Special Session at ATCM 2007

Connecting Dynamic Geometry Software with Computer Algebra System

(in alphabetical order)

Thomas F. Banchoff
Thomas_Banchoff@brown.edu
Department of Mathematics
Brown University
U.S.A.

Jen-chung Chuan
jcchuan@math.nthu.edu.tw
Department of Mathematics
National Tsing Hua University
Taiwan

Jean-Marie Laborde
Jean-Marie.laborde.cabri.com
Cabrilog
France

Phillip Todd
philt@saltire.com
Saltire Software
U.S.A.

Wei-Chi Yang
wyang@radford.edu
Department of Mathematics & Statistics
Radford University
U.S.A.

Abstract: This special session consists of five short presentations. Each author will demonstrate their examples to show why it is crucial to connect Dynamic Geometry (DG) with Computer Algebra System (CAS) in achieving the followings:
- Making Mathematics more accessible to more students.
- Motivate students to investigate more challenging problems.

Contribution from Thomas Banchoff:

For the calculus and geometry of curves and surfaces, interactive geometry software makes it possible for the instructor to illustrate phenomena in the classroom and for the student to investigate conjectures individually and in groups, and to share their experiences with the class.

Our visualization software makes it possible to enter parametric curves and surfaces as vector functions of one or two variables, and to explore the images in two- and three-dimensional linked windows as they transform under rotations and parameter changes, as well as algebraic manipulations of the vector representations.

We illustrate these features by studying the behavior of normal vectors of curves in the plane and surfaces in space and their associated curvature functions, to generate and refine conjectures and to relate algebraic proofs with geometric demonstrations. Specific examples include parallel curves and surfaces and the behavior of evolutes.

The items I would choose to emphasize the way that an instructor or student can modify an existing demonstration to illustrate specific vector functions, and to define auxiliary functions in a way that is natural for mathematicians. I want to stress the fact that we can deal with several curves and their tangent-normal-binormal apparatus at the same time, as well as curves associated with surfaces. The idea of linked windows, so that selecting one image and rotating it produces the same rotation in the other, is a crucial feature of our operation, and one that distinguishes it to some extent from other 3-dimensional systems, at least the last time I checked

Some of the things I will show include defining a curve $X(t)$ with easy to use notation, immediately being able to show its velocity and acceleration curves $X'(t)$ and $X''(t)$ computed by a combination of symbolic and numerical techniques, defining normalized vectors such as the unit tangent $T(t)$ by typing $T(t) = X'(t)/|X'(t)|$ or simply unit(X') and defining the unit normal $U(t)$ obtained by rotating $T(t)$ a quarter-turn in the counterclockwise direction by typing $U(t) = (-T\_2(t), T\_1(t)$.

We can then observe animate the points $X(t0)$ moving along the curve and show a vector $X(t0) + U(t0)$ moving along with it. We can show $U(t0)$ in separate window so that motion of $X(t0)$ along the curve is related to motion of $U(t0)$ along the unit circle. We can then define the signed curvature by typing $kg(t) = dot(T'(t), U(t))/|X'(t)|$ and we can display the graph of $kg(t)$ versus t in a separate window, with $(t0, kg(t0)$ moving along the graph as t0 goes through the domain of $X(t)$. We can then define a parameter r and investigate the parallel curve at distance r by typing $X(t) + r*U(t)$ as we change the parameter r. We can observe the set of singularities of parallel curves and carry out the algebra to find the locus of singularities, presented as the evolute curve $E(t) = X(t) + (1/kg(t))U(t)$.

We can then do the same thing for surfaces in space by typing $X(u,v)$ for the surface and $X\_u(u,v)$ and $X\_v(u,v)$ to get the partial derivatives. We get the unit normal vector by typing $N(u,v) =$ unit(cross(X_u(u,v),X_v(u,v))) and we can then explore the geometry of the Gauss mapping by looking at $X(u0,v0) + r*N(u0,v0)$ on $X(u,v) + r*N(u,v)$ in one window and $N(u0,v0)$ on $N(u,v)$ in another. The two surfaces move in a coordinated fashion as we rotate either in its window. As an analogue of extrinsic curvature in the case of curves, we define a curvature function $K(u,v)$ as the dot product with $N(u,v)$ of cross(N_u(u,v),N_v(u,v)) divided by $|cross(X\_u(u,v),X\_v(u,v))|$.

Contribution from Jen-chung Chuan:
The name ATCM contains both the words "technology" and "mathematics". The role of the mathematicians in this technological world is to check the dishonesty of the engineers. As it will be shown at the Opening Ceremony at ATCM 2007 (http://atcm.mathandtech.org) , a serious bug appears in Excel 2007, which didn't exist in the previous versions; try to enter

$$=850*77.1$$

under Excel 2007. Take a look to see what you get when enter

$$\text{int}(1/(1+x+x\^4)\^2,x=0..\text{infinity});$$

under Maple 11. Furthermore, Wei-Chi Yang also found the error of

$$\frac{d}{dx}\left(-3 \cdot \sqrt[3]{(2x-1)^2}\right)$$

under Maple 11; we suspect there are more errors from many software packages. We now see a warning sign of the software industry: Are we entering the Era of the Downfall of the Computer Revolution? If the public continues to tolerate the situation, then more and more similar stupidities are bound to multiply in future.

During the first week of medical school, students learn to fill out the death certificate. Visitors to the department of civil engineering of National Taiwan University are shown a two-inch wide gap due to construction irregularities. So, when we go back to classrooms, the first thing is to show students this bug in Excel 2007. All three samples will teach the students to be humble. Be humble, doctor cannot treat every disease. Be humble, engineer cannot build a perfect building. Be humble, applying computer to mathematics has its limitations.

Contribution from Jean-Marie Laborde
There are various examples worldwide, relate the nature of the mathematical thinking developed by users in solving problems (perceived as open), when using powerful technologies like dynamic geometry and/or CAS. Many people report about the crucial role technology can play to help them in stating the right conjecture and or in helping scaffolding towards a proof. I do not see conflicts between the Computer Algebra and Dynamic Geometry Systems because I see them as complementary.
Consider a simple example: To express the radius of the sphere through the midpoints of the edges of a regular dodecahedron as a function of its edge length *a*. It can be show that

$$r = (3+\text{V5})a/4$$

To do so one way might be to try to compute it using a regular CAS system. However, it is not so easy, because we first have to create some symbolic quantities representing the dodecahedron.
Suppose we are using now a DGS, say Cabri 3D. It is easy (in a few clicks) to create the dodecahedron. Then we can probably visualize how the Pythagorean Theorem could be applied to compute the desired radius and create a chain of intermediate computations leading to the expression (3+V5)*a*/4. Here the paper-pencil parallel computation involves some intermediate

formulas and it could be a good idea to check them at each step in comparing actual lengths displayed by the DGS and the results of the evaluation of the successive formulas for some different values of the edge length (dynamic features of DGS). This provides us with a mean to be sure about the final formula (something very important).

In addition, if we arrange for the size of the dodecahedron edge to be 1 in Cabri3D, we can turn the attribute of the display of the desired quantity, to *symbolic* and Cabri3D will then, explicitly, display the value of $(3 + \sqrt{5})/4$, see the figure below:

$$OM = (3+\sqrt{5})/4 \text{ cm}$$
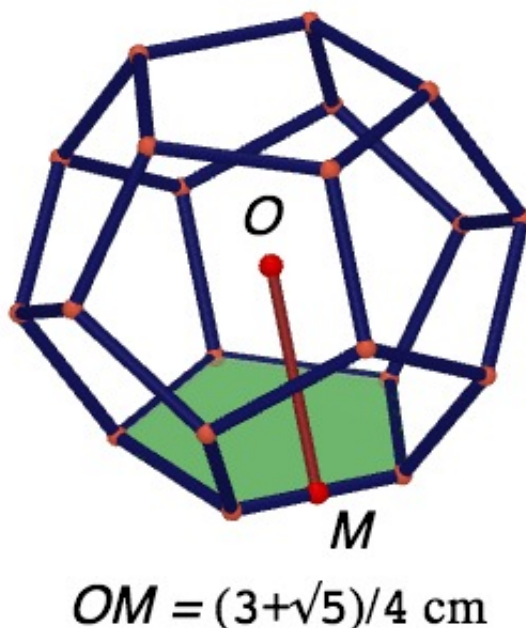
In my short presentation I will discuss some consequences of above example and story where CAS and DGS can be viewed as a mixed compound.

Contribution from Phillip Todd

I'd like to show how to solve a number of classical problems using a combination of Symbolic Geometry (www.geometryexpressions.com) and CAS. I will finish by applying the same process to a problem which was posed to me at ATCM 2006 [1]. I'll focus on the fact that this combination of tools allows a radically different approach to problem solving, where the emphasis is on model building and solution strategy and not on the mechanics of solution, nor on geometric insight. Once the problem is solved, geometric insight may be reinserted to taste. My problems will be taken from Dorrie's, "100 Great Problems of Elementary Mathematics" [2], and will include:

- Fergano's altitude base problem
- Regiomontanus' maximum problem
- Euler's tetrahedron problem
- Curvature of conic sections
- An inscribable circumscribable pentagon

Illustrative of the approach is the following solution of Regiomantus' Maximum Problem.

At what point on the earth's surface does a perpendicularly suspended rod appear longest?

A typical symbolic geometry approach to an optimization problem is to create a model, derive an expression for the objective quantity, differentiate and solve. In the figure, the earth has radius r, and a rod of length b is suspended in space distance a above the surface of the earth. The angle subtended by the rod at point E on the surface of the earth angle θ from the direction of the rod is calculated by Geometry Expressions:
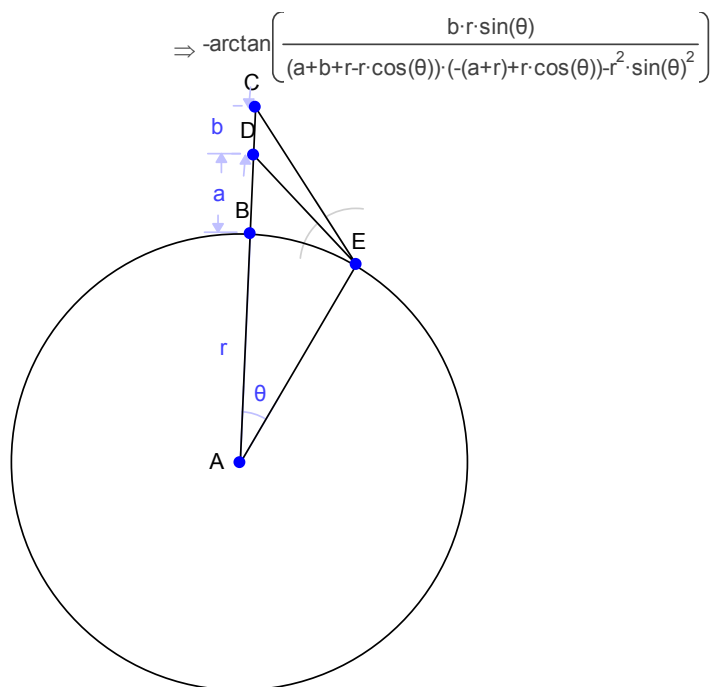
$$\Rightarrow -\arctan\left(\frac{b \cdot r \cdot \sin(\theta)}{(a+b+r-r \cdot \cos(\theta)) \cdot (-(a+r)+r \cdot \cos(\theta)) - r^2 \cdot \sin(\theta)^2}\right)$$
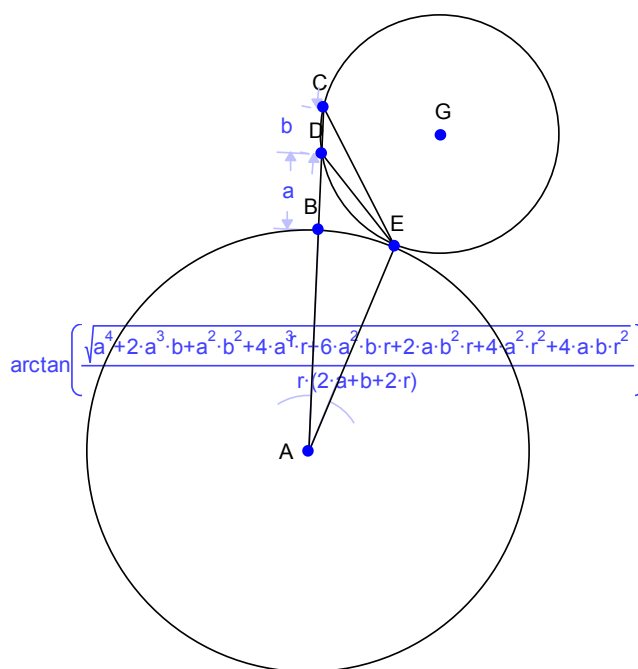

Copying into Maple, we can differentiate and solve:

$$-\arctan\left(\frac{\sin(\theta)\, r\, b}{(-a - r + \cos(\theta)\, r)\,(a + b + r - \cos(\theta)\, r) - \sin(\theta)^2\, r^2}\right)$$

> **solve(diff(%,theta)=0,theta);**

$$\arctan\left(\frac{\sqrt{4\,a^2\,r^2 + 4\,b\,a\,r^2 + 2\,b^2\,a\,r + 6\,a^2\,r\,b + a^4 + 2\,b\,a^3 + 4\,a^3\,r + b^2\,a^2}}{a^2 + b\,a + 2\,a\,r + r\,b + 2\,r^2},\right.$$

$$\left.\frac{r\,(2\,a + b + 2\,r)}{a^2 + b\,a + 2\,a\,r + r\,b + 2\,r^2}\right), \arctan\left(\vphantom{\frac{1}{1}}\right.$$

$$-\frac{\sqrt{4\,a^2\,r^2 + 4\,b\,a\,r^2 + 2\,b^2\,a\,r + 6\,a^2\,r\,b + a^4 + 2\,b\,a^3 + 4\,a^3\,r + b^2\,a^2}}{a^2 + b\,a + 2\,a\,r + r\,b + 2\,r^2},$$

$$\left.\frac{r\,(2\,a + b + 2\,r)}{a^2 + b\,a + 2\,a\,r + r\,b + 2\,r^2}\right)$$

Copying a solution back into Geometry Expressions we can visualize the solution.
Geometric insight can be had by drawing the circumcircle of CDE, and observing that it is tangent to AE.

$$\arctan\left[\frac{\sqrt{a^4+2\cdot a^3\cdot b+a^2\cdot b^2+4\cdot a^3\cdot r+6\cdot a^2\cdot b\cdot r+2\cdot a\cdot b^2\cdot r+4\cdot a^2\cdot r^2+4\cdot a\cdot b\cdot r^2}}{r\cdot(2\cdot a+b+2\cdot r)}\right]$$

References

[1] P. Todd, "An Inscribable Pentagon", American Mathematical Monthly 114:639 August/September 2007

[2] H. Dorrie, "100 Great Problems of Elementary Mathematics", Dover, New York, 1965

Contribution from Wei-Chi Yang

One of my favorite observations is DG allows us to be coordinate free and we may use its numerical capability to approximate a solution. In the mean time, we need a CAS to verify if our conjecture is true.

My conjecture is that DG makes mathematics more accessible. There are many real-life problems and abstract concepts can be explored by students before they enter colleges. On the other hand, we need a CAS to prove results analytically, which makes mathematics challenging, and this makes some motivation for students to study more mathematics in university level or beyond.

We will see from the following example that it is to create the reflection of a curve along a line, which is to generalize the idea of finding an inverse of a function. However, it is a challenging exercise to first year university student to find the solution analytically.

Example 1. We are given a line L of the form y=m*x+b and a curve S (see Figure 1 below), find the reflection of the curve S respective to the line L.
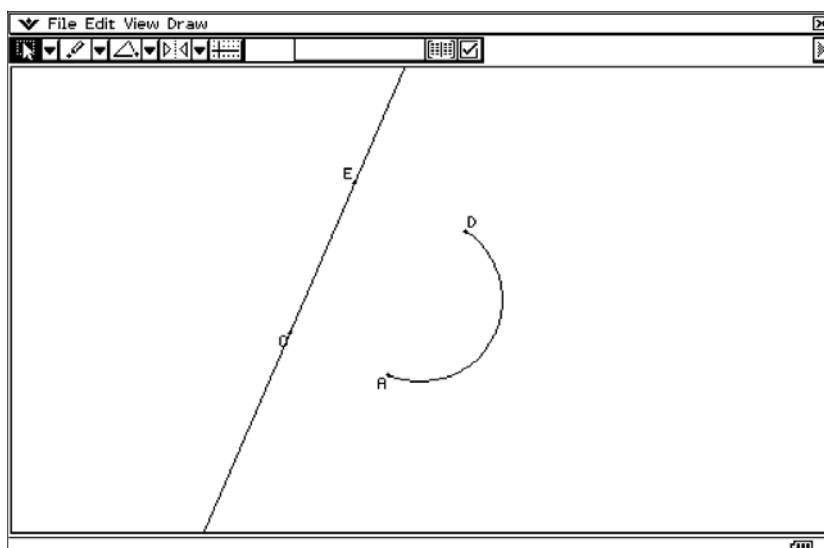
Figure 1.

The second example below shows how DG allows us to explore the concept of Riemann integral by choosing a proper coordinate system, which will simplify greatly when one wants to approximate the area bounded by curves. Again, it is a challenging exercise to prove this analytically.

Example 2. Refer to Figure 1 above, we would like to find the area bounded by the curve S, line L, line AC and line DE; here we assume AC is parallel to to DE, and AC is perpendicular to CE.

The third example below shows how we can link the idea of Lagrange Multiplers, its geometric interpretation and linear indepency of Linear Algebra together with the help of Dynamic Geometry Software packages.

Example 3. We are given three curves in the plane, see C1, C2 and C3 below in Figure 2. We need to find points A, B, and C on C1, C2 and C3 respectively so that the distance AB+AC achieves its minimum. In 3D case, we are given four surfaces in the space, represented by the orange (S1), yellow (S2), blue (S3) and purple (S4) spheres respectively in Figure 3. We want to find points A,B,C and D on S1, S2, S3 and S4 respectively so that the distance AB+AC+AD achieves its minimum.
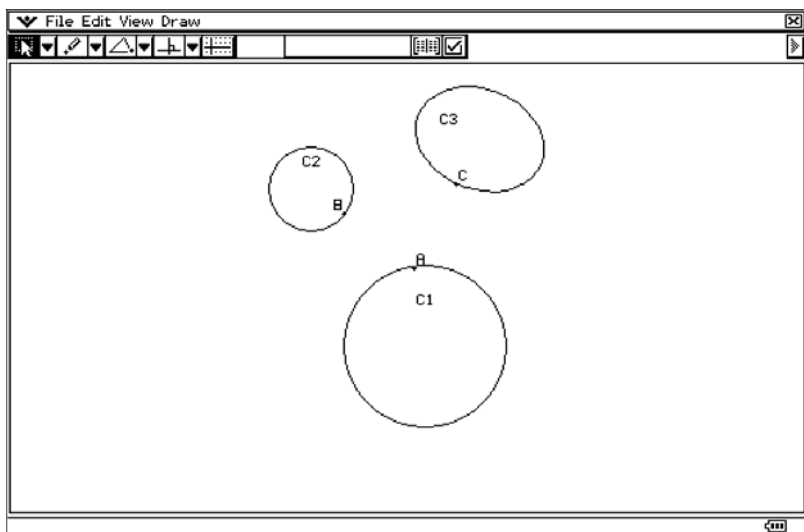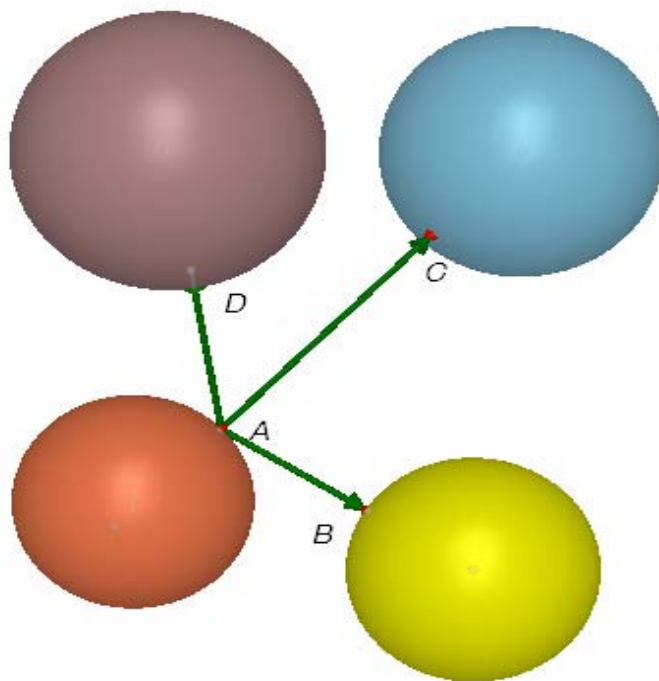
Figure 2.

Figure 3.