Exploring the Place of Hand-Held Technology in Secondary Mathematics Education

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While sophisticated technology for mathematics is available and used in many educational settings, there are still many secondary school mathematics classrooms in many countries in which student access to such facilities is either very limited or non-existent, either at home or at school. This paper focuses on secondary mathematics education for students and teachers who are without reliable and regular access to computers or to the Internet. The place of hand-held technologies, including scientific calculators, graphics calculators and integrated devices will be considered. The computational support such devices offer to students is described and evaluated. Opportunities for new approaches to teaching and learning mathematics are described. The significance of hand-held technologies for aspects of the mathematics curriculum, its evolution and its assessment will be outlined and some issues associated with effective integration of technology into the secondary school curriculum are identified.

1. Introduction

In recent years, mathematical use of computers has increased enormously in some settings, while in others it has not much changed at all. So there are still many secondary school mathematics classrooms in many countries (including affluent and industrialized countries) in which secondary school student access to technology for mathematics is very limited or non-existent, both at home and at school. This paper focuses on secondary mathematics education for those students and teachers who are without reliable and regular access to computers or to the Internet.

It is argued that there are good reasons for using hand-held technologies such as calculators to meet the needs of students, mostly deriving from the accessibility and affordability of the technology to a wide group of students. In addition, and importantly, hand-held technologies have been developed with the particular needs of secondary school mathematics education in mind, in contrast to more sophisticated technologies, which have been developed for quite different purposes and audiences.

While those less experienced with using technology in schools frequently think the main purpose is concerned with undertaking arithmetical calculations, in fact much more important issues of teaching and learning are at stake. Technology by itself is not enough: the capability of mathematics teachers and the nature of the school mathematics curriculum both need to be taken into account if secondary school mathematics is to be improved through the effective use of technology. When school mathematics curricula are dominated by external examination requirements, which is the case in many countries, hand-held technologies also take on a new significance.

The arguments in the paper draw on earlier ATCM and other papers presented by the author in the region, apply to a range of settings, and draw in part on experiences in developed countries, such as Australia and the United States. In developing countries, in which resources for education are more modest both at school and at home, the arguments for hand-held technologies are even more compelling, as they may represent the only realistic means to make progress connecting the mathematics curriculum to a modern world, already laden with technology.

2. Technology for education

Technology of many kinds is now widely available to most people throughout the industrialized world and in many parts of the developing world, especially in commerce and industry. A wander around Taipei makes this clear. It has now become a familiar part of the everyday world of citizens, parents and teachers. In addition, many technologies of potential interest to secondary school mathematics are manufactured in East Asia. Despite the widespread presence of technology, it seems that technology is not yet widely used in secondary mathematics teaching and learning in East Asian countries, such as Taiwan, China, Japan and Korea.

When considering ‘technology’ for education, it seems that many people interpret the term to refer to computer software and hardware of various kinds, and recently also to include the Internet. Although the ATCM has included aspects of other technologies, including hand-held technologies, over its entire history, the emphasis has been on computers, especially with the needs, interest and expertise of university teachers and researchers in mind.

It is much rarer for discourses regarding technology to refer to hand-held technologies, such as calculators and similar devices, although these are arguably of more importance to some parts of the school curriculum than computers. (Indeed, they are also arguably described as computers themselves, but for the present purpose, a distinction will be drawn.) It is common practice in schools and elsewhere for IT departments, policies and budgets to make no reference to calculators and similar hand-held technology devices in education, but to assume instead that the only technology of interest involves computers.

Indeed, some of the enthusiastic promotion and discussion of technology in mathematics education by both official sources and by commercial companies seems to take place under the assumption of an ideal education world. In the extreme, such an ideal world would be characterized by: (i) all students have unlimited access to modern high-speed computers; (ii) all software is free, or budgets for software are essentially unlimited; (iii) students and teachers have unlimited access to high-speed Internet lines; (iv) facilities in students’ homes match those in their schools; (v) teachers are well-educated enthusiasts in mathematics and pedagogy, with unlimited free time; (vi) curriculum constraints, including externally imposed and administered examinations, do not exist.

Although such assumptions are mostly unrealistic, they do in fact provide a useful starting point to think about and study technology in mathematics education. Proceeding on the basis of such assumptions, teams of professionals can and should develop good uses of technology, free of the shackles of the present reality. Such professionals include mathematicians, computer scientists, software developers, mathematics teachers, education researchers and others.

Few school contexts today match these idealistic assumptions, however. The virtual world that has no constraints is not the same as the present world inhabited by most students in most classrooms in most schools, in most countries (including the more affluent countries). The present paper is concerned with the real educational world in which many students, teachers and curriculum developers find themselves, in these early years of the twenty-first century. In the real world inhabited by most people today, hand-held technologies continue to be of more significance than computers, and hence are the focus of this paper.

3. A hierarchy of hand-held technologies

In this section, a four-level hierarchy of sophistication of hand-held technologies is described, in increasing order of sophistication (and thus also of price).

Arithmetic calculators

First appearing more than thirty years ago, arithmetic calculators are in common use in commercial contexts everywhere. These include shops and street markets throughout Asia, where the main function is sometimes to communicate prices, especially to tourists and others who do not speak the local language well. Basic calculators essentially provide a means of completing everyday numerical calculations, using decimals, and are very inexpensive. They are generally restricted in capabilities to the four operations of addition, subtraction, multiplication and division; many models also deal (sometimes strangely) with percentages as well. More sophisticated versions have been developed for educational use. One embellishment is to use mathematically conventional priority order for arithmetic calculations, so that 3 + 4 x 5 gives the correct result of 23 instead of 35. Another is to include operations with fractions as well as decimals.

Arithmetic calculators have been available to elementary (primary) schools for many years now, although the extent to which they have been adopted has varied between teachers and between countries. Despite the concerns of some teachers and parents, extensive research has established that these are educationally useful, and not harmful [1], [2], and few researchers are interested any longer in looking for negative effects associated with their use. However, they provide insufficient capabilities for secondary school students, whose mathematical needs extend considerably beyond mere computation.

Scientific calculators

Scientific calculators offer students slightly more capabilities than numerical calculations. Most scientific calculators provide the same facilities as an arithmetic calculator, as well as some more sophisticated ones, such as powers and roots. Table functions are also provided: values of functions that previously had to be obtained from mathematical tables are available directly from the keyboard. These include logarithmic, exponential, trigonometric and inverse trigonometric functions. Statistical calculations are available, so that means and standard deviations are calculated for data entered, and for many calculators, bivariate statistical calculations (such as correlation coefficients and linear regression coefficients) are also included. Recently, sophisticated versions have included higher level calculations of interest to secondary schools, such as those involving complex numbers, probability distributions and combinatorics. In essence, scientific calculators provide students with the capacity to undertake numerical calculations relevant to the mathematics of the secondary school.

Scientific calculators have been routinely used by secondary school students in most western countries for almost thirty years now. They are generally regarded as inexpensive items of equipment, essential for computation in mathematics and science, and are usually permitted for high-stakes examination use. They reduce the need for extensive by-hand calculation and consulting of tables of values of functions, characteristic of secondary school calculations of the previous generation.

Graphics calculators

Graphics calculators are distinguishable by their relatively large graphics screen, which accommodates several lines of display or a visual image of some kind. As well as including the capabilities of scientific calculators, graphics calculators include their own software for a range of mathematical purposes, including the representation of functions in tables and graphs, statistical displays and two-dimensional drawings. The range of mathematical capabilities varies between models, but these days can include numerical calculus, complex numbers, matrices, spreadsheets, probability simulation, sequences and series, numerical equation solving, statistical analysis and hypothesis testing, financial analysis and geometry. Some more advanced (and thus more expensive) graphics calculators also include low-level versions of Computer Algebra Systems (CAS). An important difference between scientific and graphics calculators is the possibility of students using the latter for mathematical explorations, rather than just calculations, either spontaneously or under the direction of the teacher.

In most industrialized countries, graphics calculators have been well-received in schools over the past twenty years, and are now routinely used by many students in the senior secondary school years as well as the early years of post-secondary study. As an illustration of the reception of this technology by teachers, the Australian Association of Mathematics Teachers' graphics calculator communiqué [5] described several ways in which this technology was being used in many Australian schools to good effect. In many countries, graphics calculators are permitted for use in formal external examinations, including those for selective entrance to universities. Empirical research results (eg, recently summarized in [6], [7]) have generally supported the use of graphics calculators for student learning, especially conceptual learning, and have generally suggested that students do not lose important procedural skills at the same time.

Integrated devices

In recent years, powerful new devices have been manufactured to create a new category in the hierarchy of hand-held devices for school mathematics. Good examples (among others) include the *ClassPad 300*, manufactured by Casio, and the CAS version of *TI-Nspire*, manufactured by Texas Instruments. In some respects, these devices are similar to graphics calculators, with inbuilt CAS, and include the capabilities of graphics calculators within their software suite. They are distinguished from graphics calculators in at least three important ways, however. In the first place, they contain more significant mathematical software, dealing with a wider range of mathematical concepts.(In the case of the *ClassPad 300* for example, these include a powerful computer algebra system with exact solution of differential equations, three-dimensional graphing and vectors.) In the second place, they provide significant interactivity between the various software applications. Thirdly, they have substantial storage capacities and similar operating systems in some respects to computers, so that they can almost be regarded as small computers, dedicated to teaching and learning mathematics. In these respects, such devices enable both more sophisticated mathematical ideas to be handled (in addition to less sophisticated ideas) and offer extensive opportunities for student manipulation and exploration, with teacher guidance in various forms.

These are relatively recent devices, and are included here in part to make it clear that the hierarchy does not end with graphics calculators. Although these devices have been available for only a short time, they have already attracted considerable attention, and some are permitted for use in formal examinations in some locations (such as Melbourne, Australia).

4. Educational advantages of hand-held technologies

Although some computer technologies are very powerful, there are some very good reasons for using hand-held technologies for secondary mathematics education. Five important advantages include the following:

1. They are easily portable, and can be comfortably carried in a school bag along with other materials students need. A consequence of this portability is that they can be used both at home and at school, and can be easily taken from one school classroom to another.
2. They are less expensive than computers, especially when all the software needs are taken into account (as calculators contain their own software). This cost issue has important implications for accessibility, regardless of whether costs are met by individual parents or by schools.
3. They are potentially more accessible to more students than are other forms of technology, as a consequence of the first two advantages: curriculum developers can design curricula on the assumption that students can access technology, only if it is accessible on a wide scale.
4. They can be used in formal examinations, which are of considerable importance in many educational settings. This advantage is mostly a consequence of the preceding reasons, since it is realistic to design curricula and associated examinations only for technologies that are potentially available to all relevant students; to do otherwise is likely to be regarded as unfair.
5. Most of them have been designed, and continue to be modified, for the express purpose of school mathematics education. Unlike other technologies, designed for other purposes, today's calculators are developed solely for the purposes of education, and so can be expected to be sensitive to the needs and interests of those involved, such as students and teachers.

A possibly surprising consequence of this last advantage is that, unlike other more sophisticated forms of technology, hand-held technologies are less likely to be used by mathematics and science professionals in universities than by secondary students and their teachers. At least in the developed world, the present generation of professionals in the mathematical sciences are comfortable users of computers and computer software, but have often had little experience with the comparatively recent technologies of interest to this paper.

5. A computational role

It is important to recognize that there are different roles for technology in secondary mathematics education. For example, the *Technology Principle* of the National Council of Teachers of Mathematics, widely quoted, asserts that "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." [3]. In elaborating this principle, the NCTM in the USA observed:

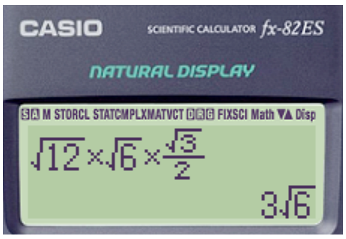
"Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect those changes. Students can learn more mathematics more deeply with the appropriate and responsible use of technology. They can make and test conjectures. They can work at higher levels of generalization or abstraction. In the mathematics classrooms envisioned in *Principles and Standards*, every student has access to technology to facilitate his or her mathematics learning." [3]

Some roles for technology concern computation, the provision of different educational experiences and influence on the school mathematics curriculum. These three roles are elaborated in an earlier paper [10]. In this section, we consider the important computational role played by hand-held technologies in secondary mathematics education

In brief, hand-held technologies can now meet all the computational needs of secondary education, providing a means to obtain reliable answers to numerical questions. This role is significant, as it potentially allows more time to be devoted to developing mathematical concepts, where previously a lot of time was required just to do computations. The centrality of calculation in mathematics was emphasized by Wong's observation that mathematics is a "subject of calculables." [8]

For students in elementary (primary) school, an arithmetic calculator allows everyday calculations with any measurements that are meaningful to them to be carried out, an important consideration if realistic applications of mathematics are to be included in the curriculum. The scientific calculator extends this capacity to large and small numbers, including those expressed in scientific notation (one of the many reasons that an arithmetic calculator is inadequate for secondary school use.) This is an important consideration for any mathematical modelling undertaken by students, whether in a mathematics class, a science class, or elsewhere. When confronted with calculation needs that could not be handled mentally, or for which reasonable approximations were insufficient, previous generations of students have been reliant on less efficient means of calculation, such as by-hand methods, or the use of logarithms and tables. A scientific calculator provides values for functions that were previously published in tables (such as trigonometric and logarithmic functions, as well as squares and square roots), and thus offers the opportunity to avoid long, tedious and error-prone calculations. Historically speaking, in many mathematics curricula, calculations have frequently became procedural ends in themselves, distracting from the important mathematical features of the work, and rarely offering much insight to students. This problem has often been exaggerated by the use of examinations emphasizing efficient use of procedural computational techniques.

As well as handling arithmetic calculations, scientific calculators also provide a means for efficiently dealing with easy, but lengthy computations, such as those associated with combinatorics (such as determining ${}^{52}C_5$, the number of poker hands possible from a standard deck of cards) or with elementary statistics (such as finding the mean and variance of a sample of 20 measurements). It is interesting that such calculations were not routinely available on early scientific calculators, but were added to later models, designed for education, almost certainly to accommodate the computational needs of secondary school students rather than 'scientists', for whom presumably the original scientific calculators were designed. It is noteworthy that recent scientific calculators also provide some exact answers as well as numerical approximations, consistent with the continuing support of computational needs. Figure 1 shows two examples from a recent scientific calculator model.



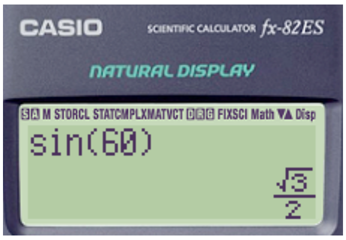


Figure 1: Exact computation on an entry-level scientific calculator

As well as providing reliable answers to computational questions, such capabilities might support student thinking and even curiosity about the mathematical ideas involved.

The scientific calculator now has a history of about thirty years in secondary schools, and has continued to undergo developments, partly fuelled by competition between rival manufacturers, and partly as a consequence of advice from mathematics teachers themselves. Over that time, they have become much easier for students to use, with more informative screens and a broader range of capabilities, which have together improved their capacity to fulfill the computational role for students.

Modern graphics calculators usually have at least the same suite of capabilities as scientific calculators (so that it is not necessary for students to have access to both kinds of devices.) A difference between the two is their relative ease and their range of computations. These vary a little between models, but Figure 2 shows some examples of evaluating expressions, calculating a logarithm, calculating with complex numbers and inverting a matrix.

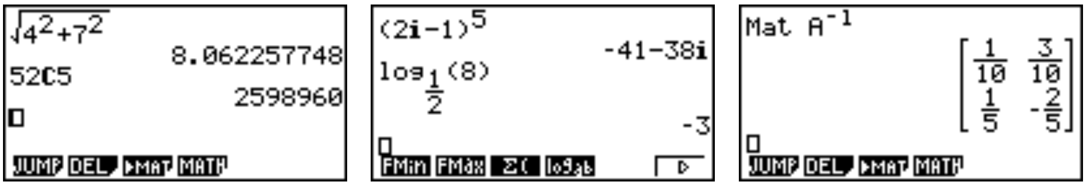


Figure 2: Some computations with a Casio fx-9860G graphics calculator

The examples illustrate routine numerical calculations that, if required to be done by hand, occupy a lot of student time. Although evaluating the logarithm involves conceptual thinking about the nature of logarithms, neither the inversion of a matrix nor the expansion of a complex power involve anything other than routine procedures, which some would argue are better left to a machine.

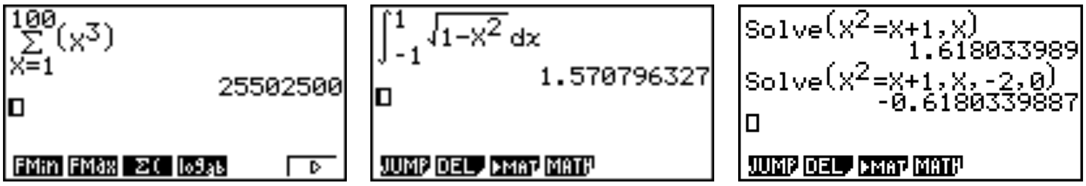


Figure 3: Further computations with a Casio fx-9860G graphics calculator

The further examples of computations shown in Figure 3 illustrate how significant numerical work can be completed on a graphics calculator, raising issues regarding the appropriate balance of mathematical concepts and skills in the curriculum.

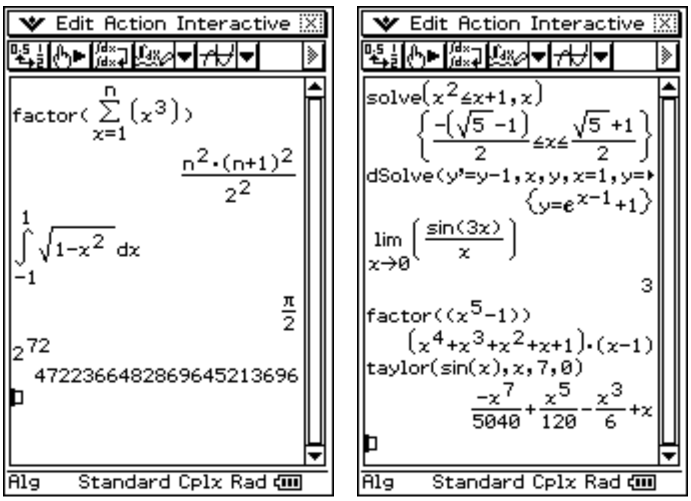


Figure 4: Some computations on a ClassPad 300

Finally, Figure 4 shows some examples of computations available on Casio's *ClassPad 300*, a good example of an integrated hand-held device in current use. The eight examples chosen for this purpose illustrate the powerful capabilities for exact computation and symbolic computation routinely available on this device. In each case, a computation has been entered on a single line, with the result displayed on the following line. Several of the examples chosen make direct use of the computer algebra system built into the *ClassPad 300* software; these show integration by parts, the solution of a quadratic inequality, factoring of simple and complicated expressions, and a Taylor series expansion. While secondary school students have previously undertaken computations of these kinds by hand, because the necessary results could not be obtained in any other way, the work involved has generally been routine and procedural in nature and has not added greatly to the quality of their mathematical thinking. It is important to recognise that sophisticated and routine activities of these kinds can now be accomplished by a few keystrokes on a hand-held device.

The fact that exact computations are available is also of significance, both for numerical work and for symbolic work. Some of the results shown in Figure 4 can be obtained numerically (although not exactly) on graphics calculators, while others can be obtained only on a CAS-capable device.

The examples in Figure 4 have been chosen from many possibilities, to illustrate and support the claim that any of the routine computational needs of secondary school mathematics can be readily obtained on an integrated device like a *ClassPad 300*.

6. An experiential role

While computation is important in mathematics, it is not the main contribution of hand-held technology to teaching and learning mathematics. The experiential role, describing the possibility of students encountering different experience, is arguably of greater significance. In this section, some examples of the ways in which hand-held technologies can offer students new experiences for learning and teachers new ways of teaching are briefly described. A major element is the possibility of provoking students to use technology to explore mathematical ideas for themselves, and thus to support cognitive development and not only procedural skill.

Scientific calculators are less powerful than graphics calculators in this respect, which perhaps accounts for the relatively little impact they have had on thinking about school curricula. The lack of a graphics screen, allowing for various representations of mathematical objects, is a significant limitation. Despite this drawback, scientific calculators can be used in intellectually productive ways, many of which are explored in [12], which contains many detailed ideas, ranging across several areas of mathematics, including algebra, functions, trigonometry, geometry, statistics, calculus and business mathematics. Some of the examples derive from the ability of the calculator to show different representations of numbers, such as fractions and decimals, powers and logarithms. Others derive from alternative approaches on a sophisticated scientific calculator to mathematical topics, such as numerical solution of equations or evaluation of integrals.

For graphics calculators, experiential opportunities are much more plentiful, as the availability of a graphics screen offers students ways of interacting with mathematical ideas that were not available to them prior to the advent of technology. There are very many examples in [14], but space here to include only a few.

Perhaps the most common ways in which graphics calculators offer students new experiential opportunities are those related to the representation of functions. A graphics calculator allows for both graphical and tabular representation of a function, in addition to symbolic representations, as Figure 5 illustrates.

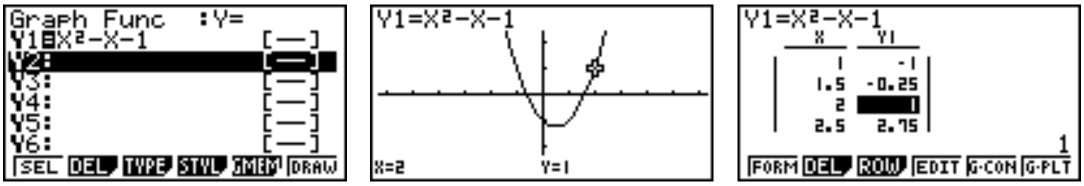


Figure 5: Three representations of a function on a Casio fx-9860G graphics calculator

Students can manipulate and explore these representations in a number of ways in order to understand better the mathematics involved. For example, they can modify the symbolic function and see almost immediately the consequences for the graph, compare the graphs of several functions at once, or a family of functions, to understand the effects of the coefficients, and can learn about functional forms (such as linear, quadratic, cubic) interactively. They can zoom in or out on a graph to study its shape and properties in detail. They can examine at close quarters the numerical values of the function, connecting th graph to the solution of equations or roots or both. They can study the intersections of graphs and connect these to the solutions of equations. They can examine the shape of a graph in detail to encounter ideas of rates of change informally. The experiences offered by these three representations provide both teachers and students with new ways of interacting with the mathematical ideas involved.

A powerful way of using these sorts of capabilities involves the derivative of a function. The idea of a derivative at a point (rather than the slope of a tangent to a function at the point) is well illustrated on the graphics screen shown in Figure 6, using an automatic derivative tracing facility built in to the Casio fx-9860G calculator. Students can get develop their intuitions about the relationship between the derivative and the shape of a graph by seeing how the derivative changes sign and magnitude at various points as the graph is traced.

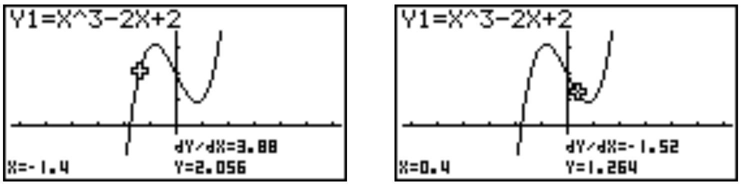


Figure 6: Using a derivative trace to explore the idea of a derivative at a point

The calculator can also be used to represent a derivative function automatically, by evaluating the derivative at each point of a function and graphing the result, as shown in Figure 7. Since both the function and its derivative are represented on the same screen, important connections between these are readily examined by students. In this case, the characteristic parabolic shape of the derivative function is important, as are the observations that the derivative function changes sign near the turning point of the cubic function.

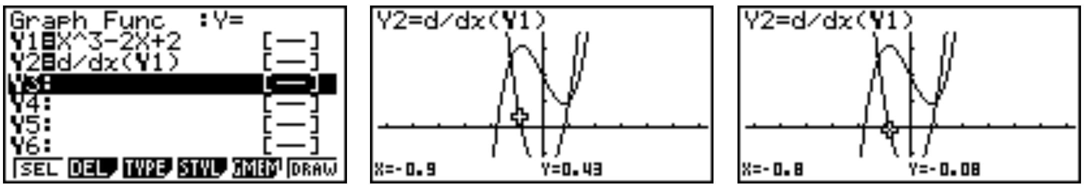


Figure 7: Simultaneous graphical representation of a function and its derivative function

Graphics calculators offer students much more opportunity to explore data and engage in statistical thinking than do scientific calculators, because data are stored in the calculator once entered, and then can be manipulated in a variety of ways. For example, data can be edited to correct errors or omissions, can be transformed (eg with a logarithmic transformation, in order to linearise an exponential relationship), can be represented in graphical displays (such as scatter plots, histograms or box plots), can be compared with ideal mathematical models and can be used to undertake standard statistical tests. Figure 8 shows representative screens for some of these sorts of operations.

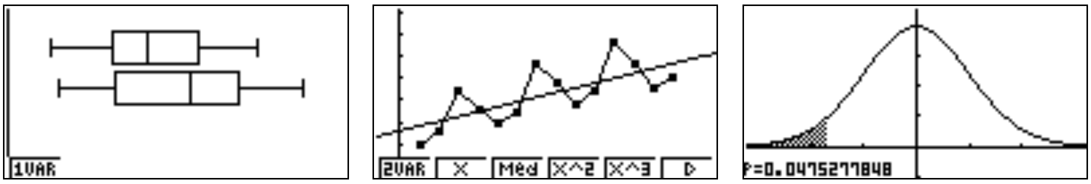


Figure 8: Examples of data analysis activities on a graphics calculator, taken from [14]

Taken together, these sorts of capabilities suggest that a graphics calculator can be regarded as device for exploratory data analysis in a range of flexible ways, supporting both descriptive statistics and inferential statistics, and allowing students an opportunity to develop important data analytic skills and understandings, using their own collected data or those of someone else.

Figure 9 shows examples of using a calculator spreadsheet to show the kinds of interactions with which students can engage in exploring the behaviour of the Fibonacci Sequence. In this case, the ratio of successive terms (in column C) converges quite quickly to φ = 1.61803…, as can be seen from the graph of column C. Students can manipulate spreadsheet elements (such as the starting value of the sequence) to see the (unexpected) effects. Such experimental activity is not practically possible without access to technology.

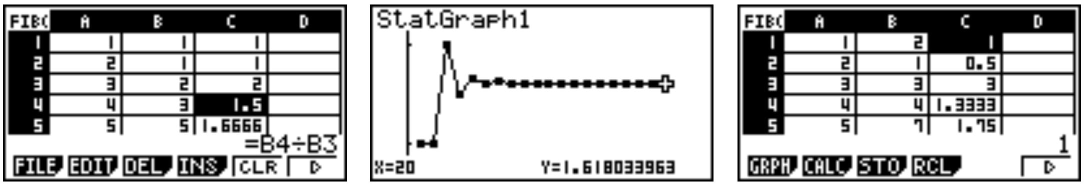


Figure 9: Exploring a series with a graphics calculator

As a geometric example of offering a different experience to students, Figure 10 shows some Casio fx-9860G graphics calculator screens concerned with plane geometry, in particular the idea of a locus. In each of the three screens, D is a point on a circle centred at A with radius AB. C is a point external to the circle and E is the mid-point of CD. When animated, the screen shows the locus of E as D traverses the circle, suggesting visually that the locus is itself a circle. Students can experiment with this situation by moving C (as suggested) or by changing the size or location of the circle; in all cases, they can examine the effects to seek invariants in the locus, with a view to understanding what is happening at a deeper conceptual level.

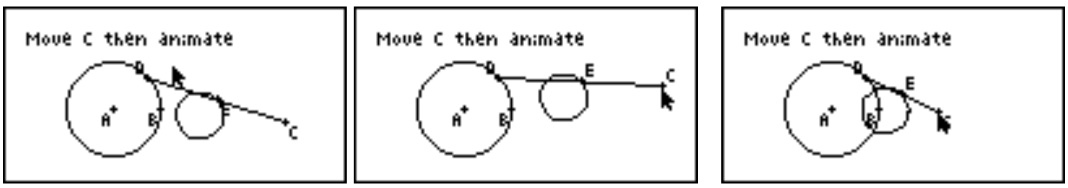


Figure 10: Experiencing geometry with a graphics calculator.

Another kind of opportunity for experimentation is made available through a calculator, using simulation, a powerful tool for both understanding probabilistic situations and for modelling contexts. The sample screens shown in Figure 11 contain histograms of simulated tosses of two dice. Students can readily perform the simulations on a calculator and then graph the results.

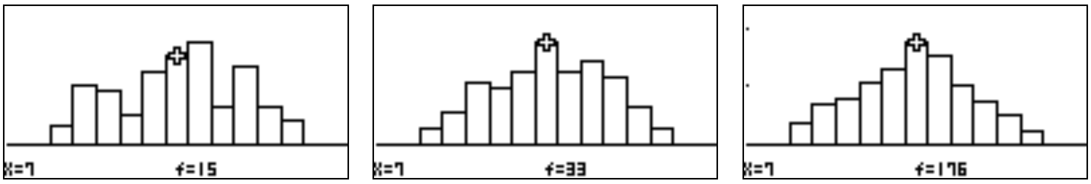


Figure 11: Simulations of the total of two dice thrown 50, 200 and 999 times respectively

While some practical work of this kind is useful, efficiency suggests that technology is needed to provide a good environment for learning about the regularities involved. In this case, a larger number of simulations results in a more symmetrical distribution, and a closer match to the theoretical result obtained on the basis of a probabilistic analysis. In a classroom, the potential for students to compare simulations offers opportunities for learning about the nature of randomness.

Opportunities for new learning experiences are even more plentiful on a more sophisticated device such as a *ClassPad 300*. To give an illustration, Figure 12 shows the reflection of the plane about line AB, with the triangle CDE and its image C′D′E′, both shown.

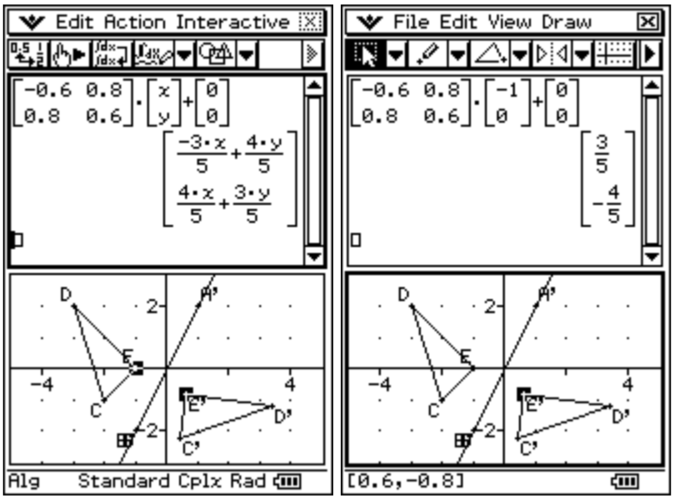


Figure 12: Experiencing reflections on a ClassPad 300

A student can manipulate the elements directly and see the results: moving line AB and changing the triangle CDE will both result in interesting effects. Powerfully, students can also drag a point and its reflection from the lower (geometry) window to the upper window to produce automatically the relevant transformation matrix, shown above. Individual points may be inserted into the general relationship to see the relationship in action; for example, the second screen above shows that the image of E(-1,0) is E′(3/5,-4/5). Such an environment offers new experiences for students (to learn about reflections and linear transformations in this case) that are not available otherwise.

In summary, this section has presented some examples, from a potentially much larger set, to elaborate the notion that hand-held technology can significantly alter the experience of learning mathematics for students, by providing them with access to tasks and opportunities not otherwise available to them. For the same reason, the opportunities for teachers to provide a different experience for students are also significantly changed.

7. Curriculum influences

The availability of hand-held technologies raises important curriculum questions for teaching, learning and assessment of secondary school mathematics, arising from the execution of the previous two roles concerning computation and experience. In this sense, the technology has become an important source of curriculum influence. There are at least three dimensions of this influence, concerned with the place of computation, the choice of mathematical content to be included in the curriculum, and the sequence in which material is presented to students.

Computation

Since most routine computational procedures can now be conducted efficiently on a calculator, curriculum developers need to decide the extent to which it is necessary, or desirable, for students to develop expertise at executing these same procedures by hand, probably less efficiently and less reliably than their calculators. Few would suggest that all procedures ought to be mastered by all students, partly because there is insufficient time likely to be available for this. Alternatively, few would suggest that hand-held technology be relied upon too heavily for computational purposes, since the development of some level of personal computational expertise is widely regarded as an important outcome of mathematics education. The issue instead is one of finding a suitable balance between these opposing views, which is not an easy matter.

Part of the problem of balance, of course, is that there is usually only a fixed amount of time available for mathematics in school; in many countries, even this fixed amount of time is reducing, as other pressures on the school curriculum.. It might be argued that only some students (not all) are expected to develop high levels of personal computational efficiency, or that by-hand expertise is left until later in the curriculum (for all, or for only some) students.

The situation can be illustrated by considering the solution of systems of linear equations, universally included in secondary school mathematics curricula. Consider the following system:

$$\{x - 2y - z = 2;\ 3x + y + 4z = 6;\ 2x - 3z = 7\}$$

Algebraic procedures involving Gaussian elimination are available to solve this system, and students have routinely been taught how to use them in secondary school. Such procedures are powerful, relatively efficient, generalizable and should always produce the correct solution; in addition, refined versions of the procedures using matrix representation and arithmetic are available, rendering the tasks even more efficient, because attention is paid only to the coefficients. On the other hand, the by-hand procedures themselves are complicated, heavily laden with elementary arithmetic and hence fundamentally error-prone. Furthermore, carrying out such procedures offers very little insight into the solution, in most cases. Developing student expertise at this sort of task requires a lot of time in class and time for practice (possibly at home) and considerable motivation and commitment by students. Using curriculum time for such tasks means that the time is not available for other tasks, since the total time available to students is always constrained.

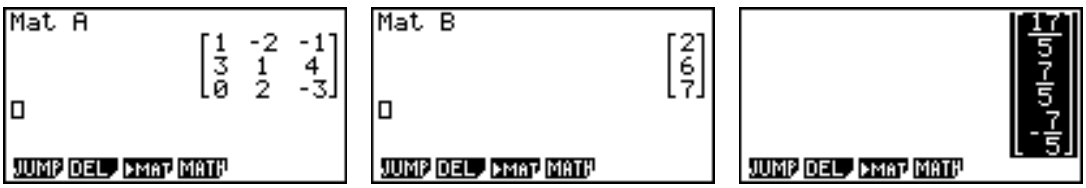


Figure 13: Matrix solution of the system of equations

On a graphics calculator, the matrices of coefficients are readily entered, and readily manipulated to produce a solution, as shown in Figure 13 on the Casio fx-9860G. To do this, students need to know both how a linear system can be represented by a matrix and how to use their calculator to do this. They also need to know how the matrix formulation can be used to represent the solution, and how to obtain this on their calculator, using the inverse of the matrix of coefficients.

The computations involved can be streamlined even further, however, on this graphics calculator, as shown in Figure 14, in which the augmented matrix of coefficients is shown and the student needs merely to execute a *solve* command to see the result, available as both a decimal and a fraction in this case, after scrolling the solution vector. While the same procedures are employed internally in this case and for the matrix formulation, this is not the case for students; this latter version of the calculator solution does not require any knowledge of matrices or their manipulations. Students merely need to recognize the problem as one that involves the solution of a system of three simultaneous linear equations and know how to enter the coefficients faithfully.

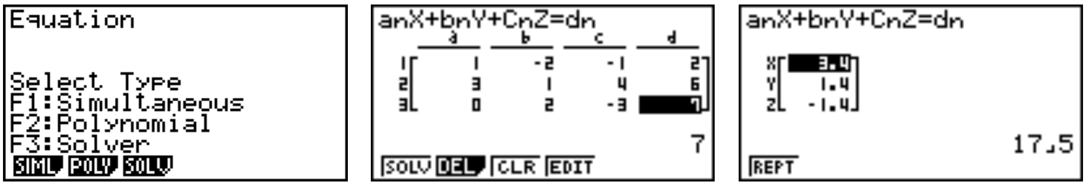


Figure 14: Direct calculator solution of the system of equations

In each of these cases, students need to both know about the idea of a system of linear equations and how to both represent and solve such a system numerically on the calculator. As for developing by-hand algebraic procedures, developing this knowledge also requires the use of curriculum time, although less time is likely to be needed with the technology.

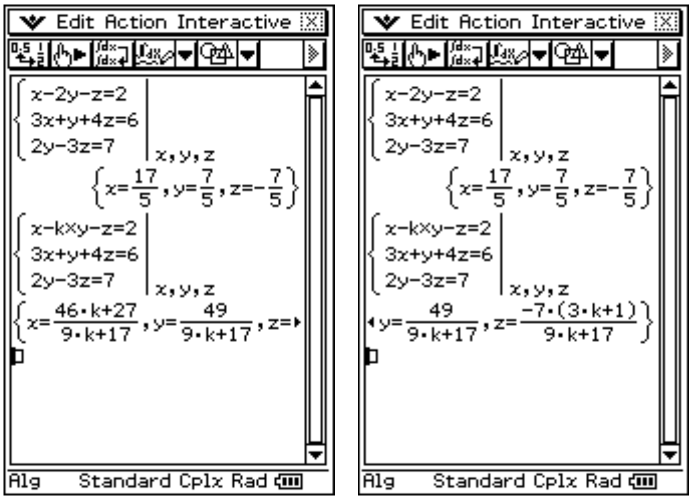


Figure 15: Solving a linear system of equations on the ClassPad 300

It might be argued that the regular use of a calculator to solve a system of linear equations might render students powerless when confronted with a more significant system, such as one that could not be solved numerically. In fact, similar procedures are available on more sophisticated devices, using a computer algebra system. To illustrate this, Figure 15 shows the solution of the linear system above, but with the first equation replaced with $x - ky - z = 2$, with the parameter k replacing the 2 as the coefficient of y in the first equation. (The result on the *ClassPad 300* occupies more then a single line, so a second screen is shown to display the result after horizontal scrolling.)

This example has been chosen to illustrate the influence that hand-held technologies might exert on thinking about the secondary school mathematics curriculum, at least as far as computation is concerned. The fundamental question that needs to be addressed concerns the extent to which students ought to be taught to imitate by hand the vast range of procedures that can be quickly, efficiently and reliably undertaken by a machine. While the answer to such a question is necessarily one of finding the appropriate balance, it seems inevitable that the balance will be shifted somewhat as a result of the influence of the technology.

Content

A second aspect of potential influence concerns the content of the curriculum, which represents an answer to the fundamental question: which aspects of mathematics are important enough to be included in the school curriculum, and which can be safely left until later, or excluded altogether?

The answers to such a question have always varied between countries and over time within a country. The answers depend to some extent of course on the amount of time that can be devoted to school mathematics in the wider school curriculum and the aspirations and interests of the students concerned. Thus, some countries have a range of courses to suit students of different kinds, with varying emphases on various aspects of mathematics, such as algebra, calculus, trigonometry, geometry, statistics, probability, and so on. In addition, there is varying emphasis given to mathematical processes, such as proof, problem solving, mathematical modelling, inductive and deductive reasoning.

Hand-held technologies influence the answers to this sort of question, as they provide access to different aspects of mathematics in a variety of ways. To continue the example of equations explored briefly in the previous section, access to hand-held technologies provides an opportunity to reduce the emphasis on some algebraic manipulations associated with solving equations, and at the same time extend the repertoire of equations and solution methods to which students are exposed. More attention might be paid in secondary school mathematics to numerical methods of solution of a wide range of elementary equations and less attention to exact algebraic methods of solutions of a very small range of elementary equations (mostly linear and quadratic). [11] Exact methods may receive more emphasis later in the curriculum and for only some students.

In a similar way, the calculus curriculum has always required that students develop substantial competence with essentially algebraic procedures, such as differentiation and integration, because progress in using calculus to solve problems requires such expertise. Although students in the past have developed the necessary expertise, for too many students this has taken so much time and energy that the (arguably more important) conceptual development has been less well developed. A good example involves integration, where students have been taught a suite of methods to handle various situations, such as integration by substitution, by parts and by partial factions. While there is some conceptual value in such procedures, their main place in the curriculum is as a means to an end, rather than because of their intrinsic mathematical interest. A hand-held device can evaluate integrals (numerically, symbolically or both) without focusing on the methods of integration employed, offering opportunities to reconsider the balance between the idea of integration and its many uses and the routines associated with evaluating integrals.

In other cases, the nature of school mathematics can be reconsidered because of the availability of hand-held technology. In the statistics curriculum, attention can be directed to statistical thinking and choices by students to represent data in various ways, rather than on the tedium of calculation or construction of graphical representations. A graphics calculator essentially provides students with a suite of data analysis techniques, so that attention might shift towards how, when and why to make use of these, and more attention can be directed at previously neglected aspects such as designing the collection of data to answer questions of interest and interpretation of statistical results.

Sequence

Another form of curriculum influence involves the sequence of ideas in the curriculum, which is also potentially influenced by the availability of hand-held technology. When thinking about the curriculum, under the assumption that technological support is available, some concepts appear earlier than might have been previously expected.

An elementary example of this on a calculator is when a subtraction results in a negative number or a multiplication results in a number expressed in scientific notation (before students have studied these. Similarly, a graphics calculator may produce results which suggest that some mathematical ideas may be introduced into the curriculum earlier than might otherwise have been the case. Two examples are shown in Figure 17. Both the evaluation of the square root of a negative number and the solution of the cubic equation $x^3 - 2x^2 + 2x = 4$ result in complex numbers.

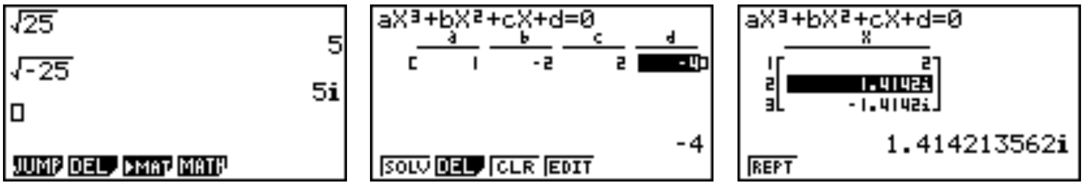


Figure 17: Unexpected appearance of complex numbers on a graphics calculator

More significantly, other aspects of the sequence of the mathematics curriculum may be affected by the availability of hand-held technology. Some of these are concerned with the capacity of a graphics calculator to display graphs of functions and allow them to be interrogated by the person using the calculator. The shapes of various families of functions (such as linear, quadratic, cubic and exponential) will be accessible to students at an earlier stage than previously, and without the necessity of tedious and extensive plotting of points. This might be expected to lead to a consideration of the nature of different kinds of functions earlier than previously.

In a similar vein, having ready access to a graph of a function changes some of the rationale for traditional approaches to the calculus. Before the technology was accessible, early approaches to the calculus focused attention on sketching curves in order to understand their shape, including their asymptotic shape, and also to identify key aspects such as local extrema. Some understanding of differential calculus and some expertise at finding and using derivatives was needed in order to consider such mathematical ideas. Similarly, the concept of the area under a curve was not introduced before the study of integration. However, as the screens in Figure 18 suggest, the mathematical ideas can be extracted from calculator graphs, which are accessible to students long before the study of calculus is introduced.

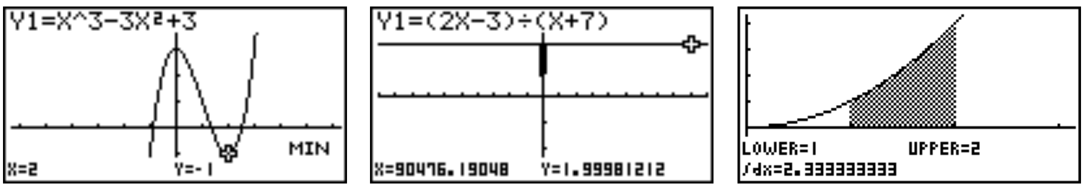


Figure 18: Early graphical introduction to local extrema, asymptotic behaviour and integration

Changes of this kind may serve to help us reconsider the role of the calculus in school: not merely as a means to answering questions about functions, but a means of doing so *exactly*, rather than with numerical approximations. Excellent numerical approximations are available to students well before the formal study of calculus, thus influencing the way in which we regard the calculus itself later in the sequence.

8. Issues for resolution

In this section of the paper, some key issues associated with the use of hand-held technologies in secondary school mathematics are identified and briefly explored.

Integration of technology

A major advantage of hand-held technology over other forms of technology for secondary school mathematics is the possibility of it being integrated into the curriculum, rather than being regarded as an addition of some kind. It is important that there be coherence in the place of technology for each of teaching, learning and assessment; hand-held technology offers the best prospects for this.

In many contexts, such as those in Australia [4], a key aspect of the integration of technology concerns its role in external examinations, especially high stakes examinations at the end of secondary school used for selection into universities. When technology is permitted for use in examinations, it is much more likely to be a part of the teaching and learning practices of schools, for obvious reasons. In the same way, when technology is not permitted in examinations, schools and teachers are understandably reluctant to use it in their teaching and learning activities. The small size, portability, relatively low cost and (perhaps ironically) limited mathematical power together render hand-held technologies much more likely to be permitted into formal assessment programs than, say, computers or the Internet. All of these characteristics enhance the likelihood of coherence and integration.

If technology is integrated into the curriculum, both curriculum developers and textbook manufacturers can develop materials that make sound use of it. Without widespread integration, it seems unlikely that the necessary curriculum changes (some of which were suggested in the previous section) can be seriously undertaken.

An additional aspect of integration concerns teachers, most of whom are unlikely to have extensive experience themselves with hand-held technologies. As noted earlier, graphics calculators are much more likely to be found in secondary schools than universities, so that even recent graduates may not have a lot of experience with using them as learning tools. Integration of technology requires that teachers be supported adequately, as elaborated later in this section.

Curriculum balance

Hand-held technologies of the kinds considered in this paper are frequently misunderstood as having *only* a computational role, especially and ironically, it seems, by people who do not make much use of them. Such a view is understandable, particularly when ‘calculators’ are naturally interpreted as devices whose main purpose is to ‘calculate’. However, this is a rather limited view of the many ways in which learning can be supported, some of which are described earlier with others described elsewhere (eg [14], [4], [5]).

A fundamental issue at stake in this respect concerns the significance of formal algorithms and procedures in mathematics. While none would doubt the importance of these, educators are increasingly questioning the balance of conceptual and procedural thinking in curricula. Extensive memorization and development of by-hand algebraic and arithmetical skills would seem to be both less necessary and less defensible in the opening years of the 21st century than they were a few decades ago, before technologies of the kinds discussed here were first developed.

In a similar way, formal assessment mechanisms, such as external examinations, seem now more likely to encourage conceptual development and careful mathematical thinking than smooth repetition of memorized procedures, so that the changing roles of a calculator to foster learning need to be considered in such a context.

Motivation

Reference has already been made to the role of external examinations as a source of influence, widely recognized in many countries (both East and West) as a key agent in directing the activities of both teachers and their students. When hand-held technologies are integrated into examinations, the motivation to use them thoughtfully and efficiently is considerable.

In addition, many teachers report that hand-held technologies themselves can be intrinsically motivating, as they offer students a responsive environment in which to experiment with mathematical ideas and explore connections between them. Many of today's students are accustomed to environments that are awash with technologies, and consequently are more inclined and less anxious than many of their teachers to experiment with them. As noted earlier, calculator display screens can themselves provoke students to explore new aspects of mathematics, when surprising results are given (such as the new kinds of numbers shown in Figure 17). The powerful range of software built into modern devices offers a platform upon which many interesting learning ideas and activities can be developed to motivate learning. (eg, see [12], [13], [14])

The work of the teacher

Effective use of hand-held technology requires teachers who are themselves competent and confident users of the technologies concerned. This is of critical importance: nothing important changes in mathematics classrooms without the teacher changing in some way. [9] Supporting the professional development of teachers for this task takes both time and effort. While publications such as [14] and websites such as [13] are important components of the professional development involved, experience in Australia suggests that hands-on time in workshops, together with the support of colleagues in a school are also key elements [4], [5].

By their very nature, hand-held technologies are personal devices that lend themselves to individual work or shared work among two or three students. Their use thus has some implications for pedagogy, with more emphasis on individual and small group work in a classroom than on whole-of-class instruction. Many teachers need help to develop expertise in such an environment, which differs from many traditional ways of teaching mathematics.

Of course, there continues to be a place for whole class instruction. A means of projecting a calculator for the whole class to see is useful; both overhead projection panels and emulators displayed via a computer projector can be used productively in classrooms. Some classroom time needs to be devoted to making sure that students can use their technology well, including thoughtfully deciding when not to use it at all. Similarly, productive class discussions can be provoked by a calculator display visible to all members of a secondary school classroom at once.

An interesting and recent development offers an opportunity to creatively mix both individual and collective work, through the connection of individual student graphics calculators to a computer projector, or through networking of calculators, so that the work of one student can be the subject of discussion by other students in a class, or engaged with by the teacher directly.

Changing the curriculum

Finally, it has often been recognized that it is very difficult to bring about deep change in mathematical curricula. There are many understandable reasons for this: a natural conservatism of teachers, especially older teachers; a reluctance to remove from the curriculum anything that the current generation of university and high school teachers themselves learned as students; the difficulties and anxieties associated with adjusting well-developed and well-understood practice; the reluctance to risk existing practices for new alternatives that have not yet stood the test of time; the extreme difficulty many teachers have to find any time to acquire new skills and take on new challenges, in the context of doing a job that is already very demanding; the ever-present threat in many countries of external examinations, which discourage adventurous and innovative teaching practices. In such circumstances, and with the history of curriculum development in mind, we should be cautious about expecting too much curriculum change too quickly.

Alternatively, if curricula do not undergo a process of gradual change, they risk a process of having to make very large changes periodically, which is much more difficult. Hand-held technologies offer an opportunity for teachers themselves to explore some new opportunities for teaching and learning in their classrooms, rather than relying entirely on external influences, such as curriculum and assessment authorities. It is clear that many teachers have found this opportunity valuable [eg 4], laying important groundwork for others to change and for larger-scale changes to be contemplated. From the perspective of curriculum developers, there seems to be more likelihood that technology can bring about curriculum change if it takes the form of hand-held technology than other variations, as these seem to have more prospects of being widely accessible. It seems most unlikely that large-scale curriculum change is possible, at least as far as technology is concerned, unless universal access to the technology can be contemplated. This probably explains why it is easier to change curricula on a local level (such as a single classroom, a single teacher, or a single school) than on a global level (such as a school district, a province or a country).

9. Conclusions

In this paper, we have surveyed the place of hand-held technologies in secondary mathematics education, relying on an analysis of the devices themselves, their mathematical capabilities and some educational consequences of using them. If secondary mathematics education is to be responsive to the changing technological world of the 21st century, there appear to be good prospects for using these kinds of devices to do so. While this is already the case in developed and affluent countries such as Australia and the USA, the arguments seem also to be relevant to less affluent and less developed communities, including many countries in Asia, Africa and South America.

By its nature, curriculum change is slow–much slower than technological change; however, hand-held technologies offer considerable promise to support sound curriculum change, provided teachers are given enough help and attention is paid to the curriculum constraints that militate against change, such as the expectations of universities and high stakes external examination systems. They continue to offer more hope for real change than do some more sophisticated technologies.

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