGS software are microworlds designed to facilitate the teaching and learning of Euclidean geometry. The Geometer's Sketchpad, is a highly visual dynamic tool for exploring and discovering geometric properties. Many researchers have conducted studies, using the van Hiele model as descriptor for their analysis and concluded that students who used the Sketchpad displayed (see [29]): more positive reactions when testing conjectures and constructions (see [13]); achieved significantly higher scores on a test containing concepts (see, [9]). Dixon concludes that students who were taught about the concepts of reflection and rotation in a GSP environment significantly outperformed their peers who had received traditional instruction in the content measures of these concepts); achieved significantly higher scores between the pre- and post-tests (see, [37] and [1]). Hollebrands (see [15]), examined the ways in which the Geometer's Sketchpad, mediated students' understanding of geometric transformations; her study investigated the ways in which they used technological affordances and the ways they interpreted technological results in terms of figures and drawings. She declares that "students learning geometric transformations in a technological context may develop understandings that are influenced by their interactions with the technological tools."

4. The role of LVAR to theoretical thinking

Researchers around the world agree that learning is a complex process, being both constructivist as it depends on active individual construction, and sociological, since it becomes part of culture having sociocultural aspects. The arrival of computers led to a good deal of hope being invested in the autonomous cognitive activity a learner could develop when presented with specific tasks and activities (see [2]). The general framework owes much to Piaget's approach: faced with a sufficiently problematic context, the learner has to negotiate gaps in or inconsistency problems with his/her knowledge. As the students become familiar with the technological tools, they control their world, and their cultures and modes of knowing thanks to their acquired competence (see [6]). Since tools exert an influence over the technical and social way in which they conduct an activity, they are considered essential to their growth and development.

Dynamic geometry softwares are *representational infrastructures* (see [19]) that may be used to make changes both in geometry and the expression of geometric relationships, consequently in the teaching and learning of mathematical concepts. These systems can play an intermediary/mediatory role organizing students' thought processes, so they can build an internal representation based on an external model (see [17]). In Geometer's Sketchpad v4 DGS environment, LVAR are interpreted as "encoding the properties and relationships for a represented world consisting of mathematical structures or concepts" (see [33]) in line with Goldin and Janvier (see [12]): a) "a physical situation, or situation in the physical environment", can be described /modelled mathematically embodying mathematical ideas; b) a combination of "syntactic and structural characteristics" enhanced by selected different interaction techniques facilitated by the DG Sketchpad v4 enviroment where the problem is transferred or a geometrical theory is discussed; c) a formal mathematical proof, "usually obeying axioms or theorems or conforming to precise definitions, --including mathematical constructs that may represent aspects of other mathematical constructs"; d) "an internal, individual cognitive configuration, inferred from behavior or introspection, describing some aspects of the processes of mathematical thinking and problem solving".

The goals in developing LVAR were to (1) provide DGS –based problems that are adaptations of variations and extensions of existing activities, (2) getting students to solve problems individually or in a classroom orchestrated process, which developed mathematical understanding and formal mathematical proof (3) provide experiences that are more effectively presented by selected interaction techniques facilitated by the DGS environment than by other didactic materials and (4) provide this experience in the context of the figurative or drawing design mode (see [23]), by means of which pupils develop their aesthetic sense and acquire actual cognitions in geometry.

This idea is in accordance with Parzysz (see [28]) drawings and figures. He defined a drawing as a representation of a geometrical object and a figure as the "text defining it [the geometrical object]". Hollebrands (see [15]) writes that "building on Parzysz's ideas, Laborde defines drawing as that which refers to the material entity (the physical drawing) while figure refers to the set of discursive representations and diagrams referring to the geometrical referent (the theoretical object)". LVAR in the Dynamic Geometry environment play an ambiguous complementary role exactly as Laborde (see [20]) reports for the diagrams in the plane geometry: "on the one hand, they refer to theoretical geometrical properties, while on the other, they offer *spatiographical* properties that can give rise to a student's perceptual activity". In the same way, LVAR exactly link the material digital entities on the screen with the theoretical mental referent which can be worked on.

Sedig and Sumner (see [33]) have distinguished between basic and task-based interactions with VMRs. To achieve pupil interaction using LVAR, the researcher used a diverse set of interaction techniques including "animating" a point on its path, 'tracing" a segment, "hiding and showing" action buttons, and "linking" or "presenting" action buttons. In so doing, she successfully linked both the steps in constructional and transformational actions and the various sequential phases in the proof. According to Lagrange (see [22]) "A technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." Through LVAR the teacher can guide the students by means of elucidation or questions eliciting conclusions which form a step-by-step visual proof. The software's successive pages also play a significant role, and can be seen as a vivid section in a book revealing the various stages in the proof. The sequence of increasingly sophisticated construction steps could thus correspond to the numbering of the action buttons which allows student to interact with the tool when they want to, or when they are encouraged to do so by their teacher in class.

The theoretical framework includes the notions of instrumental genesis (see [36]) and the distinction between phases of instrumentation and instrumentalisation (see for instance [2], [3], [4] and [14]), which are fundamental in teaching in computer environment. During the instrumental genesis both the phases (instrumentation and instrumentalization) coexist and interact. Then the user structures that Rabardel (see [31]) calls utilization schemes of the tool/artefact. Utilization schemes are the mental schemes that organize the activity though the tool/artefact. This process involves many studies, among them, for example the one of Artigue (see for instance [4]), based on the research of Verillon & Rabardel (see [31]) about the ways by which an artefact becomes an instrument for a student. From Trouche's point of view, "instrumental geneses are individual processes, developing inside and outside classrooms, but including of course social aspects" (see Figure 3) (personal e-mail correspondence with Luc Trouche on April 4, 2008 quoted in [29]). Trouche supports that "an artefact is tranformated thus through instrumental geneses, oriented by finalized actions, assisted by instrumental orchestrations, into an intrument".

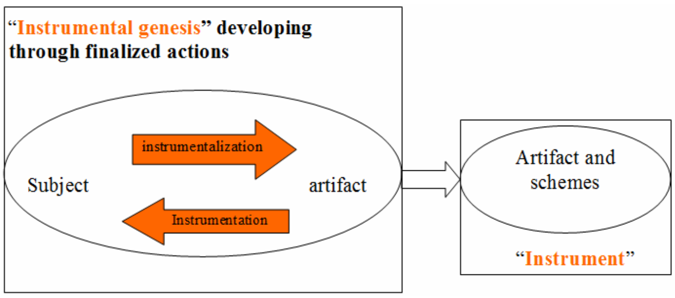

Figure 4.1: The schema of instrumental approach (personal e-mail correspondence with Pr. Luc Trouche on April 2, 2008) based on Trouche's (see [34]) instrumental approach ,quoted in [29])

Artigue (see [4]) supports that "an instrument is thus seen as a mixed entity, constituted on the one hand of an artefact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appro-priation of socially pre-existing schemes." Artigue (see [3]) reports on the 'genesis of reflection about instrumentation issues, and the dialectics between conceptual and technical work in mathematics'. Reflecting on dynamic diagrams constitutes the conscious representation of actions or mental processes and then considering their results or composition. The students act on dynamic diagramms (for instance LVAR) to construct their knowledge or to investigate the problem solution and they interplay with the dynamic diagramms to express their thoughts. In that case they use dynamic diagramms as tools/ artefacts with which they shape their thoughts. Noss and Hoyles (see [24]) argue that during instrumental genesis in a computing environment 'students' activity is shaped by the tools'/in our case the dynamic LVAR, 'while at the same time they shape' the dynamic LVAR 'to express their arguments'. During the construction/ action of a diagramm the student structures an internal invisible side of the representation which is a part of the process on the external representation /model.

LVAR give to the users-pupils the affordance to improve/ facilitate their understanding, and to transit to a higher van Hiele level by acting in a auxiliary/ complementary manner, assimilating /accommodating students prior knowledge, or as a confirmation of the pupil's thought processes /mental approach. As Kaput (see [17], [18]) writes "a representational framework for mathematical cognition and learning is consistent with constructivism". During the interaction with LVAR students interplay with the spatiographical features of the diagrams and their spatial characteristics and construct a deep understanding of their properties referring to the theoretical object from the "reflection shaped by the tools and the language operationalized by them" (see [25]). Meaning that through LVAR and the operationalization of reflective abstraction, formed or structured previously abstract items of mental operations can become the content in future acts of abstraction (see [38]). According to Hollebrands (ibid.) "students are usually asked to work on material drawings, the spatial graphical features of the signifier, but they are expected to reason about figures, the signified. ...To reason more formally about geometrical properties, rather than just about the spatial characteristics of diagrams, students need to have deep understandings of those properties. Deeper understandings may be indicative of a student who engages in reflective abstraction and possesses an object conception of a concept (see [15])."

5. What are the different modes of the LVAR? What is the relation between LVAR and the phases of the van Hiele model?

The next section presents the different LVAR modes. Screenshots of the sequential representations of two modeled problems in the software are presented, correlated with excerpts from dialogues recorded during the research process, as student participants constructed their solutions to the problem. Example 1, 2 are parts of the solutions of the problems 1, 2 representing the different LVAR modes.

The first problem is a revision of the problem created by George Gamow (see [11]) involving pirates and buried treasure. Gamow's problem hinges on a treasure map found in an old man's attic. Here is the revision provided by the researcher (see [29] and [30]): "In the Odyssey, Homer (c74-77) mentions that the pirates also raided Greek islands. The pirate in our story has buried his treasure on the Greek island of Thasos and noted its location on an old parchment: "You walk directly from the flag (point F) to the palm tree (point P), counting your paces as you walk. Then turn a quarter of a circle to the right and go to the same number of paces. When you reach the end, put a stick in the ground (point K). Return to the flag and walk directly to the oak tree (point O), again counting your paces and turning a quarter of a circle to the left and going the same number of paces. Put another stick in the ground (point L). The treasure is buried in the middle of the distance of the two sticks (point T)." After some years the flag was destroyed and the treasure could not be found through the location of the flag. Can you find the treasure now or is it impossible?"

The second problem is the following: “A power plant is to be built to serve the needs of the cities of A (Athens), B (Patras) and C (Thessaloniki).Where should the power plant be located in order to use the least amount of high-voltage cable that will feed electricity to the three cities?”(see[26])

The researcher carried out the didactic experiment having in mind the didactic approach of Dina Van Hiele-Geldof which: 1) “Promotes a building up of geometry by way of structure 2) First directs the thinking activity of the pupils to the analysis of structure prior to the formation of associations 3) Concurrent with that, provides an opportunity for the pupil to develop thinking focused on structuring” (see [10]).

5.1 Mode A-the inquiry/information mode

A part of the problem requires the use of an action button (animation, for example, or the trace command) so that the result can be seen (becomes perceptible) during the investigation. The original diagram is transformed / converted into a "diagram in motion" reinforcing the original image since the stimulus received from the visual representation leaves the properties of the figure unaltered despite the transformation it undergoes.

In this phase of the problem, the students familiarize themselves with the field under investigation using the isolated parts of the diagrams which lead them to discover a certain structure through their interaction with the diagram or during discussions. “Reflection upon the manipulation of material objects, by taking the relations between those shapes as an object of study, can lead to geometry” (Dina van Hiele in [10])

First problem’s example: When the students interact with the linking visual representations they can visualize the sequential steps of all the visual representations that appear during the animation of point F as the segment KL is being traced. Besides the students verify visually that the distances KT,TL remain equal as proint F is moved on its path PO and that T remains the midpoint of KL for every point F. Namely the depicted representations display spatial-graphical shapes and their relations.

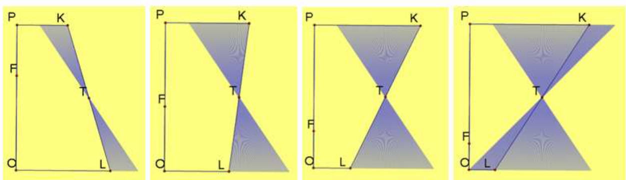

Figure 5.1.1: first problem’s example- Sequential phases of the figure while point F is animated and KL is traced

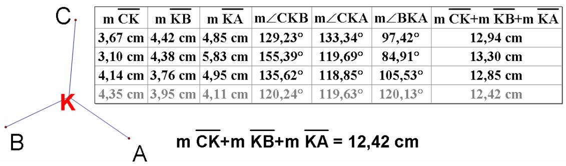

Figure 5.1.2: second problem’s example- Sequential phases of experimentations with the lengths of the segments and the angles

This process results in a connection between the ‘spatio-graphical’ and the ‘theoretical field’ as Laborde (see [20]) describes. The pupils react to this visual stimulus and respond instantaneously. Their responce is a result of the reaction which occurs to the visual stimulus. This is to say that the pupil mentally transforms the meaning of congruence / equality of the segments perceived visually in the diagram to the meaning of symmetry. That means that the pupil constructs an instrument out of his/her interaction with the tool which also includes an instrumented action scheme relating to the meaning of symmetry. The depicted representations lead the students to globally recognize a more sophisticated representation which reflects the shape of a square. At the same time this process results in the pupil visually connecting the meaning of the square with the equality of its sides and moreover the equality of its diagonals –i.e a relationship between the two meanings. Students working on figure 5.1.1 (see [29]) discuss:

218. Researcher : Which is the position of point T as we drag point F? *(RVR)*
*219. M_4: it is the **symmetry centre** of the shape*
220.R: Can you conjecture what kind of quadrilateral is shaped?
*221. All the pupils: it seems **a square*** *(RVR)*

Second problem's example: The students investigate the modifications of the calculations/sum of the segments to identify the different positions of point K. Changing the position of point K by dragging it is dynamically linked to the changes/ modifications in the resultant angles in the table and the upcoming change/ modification in the sum of the segments. This process afford /encourages students to observe that the minimal sum is observed when the angles are at 120^o.(Figure 5.1.2)
The students are usually led to draw rough conclusions regarding the position of the point under investigation; for instance, that it is the circumcenter of the triangle. The construction of the circumcenter and the measurements cognitive conflicts in the students. The addition of a new line in the table for new measurements every time point K is dragged can lead students to posit conclusions which converge on the angles between the segments being equal to 120 degrees. During this process we have a reversible (bi-directional) transformation of a) the geometrical into an algebraic model, and of b) the algebraic conclusions drawn from comparisons between on-screen dragging into the geometrical representation.
'Students are able to perceive structure in almost any material however unordered it may be and that this structure can be perceived in the same way by different students. This allows them to discover the intrinsic ordering in the material that is presented to them. For example the knowledge of shapes is developed through manipulation of material objects."(Dina van Hiele, see [10])

5.2. Mode B- the directed orientation mode

The sequential constructional phases of the problem are displayed as a global shape which is gradually added to when action buttons are pressed. The steps in the construction of the diagrammatic reconstruction which are displayed by pressing the action buttons are linked to suitable questions and their answers.
According to Olivero (see [27]) "The possibility of hiding and showing elements ...is a powerful tool of dynamic geometry software, because according to what is left visible the focus can shift to different elements. Hiding or showing elements of a configuration at stake changes the nature of the figure to explore because what is visible changes and therefore the potential elements of the focusing process change too. What students see on the screen influences the construction of conjectures and proofs and choosing what they want to see on the screen influences the proving process".

In concrete terms, the sequential constructional steps of the solution to the problem emerge step-by step. The process has the following advantages: the student can recall/redisplay the correct answer in his question or the teacher's question which appears when he clicks on the appropriate button; the process can be repeated as many times as the student wants, which saves time in a proving process (or there is not time consuming during the proving process).
The students discover an important part of the solution to the problem on the same page of the software by means of the gradual display of increasingly complex questions which are connected to the revealing/concealing of parts of the configuration of the problem and which cognitively connect parts of the solution. Concretely, during this process the students are led to cognitively connect additional, complementary, transformational reconstructions of the problem configuration and actions aimed at externalizing the student's thoughts by means of suitable chain questions which guide them towards the solution to the problem.

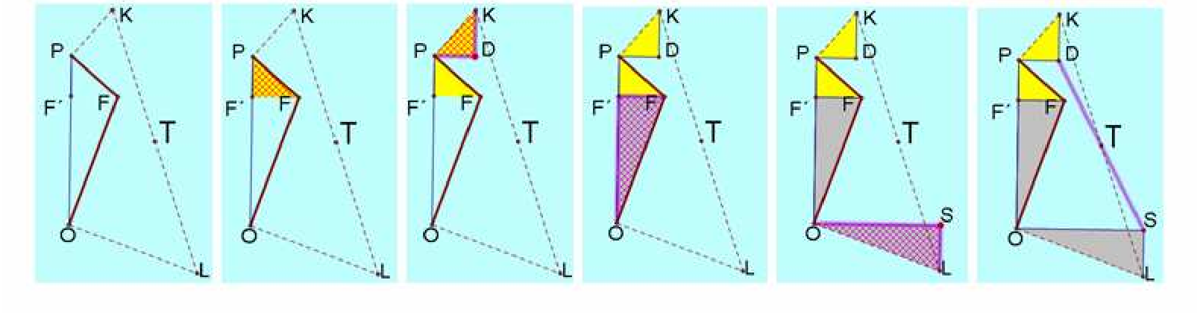

Figure 5.2.1: first problem's example -the sequential phases of the LVAR

First problem's example: The students progressively observe the rotations by 90 degreees of the similarly colored triangles and the construction of segment QS (Figure 5.2.1). The students are led to shape an instrumented action scheme relating to the rotation of the segments PF and FO. The dialogue that follows is indicative of the construction of a section of the proof by the pupil (see [29])

214.*M_2 : PQSO is a trapezium* ***because*** *PQ and SO are perpendiculars to PO as we concluded from the rotation for* 90^o. *...**we must prove** that T is the midpoint of any segment that can be. Will we join K and S? ....**If we prove** that KL, QS are the diagonals of a parallelogram **then** the diagonals are dichotomized.... .* ... (Figure 5.2.1)

216. M_1: ***if*** *we prove that these are parallel lines* ***then*** *the quadrilateral is a parallelogram* ***because*** *these are equal,* ***so*** *the diagonals will be intersected,* ***so*** *the diagonals will be dichotomized*

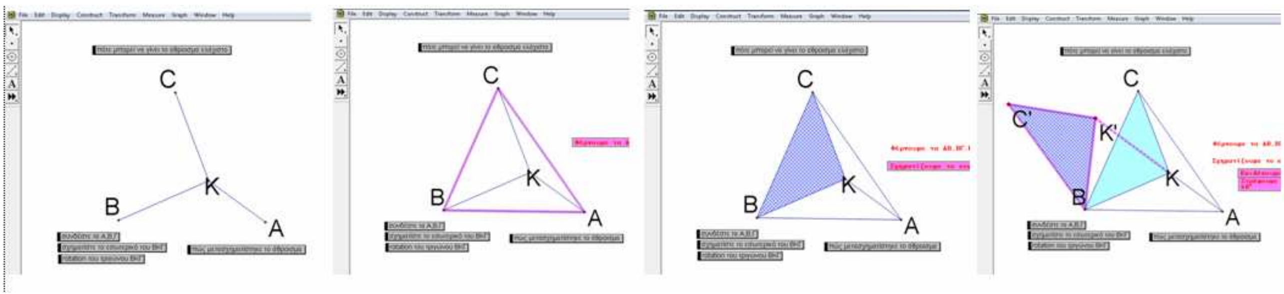

Figure 5.2.2: second problem's example -the sequential phases of the LVAR

Second problem's example: The action buttons provide the student with a sequence of progressive instructions: "Connect points A, B, C"; "Construct the interior of the triangle KBC"; "rotate the triangle KBG", or "How has the sum been transformed?".

By pressing the button, the student can see the following executing simultaneously: A constructional process on the diagram on the screen, and a calculational process in which the sum of the segments is transformated. Use and manipulation of the action buttons makes it possible to link the following forms of representations--figurative/iconic, symbolic and verbal —which appear almost simultaneously on screen. The questions on the buttons point out that the process does not substitute the teacher but facilitates him, since the teacher initially prompts the pupils to explore /experiment and intervenes with a question essential for the comprehension of the transformation. For example in the question "how can we display the sum of the segments on a line as collinear points?" the students could be guided by pressing the first button which will display the rotation of the triangle by 60^{o}. (Figure 5.2.2). “The empirical experiences are broadened though manipulations. These manipulations have been sufficiently mastered by the pupils and they are accompanied by a more concious perception in a geometric sence.”(Dina van Hiele , in [10])

During this process: a) a geometrical object is transformed into a new geometrical object emanating from the rotation. This process leads to the transformation of the sum of the three segments AK, KK΄ and K΄B΄ on a crooked line, and followed by b) a mental transformation. That is to say the process begins in the spatiographical and leads to the theoretical field. As Olivero writes : “A condition that can help the *focusing process* is the possibility of having a field of experience which allows students to manipulate, interact, and change the objects they deal with: such an empirical experience is likely to evoke theoretical elements.” (see [27]). In this particular phase, the students grow familiar with the basic links in the nexus/network of relations that take shape. In other words, with the structure of the subject that concerns the configurations and the required vocabulary or the properties and the relations. Throughout his/her teaching, the teacher organizes the activities for the special cases or actions that are expected from the students. The teacher can also simultaneously prepare the transformation in the iconic and symbolic representation, highlighting the different steps / strands in the solution in different colors so the reaction evoked from the stimulus on the screen is direct.

5.3 Mode C – the explicitation mode

Transformations in increasingly complex linked dynamic representations of the same phase of the problem modify the on-screen configurations simultaneously when, for example, a point is moved or has its orientation changed using the dragging or other tool. According to Dina van Hiele (see, [10]) “The material has to be representative in the sence that it allows the context to become clear. A figure undergoes a metamorfosis as a result of the manipulations followed by a phenomenological analysis and an expliciting of its properties: it becomes what we call a geometric symbol” (Dina van Hiele in Fuys et al. in [10]**)**

First and second problem’s examples: The successive phases of the constructional steps have been achieved using transformational processes like the use of the translation command (Figure 5.3.1, 5.3.2). The students can observe the processes that previously emerged progressively being modified simultaneously by dragging a point of the original configuration or of the translated images.

The process leads the student to construct an infinite class of transformational processes of the same geometrical object on screen, and consequently to a generalization of the conclusions they have been led in previous phases of the solution.

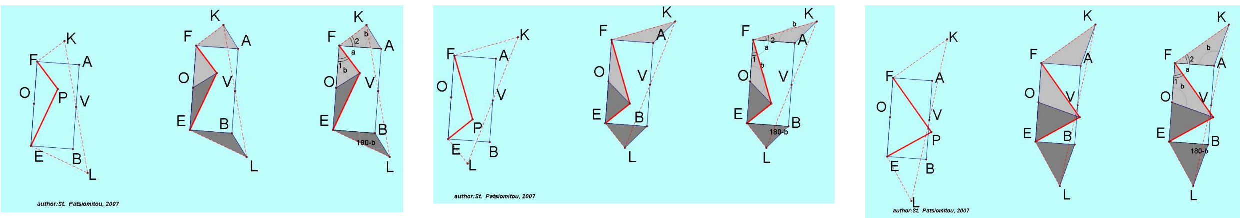

Figure 5.3.1: the transformed phases of the LVAR (problem 1)

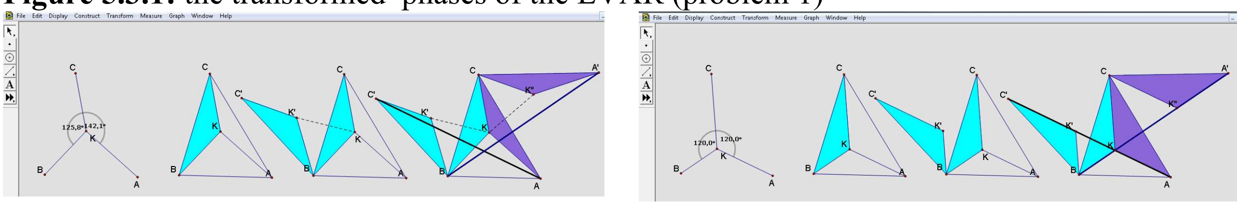

Figure 5.3.2 : the transformed phases of the LVAR (problem 2)

According to Dina van Hiele "The results of the manipulation of material objects are now expressed in words. The figures aquire geometric properties- so the goal of explicitation is to establish properties of figures. As a result the shape becomes less important and the figure become a conglomerate of properties" (see [10] **)**

5.4 Mode D –the free orientation mode

Every phase in the solution can be displayed side by side on the same page of the software by pressing the action button which presents the global configuration rather than complementary parts of the configuration.

The student can focus his observation in what extra information is presented in the next emerging iconic form of the representation. The emerging additional representations can be dragged independently; for example dragging the vertices of the triangle in configuration 3, leaves configurations 1, 2, 4, 5 and 6 unmodified. (figures 5.4.1, 5.4.2). "Showing construction lines, together with dragging the figure, will help the students to keep in mind the properties of the construction. Hiding some elements may be useful when wanting to focus on some particular configuration" (see [27]). The students are led to a proof that confirms their initial reasoning, conjectures and exploratory processes. We could call this the intermediary phase between the guided phase and free orientation.

Meaning that the students appropriate the processes and connect them conceptually. They are thus led to discover actions in the software in order to be led to the subsequent free orientation phase.

The explanation phase is the phase in which procedural is transformed into conceptual knowledge—meaning into proof; the phase in which process is transformed into meaning

First problem's example: For example, the first thing to appear in the shape, are the outlines of the figure that results from rotating segments PF, FO by 90 degrees. Next, the equal triangles are highlighted in the same color and then the equal segments or equal angles. The action button under each configuration helps the students gain an overall grasp of the modifications to the shapes in the new configuration. (see [29])

For instance, the student is led to produce the following discussion, after all six images have been revealed to her (figure 5.4.1) : *232. M_7: the segments MK and PF′ are equal,* ***because*** *the triangles MKP και FF'P are congruent as they* ***are right triangles****,(mental scheme) they have KP = PF , and angle < MKP* ***is equal to*** *<FPF′ angle – because <KPF′* ***angle is external*** *to the triangle MKP so it is constituted from an angle of 90^o and the angle <FPF′ so it is equal* ***with the opposite*** *angles <(MKP+ 90^o)* (Figure 5.4.1)

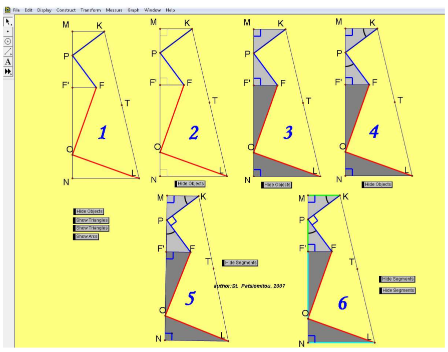
Figure 5.4.1: first problem's example

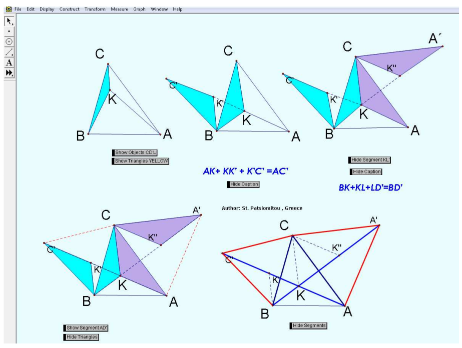

Figure 5.4.2:second problem's example

Second problem's example:

On the screenshot we can see the emerging representations in the global diagram in which the student can recall key steps in the solution of the problem (Figure 5.4.2). It is essential that the student can display every steps in the solution gathered together on the same screen, allowing them to see the progressive changes globally. A difficult problem is thus simplified through the use of pictures. The free orientation phase contains the translation of the proving process into condensed actions in the software.

‘The field of investigation is for the most part known, but the student must still be able to find his way ther rapidly.’(Dina van Hiele in [10]).The students can use their creativity to pose open goals with multiple steps and alternative solutions, thereby extending their knowledge to what they have seen before. One could call this the second phase of directed orientation in which the students learn to find their way through the network of relations assisted by their extant knowledge. For example, the proving process leads to a solution that uses Fermat’s problem, which requires the construction of the circumscribed circles of the equilateral triangles with a view to finding their intersection, which is the solution to the modelled problem. (Figure 5.5.2)

This means continuous transformations between the theoretical and spatiographical fields.

5.5 Mode E –the integration mode

Successive configurations on different pages that are connected conceptually/ cognitively and not necessarily constructionally, compose the solution to the problem (see [26]) in global terms as a series of steps. This process is linked to the strategies for solving the problem or foreseeing the different strands in the solution relating to individual thought processes or different goals. This process can help the students progress through the successive steps in the solution to completion.

The students have the possibility via this process to progress understanding the successive steps of the solution to be led to the integration. During this phase the student “must still acquire an overview of all the methods which are at his disposal. Thus he tries to condence into one whole the domain that his thought has explored. At this point the teacher can aid this work by furnishing global surveys.It is important that these surveys do not present anything new to the student; they must only be a summary of what the student already knows. ” Dina van Hiele (see [10] **)**

Meaning that the information with which they became familiar in the new network of evoked geometrical objects and their interrelationships is reviewed and summarized.

Second problem’s example (Figures 5.5.1, 5.5.2, 5.5.3): The students are guided to (their)/an interpretation of the process in the modelled problem. At this stage, if the students are to guided correctly, they must have examined every previous step successfully. For example, students can apply the custom-tool “construction of the circumscribed circle” to the sides of the equilateral triangles, so that it is in the right place for point K, which is the solution and the interpretation of the solution to the real problem.

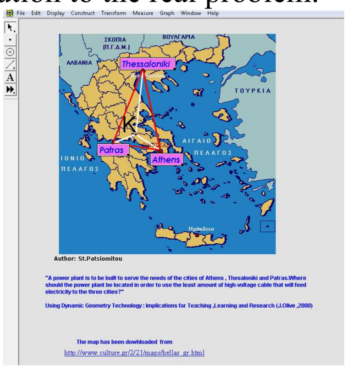

Figure 5.5.1

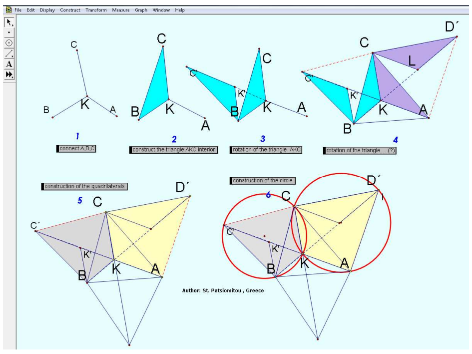

Figure 5.5.2

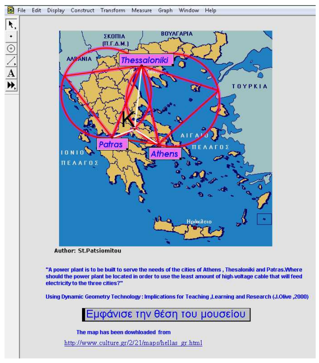

Figure 5.5.3

This process is a combination of advanced actions in the software and the proving process or strict justification; meaning that a software process has been transformed into a theoretical process by condensing the steps and using the custom tools facilitated by the Sketchpad software to prove that the point whose location they have to find is the point at which the circumscribed circles intersect.

6. Conclusion

According to Laborde (see [20]) it seems that "Dynamic Geometry environments break down the traditional separation between action (as manipulation associated to observation and description) and deduction (as intellectual activity detached from specific objects) and reinforce the moves between the spatial and the theoretical domains." When the instrumental genesis occurs, transformations of linking representations globally or on the objects in the LVAR (i.e artefacts or tools in the software) reflect on the assimilation or the accommodation of the situation by the subject. The student's development of geometrical thought takes place through the interaction with the LVAR in relation to the progressive adaptation of the schemes of use.

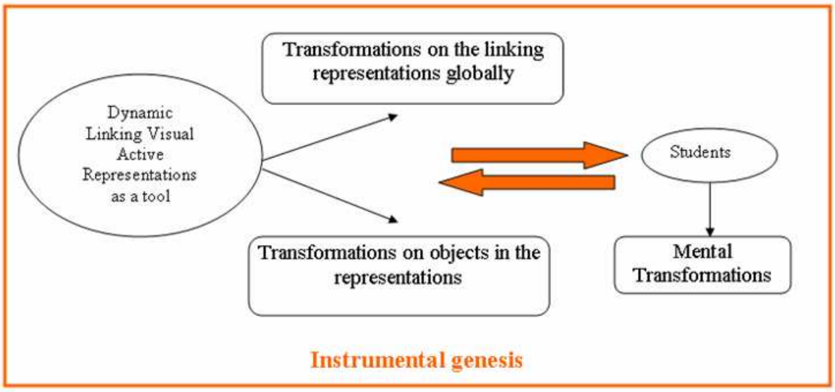

Figure 6.1

Therefore, it appears that the use of LVAR in Sketchpad dynamic geometry environment proving process can organize the problem solving situation using as tools the interaction techniques facilitated by the software, and the structuring and restructuring of the user's instrumental schemes it evokes as the activity unfolds. As the LVAR'S composition changes, there is a transformation of the user's verbal formulations due to rules subjacent to the user's organized actions. Consequently, the scheme of use associated with the constructed instrument changes, leading the students to pass from an empirical strategy to a theoretical geometrical one or to pupils mental transformations (Figure 6.1).

Mathematical properties can be described in terms of transformations which may be represented through several types of manipulative activities. In the case of modeling a problem in the DGS environment, this process can be achieved through interaction techniques in the software during the problem-solving process. Initially the students perform actions upon semi-predesigned LVAR. But eventually when the LVAR as objects become distinct images, students are able to perform mental transformations upon these images, in a cognitive operation which builds upon actions but goes beyond them. During the interaction with LVAR two different developments occur simultaneously. One is visuo-spatial, using processes on the screen to do things (i.e rotation) that are completed, between a preimage (the original figure before transformation) and an image (the corresponding figure after the transformation), and the other is conceptual: using concepts (i.e properties of the figures, interrelationships between figures, theorems etc.) thoughts becomes verbalized. In other words the interaction with LVAR becomes a versatile approach between visual and mental objects (see [34]). The process of proof is developed using verbal formulations and geometrical relationships which become conceptualized in the proving process. Students use verbal formulations to exchange their ideas meaning that they transform their mental objects into a language mapping, corresponding to motion transformations on the software pages. *Semperasmatically*: actions on LVAR (or interaction with LVAR) leading to proof, also leads to the development of geometrical thinking. Pupils can develop their level of thinking by proceeding through increasingly complex , sophisticated and integrated figures and visualizations to a more complex linked representation of problem, and thereby moving instantaneously between the successive Linking Visual Active Representations only by means of mental consideration, without returning to previous representations to reorganize his/her thoughts (see [29]).

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